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2-Dimensional complete self-shrinkers in $\mathbf{R}^3$

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Abstract It is our purpose to study complete self-shrinkers in Euclidean space. By making use of the generalized maximum principle for $\mathcal{L}$-operator, we give a complete classification for 2-dimensional complete self-shrinkers with constant squared norm of the second fundamental form in $\mathbb{R}^3$. Ding and Xin (Trans Am Math Soc 366:5067–5085, 2014) have proved this result under the assumption of polynomial volume growth, which is removed in our theorem.

Keywords Mean curvature flow · Complete self-shrinkers · The generalized maximum principle

Mathematics Subject Classification 53C44 · 53C40

1 Introduction

Let $X : M^n \to \mathbb{R}^{n+1}$ be an n-dimensional hypersurface in the $n+1$-dimensional Euclidean space $\mathbb{R}^{n+1}$. If the position vector X evolves in the direction of the mean curvature H, then

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it gives rise to a solution to mean curvature flow:

$$X(\cdot, t) : M^n \to \mathbb{R}^{n+1}$$

satisfying $X(\cdot, 0) = X(\cdot)$ and

$$\frac{\partial X(p,t)}{\partial t} = H(p,t), \quad (p,t) \in M \times [0, T), \tag{1.1}$$

where $H(p,t)$ denotes the mean curvature vector of hypersurface $M_t = X(M^n, t)$ at point $X(p,t)$. The Eq. (1.1) is called the mean curvature flow equation. The study of the mean curvature flow from the perspective of partial differential equations commenced with Huisken's paper [10] on the flow of convex hypersurfaces.

One of the most important problems in the mean curvature flow is to understand the possible singularities that the flow goes through. A key starting point for singularity analysis is Huisken's monotonicity formula because the monotonicity implies that the flow is asymptotically self-similar near a given type I singularity and thus, is modeled by self-shrinking solutions of the flow.

An n-dimensional hypersurface $X : M \to \mathbb{R}^{n+1}$ in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ is called a self-shrinker if it satisfies

$$H + \langle X, N \rangle = 0,$$

where H and N denote the mean curvature and the unit normal vector of the hypersurface, respectively. It is known that self-shrinkers play an important role in the study of the mean curvature flow because they describe all possible blow up at a given singularity of the mean curvature flow. For classifications of complete self-shrinkers, Abresch and Langer [1], Huisken [11,12] and Colding and Minicozzi [6] have obtained very important results. In fact, Abresch and Langer [1] classified closed self-shrinker curves in $\mathbb{R}^2$ and showed that the round circle is the only embedded self-shrinkers. Huisken [11,12] and Colding and Minicozzi [6] have proved that if $X : M \to \mathbb{R}^{n+1}$ is an n-dimensional complete embedded self-shrinker in $\mathbb{R}^{n+1}$ with $H \geq 0$ and with polynomial volume growth, then $X : M \to \mathbb{R}^{n+1}$ is isometric to either $\mathbb{R}^n$, the round sphere $S^n(\sqrt{n})$, or a cylinder $S^m(\sqrt{m}) \times \mathbb{R}^{n-m}$, $1 \leq m \leq n-1$.

A natural problem is whether there exist complete self-shrinkers with volume growth faster than polynomial growth. In fact, the theorem 5.1 in [9] by Halldorsson has proved that there exist complete self-shrinker curves Γ in $\mathbb{R}^2$, which is contained in an annulus around the origin and whose image is dense in the annulus. On the other hand, Ding and Xin [7] and Cheng and Zhou [5] have proved that a complete self-shrinker has polynomial volume growth if and only if it is proper. Since these complete self-shrinker curves Γ are not proper, for any integer $n > 0$, $\Gamma \times \mathbb{R}^{n-1}$ is a complete self-shrinker in $\mathbb{R}^{n+1}$, which does not have polynomial volume growth. Thus, we have the following:

Proposition 1.1 *For any integer $n > 0$, there exist n-dimensional complete self-shrinkers without polynomial volume growth in $\mathbb{R}^{n+1}$.*

In [2], Cao and Li have proved that if an n-dimensional complete self-shrinker $X : M \to \mathbb{R}^{n+1}$ with polynomial volume growth satisfies $S \leq 1$, then $X : M \to \mathbb{R}^{n+1}$ is isometric to either $\mathbb{R}^n$, the round sphere $S^n(\sqrt{n})$, or a cylinder $S^m(\sqrt{m}) \times \mathbb{R}^{n-m}$, $1 \leq m \leq n-1$, where S denotes the squared norm of the second fundamental form (cf. [14]). Furthermore, Ding and Xin [8] have studied 2-dimensional complete self-shrinkers with polynomial volume growth and with constant squared norm of the second fundamental form (cf. [4] and [8] for any dimension). They have proved.

