

COMPLETE HARMONIC STABLE MINIMAL HYPERSURFACES IN A RIEMANNIAN MANIFOLD

QING-MING CHENG¹ AND YOUNG JIN SUH²

Dedicated to Professor Susumu Ishikawa for his 60th birthday

ABSTRACT. In this paper, we will introduce the notion of harmonic stability for complete minimal hypersurfaces in a complete Riemannian manifold. The first result we proved is that a complete harmonic stable minimal surface in a Riemannian manifold with non-negative Ricci curvature is conformally equivalent to either a plane \mathbf{R}^2 or a cylinder $\mathbf{R} \times S^1$, which generalizes a theorem due to Fischer-Colbrie and Schoen [12].

The second one is that an $n \geq 2$ -dimensional complete harmonic stable minimal hypersurface M in a complete Riemannian manifold with non-negative sectional curvature has only one end if M is non-parabolic. The third one which we have proved is that there exist no non-trivial L^2 -harmonic one forms on a complete harmonic stable minimal hypersurface in a complete Riemannian manifold with non-negative sectional curvature. Since the harmonic stability is weaker than stability, we obtain a generalization of a theorem due to Miyaoka [20] and Palmer [21].

1. INTRODUCTION

The investigation of complete minimal immersed hypersurfaces in a Riemannian manifold has flourished in the last century and a much better understanding of their global geometric and topological structures has been obtained. In the following, we will agree that a minimal hypersurface means an oriented minimal immersed hypersurface. The classical Bernstein theorem asserts that an entire minimal graph in a 3-dimensional Euclidean space \mathbf{R}^3 must be a plane. As a generalization of

2000 Mathematics Subject Classification 53C42

Key words and phrases: a complete minimal hypersurface, Ricci curvature, sectional curvature, L^2 -harmonic 1-form, harmonic index, harmonic stability and stability.

¹ Research partially Supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology, Japan.

² The author's research was supported by grant Proj. No. R14-2002-003-01001-0 from Korea Research Foundation, Korea 2006.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

Bernstein theorem, Almgren [1], De Giorgi [9], Fleming [13] and Simons [25] proved that an entire n -dimensional minimal graph in an $n+1$ -dimensional Euclidean space \mathbf{R}^{n+1} ($n \leq 7$) must be a hyperplane. But, for $n \geq 8$, Bombieri, De Giorgi and Guisti [3] found nonlinear entire minimal graphs in \mathbf{R}^{n+1} .

Since minimal graphs are area-minimizing, it is natural to consider stable minimal hypersurfaces in \mathbf{R}^{n+1} . It is well known that do Carmo and Peng [10], in 1979, proved that a complete stable minimal surface in \mathbf{R}^3 must be a plane (cf. Pogorelov [22]). At the same time, Fischer-Colbrie and Schoen [12], independently, showed that a complete stable minimal surface M in a complete 3-dimensional Riemannian manifold N with nonnegative scalar curvature must be either conformally a plane or conformally a cylinder $\mathbf{R} \times S^1$. In particular, when $N = \mathbf{R}^3$, they obtained that M must be a plane.

On the other hand, from the generalized Bernstein theorem, it is asked by Yau [29] whether one can prove that a complete stable minimal hypersurface in \mathbf{R}^{n+1} ($n \leq 7$) is a hyperplane. Although much hard work on this problem was done, it remains still open. For example, Shen and Zhu [24] proved that a complete stable minimal hypersurface in \mathbf{R}^{n+1} with finite total curvature must be a hyperplane. The first author and Wan in [8] proved that a complete minimal hypersurface in \mathbf{R}^4 with constant scalar curvature is a hyperplane.

In 1997, Cao, Shen and Zhu [4] found a topological obstruction for complete stable minimal hypersurfaces in \mathbf{R}^{n+1} , namely, they proved that a complete stable minimal hypersurface in \mathbf{R}^{n+1} ($n \geq 3$) must have only one end. Here we should remark that the condition $n \geq 3$ is essential in the proof of their theorem. Furthermore, Li and Wang [17] generalized their theorem to complete minimal hypersurfaces with finite index in \mathbf{R}^{n+1} . They also gave an estimate of the number of ends of such hypersurfaces. But their estimate of the number of ends depends on the geometric structure on a compact subset in the hypersurface.

In [19], Mei and Xu introduced a notion of harmonic stability and harmonic index for a complete minimal hypersurface in \mathbf{R}^{n+1} . For a complete minimal hypersurface in \mathbf{R}^{n+1} , ($n \geq 3$), they proved that the condition of harmonic stability is weaker

than the condition of stability. Under the weaker condition, they proved that the theorem of Cao, Shen and Zhu still holds.

In fact, they proved that the number of ends for an n -dimensional, ($n \geq 3$), complete minimal hypersurface M with finite harmonic index $h(M)$ in \mathbf{R}^{n+1} is not bigger than $h(M) + 1$. Thus, we know that, in order to give an estimate of the number of ends, the harmonic index $h(M)$ is of advantage to the index.

We should remark that the condition $n \geq 3$ is essential in the proof of theorems due to Mei and Xu. Though in [19] they want to use the Sobolev type inequality, but for $n = 2$ the Sobolev type inequality does not hold again. On the other hand, it is natural to extend the theorems of do Carmo and Peng [10] and Fischer-Colbrie and Schoen [12] to the case of harmonic stability.

In this paper, we introduce the notion of harmonic stability for complete minimal hypersurfaces in a complete Riemannian manifold. In section 3, we will consider the case $n = 2$ and generalize a theorem due to do Carmo and Peng [10] and Fischer-Colbrie and Schoen [12] to the case of harmonic stability. Moreover, in section 5, we will study its general case and prove that a complete harmonic stable minimal hypersurface M in a complete Riemannian manifold with non-negative sectional curvature have only one end if M is non-parabolic.

When the ambient manifold is \mathbf{R}^{n+1} , from [19](see also [4]) the harmonic stability yields that M is non-parabolic. Moreover, in section 4, we will prove that there does not exist any non-trivial L^2 -harmonic one forms on a complete harmonic stable minimal hypersurface in a complete Riemannian manifold with non-negative sectional curvature. Since the harmonic stability is weaker than stability, we also obtain a generalization of a theorem due to Miyaoka [20] and Palmer [21].

2. PRELIMINARIES

Let M be a minimal hypersurfaces in a complete Riemannian manifold N^{n+1} . We define a bilinear form as follows:

$$(2.1) \quad I(X, Y) = \int_M \{S\langle X, Y \rangle + \text{Ric}_N(\nu, \nu)\langle X, Y \rangle - \langle \nabla X, \nabla Y \rangle\} dM,$$

for any $X, Y \in \Gamma_c(TM)$, which is a set of tangent vector fields with compact support in M , where S denotes the squared norm of the second fundamental form

A , ∇ is the induced connection, $\text{Ric}_N(\nu, \nu)$ denotes the Ricci curvature of N in the direction of the unit normal vector ν to M . The harmonic index of M , which is denoted by $h(M)$, is defined as the maximal dimension of the vector spaces on which $I(\cdot, \cdot)$ is positive definite. If the harmonic index $h(M)$ is zero, M is called harmonic stable.

A minimal hypersurface M in a complete Riemannian manifold N^{n+1} is called stable if

$$(2.2) \quad \int_M \{Sf^2 + \text{Ric}_N(\nu, \nu)f^2\}dM \leq \int_M |\nabla f|^2 dM,$$

holds for any $f \in C_0^\infty(M)$ (cf. Schoen and Yau [23]).

When the ambient space $N^{n+1} = \mathbf{R}^{n+1}$, Mei and Xu in [19] proved that a complete stable minimal hypersurface must be harmonic stable. In the following, we shall prove that the assertion is also true for any ambient space. Although the proof is similar to one in [19], for completeness, we also write it out.