Theorem DX *Let $X : M \to \mathbb{R}^3$ be a 2-dimensional complete self-shrinker with polynomial volume growth in $\mathbb{R}^3$. If the squared norm S of the second fundamental form is constant, then $X : M \to \mathbb{R}^3$ is isometric to one of the following:*

(1) $\mathbb{R}^2$,
(2) *a cylinder* $S^1(1) \times \mathbb{R}$,
(3) *the round sphere* $S^2(\sqrt{2})$.

In this paper, we want to remove the assumption of polynomial volume growth in the above theorem of Ding and Xin and to prove that the above result of Ding and Xin holds by making use of a different method.

Theorem 1.1 *Let $X : M \to \mathbb{R}^3$ be a 2-dimensional complete self-shrinker in $\mathbb{R}^3$. If the squared norm S of the second fundamental form is constant, then $X : M \to \mathbb{R}^3$ is isometric to one of the following:*

(1) $\mathbb{R}^2$,
(2) *a cylinder* $S^1(1) \times \mathbb{R}$,
(3) *the round sphere* $S^2(\sqrt{2})$.

2 Proof of Theorem 1.1

Let $X : M^2 \to \mathbb{R}^3$ be a 2-dimensional surface in $\mathbb{R}^3$. We choose a local orthonormal frame field $\{e_A\}_{A=1}^3$ in $\mathbb{R}^3$ with dual co-frame field $\{\omega_A\}_{A=1}^3$, such that, restricted to M^2, e_1, e_2 are tangent to M^2. Hence, we have

$$dX = \sum_{i=1}^{2} \omega_i e_i, \quad de_i = \sum_{j=1}^{2} \omega_{ij} e_j + \omega_{i3} e_3.$$

We restrict these forms to M^2, then

$$\omega_3 = 0 \tag{2.1}$$

and

$$\omega_{i3} = \sum_{j=1}^{2} h_{ij} \omega_j, \quad h_{ij} = h_{ji},$$

where h_{ij} denote components of the second fundamental form of $X : M^2 \to \mathbb{R}^3$. Take e_1, e_2 such that, at any fixed point,

$$h_{ij} = \lambda_i \delta_{ij},$$

where λ_1 and λ_2 are the principal curvatures of $X : M^2 \to \mathbb{R}^3$. Thus, the Gauss curvature K and the mean curvature H are given by

$$K = \lambda_1 \lambda_2, \quad H = \lambda_1 + \lambda_2.$$

For a smooth function f, the $\mathcal{L}$-operator is defined by

$$\mathcal{L} f = \Delta f - \langle X, \nabla f \rangle, \tag{2.2}$$

where Δ and ∇ denote the Laplacian and the gradient operator on the self-shrinker, respectively. In order to prove our results, the following generalized maximum principle for $\mathcal{L}$-operator on self-shrinkers is very important, which is proved by Cheng and Peng in [3]:

Lemma 2.1 (Generalized maximum principle for $\mathcal{L}$-operator) *Let $X : M^n \to \mathbb{R}^{n+p}$ $(p \geq 1)$ be a complete self-shrinker with Ricci curvature bounded from below. Let f be any C^2-function bounded from above on this self-shrinker. Then, there exists a sequence of points $\{p_k\} \subset M^n$, such that*

$$\lim_{k\to\infty} f(X(p_k)) = \sup f, \quad \lim_{k\to\infty} |\nabla f|(X(p_k)) = 0, \quad \limsup_{k\to\infty} \mathcal{L} f(X(p_k)) \leq 0.$$

Proof of Theorem 1.1 Since $X : M^2 \to \mathbb{R}^3$ is a complete self-shrinker, we have

$$H + \langle X, N\rangle = 0. \tag{2.3}$$

By a simple calculation, we have

$$\frac{1}{2}\mathcal{L}S = \sum_{i,j,k} h_{ijk}^2 + S(1-S),$$

where $S = \sum_{i,j=1}^2 h_{ij}^2$ is the squared norm of the second fundamental form and h_{ijk} denote components of the first covariant derivative of the second fundamental form. Since S is constant, we have

$$\sum_{i,j,k} h_{ijk}^2 + S(1-S) = 0. \tag{2.4}$$

If $S = 1$, then we know $h_{ijk} \equiv 0$. Hence, $X : M^2 \to \mathbb{R}^3$ is isometric to the round sphere $S^2(\sqrt{2})$ or the cylinder $S^1(1)\times\mathbb{R}$ from the results of Lawson [13]. If $S < 1$, from the theorem of Cheng and Peng [3], we know that $X : M^2 \to \mathbb{R}^3$ is isometric to $\mathbb{R}^2$.