Proposition 1. *An n -dimensional complete stable minimal hypersurface M in a complete Riemannian manifold N^{n+1} is harmonic stable.*

Proof. If M is not harmonic stable, then there exists a vector field $X \in \Gamma_c(TM)$ such that

$$(2.3) \quad \int_M \{S|X|^2 + \text{Ric}_N(\nu, \nu)|X|^2\}dM > \int_M |\nabla X|^2 dM$$

is satisfied. Let $\epsilon > 0$ be a positive real number. Since X has compact support, we know that there exist a $r > 0$ such that $\text{supp} X \subset B_p(r)$, where $p \in \text{supp} X$ is a fixed point and $B_p(r)$ is a geodesic ball with radius r centred at p . We choose a smooth cut off function φ such that

$$(2.4) \quad \begin{cases} \varphi = 1, & \text{in } B_p(r), \\ \varphi = 0, & \text{in } M \setminus B_p(r+1), \\ |\nabla \varphi| \leq 2, & \text{on } M. \end{cases}$$

We consider a function $f_\epsilon = \varphi(|X|^2 + \epsilon)^{\frac{1}{2}} \in C_0^\infty(M)$. Since M is stable, we have

$$(2.5) \quad \int_M \{Sf_\epsilon^2 + \text{Ric}_N(\nu, \nu)f_\epsilon^2\}dM \leq \int_M |\nabla f_\epsilon|^2 dM$$

holds. From

$$(2.6) \quad \nabla f_\epsilon = \nabla \varphi(|X|^2 + \epsilon)^{\frac{1}{2}} + \frac{1}{2} \varphi \frac{\nabla |X|^2}{(|X|^2 + \epsilon)^{\frac{1}{2}}},$$

we have

$$(2.7) \quad |\nabla f_\epsilon|^2 = |\nabla \varphi|^2(|X|^2 + \epsilon) + \varphi \nabla \varphi \cdot \nabla |X|^2 + \frac{1}{4} \frac{(\varphi \nabla |X|^2)^2}{(|X|^2 + \epsilon)}.$$

Hence,

$$\begin{aligned} 0 &\geq \int_M \{S(|X|^2 + \epsilon)\varphi^2 + (|X|^2 + \epsilon)\varphi^2 \text{Ric}_N(\nu, \nu)\}dM \\ &\quad - \int_M (|X|^2 + \epsilon)|\nabla \varphi|^2 dM - \int_M \varphi \nabla \varphi \cdot \nabla |X|^2 dM - \int_M \frac{1}{4} \frac{(\varphi \nabla |X|^2)^2}{(|X|^2 + \epsilon)} dM \\ &\geq \int_M \{S|X|^2 \varphi^2 + |X|^2 \varphi^2 \text{Ric}_N(\nu, \nu)\}dM - \int_M \varphi^2 |\nabla X|^2 dM \\ &\quad + \epsilon \int_M \{S\varphi^2 + \varphi^2 \text{Ric}_N(\nu, \nu)\}dM - \epsilon \int_M |\nabla \varphi|^2 dM. \end{aligned}$$

Here we used $|X|^2 |\nabla \varphi|^2 = 0$ from the definition of φ and the inequality $\langle \nabla X, X \rangle \leq |\nabla X| |X|$. Since ϵ is arbitrary and $\varphi = 1$ on support set of X , by taking sufficiently small $\epsilon > 0$, we have, from (2.3),

$$\begin{aligned} &\int_M \{S|X|^2 + |X|^2 \text{Ric}_N(\nu, \nu)\}dM - \int_M |\nabla X|^2 dM \\ &\quad + \epsilon \int_M \{S\varphi^2 + \varphi^2 \text{Ric}_N(\nu, \nu)\}dM - \epsilon \int_M |\nabla \varphi|^2 dM > 0. \end{aligned}$$

Thus, it is a contradiction. Therefore, we infer that M is harmonic stable. \square

Let M be a complete Riemannian manifold. For two curves $c_1, c_2 : [0, \infty) \rightarrow M$, they are called cofinal if, for every compact set $K \subset M$, there exists some $t > 0$ such that $c_1(t_1)$ and $c_2(t_2)$ lie in the same connected component of $M \setminus K$ for all $t_1 > t$ and $t_2 > t$. An end of M is defined as an equivalent class of cofinal curves.