Next, we prove that $S \leq 1$ holds. By a direct computation, we have

$$\frac{1}{2}\mathcal{L}|X|^2 = 2 - |X|^2. \tag{2.5}$$

Since S is constant, we know that the Gauss curvature satisfies

$$K = \lambda_1\lambda_2 \geq -\frac{\lambda_1^2 + \lambda_2^2}{2} = -\frac{S}{2}.$$

Therefore, the Gauss curvature is bounded from below. Since $-|X|^2 \leq 0$ is bounded from above, we can apply the generalized maximum principle for $\mathcal{L}$-operator to the function $-|X|^2$. Thus, there exists a sequence $\{p_k\}$ in M^2 such that

$$\lim_{k\to\infty} |X|^2(p_k) = \inf |X|^2, \quad \lim_{k\to\infty} |\nabla |X|^2(p_k)| = 0, \quad \liminf_{k\to\infty} \mathcal{L}|X|^2(p_k) \geq 0. \tag{2.6}$$

From (2.5) and (2.6), we have

$$\inf |X|^2 \leq 2. \tag{2.7}$$

Since $|\nabla|X|^2| = \sum_{i=1}^2 \langle X, e_i\rangle^2$ holds, we have from (2.6)

$$\lim_{k\to\infty} |\nabla |X|^2(p_k)| = \lim_{k\to\infty} \sum_{i=1}^2 \langle X, e_i\rangle^2(p_k) = 0.$$

Hence, we get from (2.3)

$$\inf |X|^2 = \lim_{k\to\infty} H^2(p_k), \quad \lim_{k\to\infty} |\nabla H|(p_k) = 0. \tag{2.8}$$

Since S is constant, from the definition of the mean curvature H and (2.3), we obtain, for $j=1,2$,

$$\begin{aligned}&\lim_{k\to\infty}\big(h_{11j}(p_k)+h_{22j}(p_k)\big)=0,\\&\lim_{k\to\infty}\big(\lambda_1(p_k)h_{11j}(p_k)+\lambda_2(p_k)h_{22j}(p_k)\big)=0.\end{aligned}\tag{2.9}$$

Since S is constant and from (2.4), we know that $\{\lambda_j(p_k)\}$ and $\{h_{iij}(p_k)\}$ are bounded sequences. Thus, we can assume

$$\lim_{k\to\infty} h_{iij}(p_k)=\bar h_{iij},\quad \lim_{k\to\infty}\lambda_j(p_k)=\bar\lambda_j,$$

for $i,j=1,2$. From (2.9), we obtain

$$\begin{cases}\bar h_{11j}+\bar h_{22j}=0,\\ \bar\lambda_1\bar h_{11j}+\bar\lambda_2\bar h_{22j}=0.\end{cases}\tag{2.10}$$

If $\bar\lambda_1\neq\bar\lambda_2$ is satisfies, according to (2.10), we infer

$$\bar h_{iij}=0$$

for $i,j=1,2$. According to Codazzi equations, we have

$$\sum_{i,j,k}\bar h_{ijk}^2=0.$$

From (2.4), we have $S=1$ or $S=0$. Hence $S\le 1$.

If $\bar\lambda_1=\bar\lambda_2$ holds, we have

$$S=\bar\lambda_1^2+\bar\lambda_2^2=\frac{(\bar\lambda_1+\bar\lambda_2)^2}{2}=\frac{\lim_{k\to\infty}H^2(p_k)}{2}.$$

According to (2.7) and (2.8), we have

$$S\le 1.$$

Hence, $S=0$ or $S=1$. According to the theorem of Lawson [13], we know that M^n is isometric to the round sphere $S^2(\sqrt{2})$, the cylinder $S^1(1)\times\mathbb{R}$ or $\mathbb{R}^2$. $\square$

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