An end E of a complete Riemannian manifold M is called non-parabolic if E admits a positive Green's function with Neumann boundary condition (see Li and Tam [15]). From an assertion of Li and Tam, we know that an end E is non-parabolic if and only if there exists a non-constant bounded harmonic function on E with its infimum occurring at infinity.

The following Proposition 2 will be used in order to prove our theorem 5.1 . Its proof can be found in [18].

Proposition 2. *Let M be an n -dimensional complete Riemannian manifold and F is an end of M given by an unbounded connected component of $M \setminus B_p(1)$. Assume that f is a subharmonic function defined on F which does not achieve its maximum on ∂F . Let us define $F(r) = B_p(r) \cap F$ and $s(r) = \sup_{\partial B_p(r) \cap F} f$. For any sequence $\{r_i\}$ with $r_i \rightarrow \infty$, there exist a subsequence, which is also denoted by $\{r_i\}$, and a sequence of positive constants $\{c_i\}$ such that the solutions $\{u_i\}$ to the boundary value problem*

$$(2.8) \quad \begin{cases} \Delta u_i = 0, & \text{in } F(r_i), \\ u_i = 0, & \text{on } \partial F, \\ u_i = c_i, & \text{on } \partial F(r_i) \setminus \partial F, \end{cases}$$

converges to a positive harmonic function u on compact subsets of F with boundary value $u = 0$ on ∂F . Furthermore, the sequence $\{c_i\}$ is bounded by $0 < c_i \leq Cs(r_i)$ for some constant $0 < C < \infty$ and

$$(2.9) \quad \int_{F(r_i)} |\nabla u_i|^2 dM = c_i.$$

Let N^{n+1} be a complete Riemannian manifold with non-negative sectional curvature. For a fixed point $p \in N$, let $\gamma : [0, \infty) \rightarrow N$ be a normal geodesic ray emanating from p . The Busemann function β_γ with respect to γ is defined by

$$(2.10) \quad \beta_\gamma(x) = \lim_{t \rightarrow \infty} (t - d(x, \gamma(t))).$$

We know that β_γ is bounded. The Busemann function β with respect to the point p is defined by

$$(2.11) \quad \beta(x) = \sup_{\gamma} \beta_\gamma(x).$$

It is well known that the Busemann function $\beta(x)$ is a convex exhaustion function of N and satisfies $|\nabla\beta| \leq 1$.

Let M be an n -dimensional minimal hypersurface in N^{n+1} , then the restriction of the Busemann function β onto M is a subharmonic function with respect to the induced metric (See Li and Wang [18]).

The following Lemma 1 will be used many time in order to prove our theorems in this paper (See Cheng and Nakagawa [7]).

Lemma 1. *Let M be an n -dimensional hypersurface in a Riemannian manifold N^{n+1} . Then, at any point $p \in M$, for any unit vector $v \in T_pM$, we have*

$$\begin{aligned} Ric_M(v, v) &\geq \sum_{\alpha=2}^n K_N(v, e_\alpha) \\ &+ 2(n-1)H^2 - \frac{n-1}{n}S - \frac{n-2}{n}\sqrt{\frac{n-1}{n}}\sqrt{n^2H^2(S - nH^2)}, \end{aligned}$$

where H and S denote the mean curvature and the squared norm of the second fundamental form A of M , respectively, and $\{e_1 = v, e_2, \dots, e_n\}$ is an orthonormal frame of T_pM and K_N denotes the sectional curvature of the ambient manifold N^{n+1} .

Proof. Since M is a hypersurface, at any point $p \in M$, we can choose an orthonormal frame $\{e_1, e_2, \dots, e_n\}$ such that

$$h_{ij} = \lambda_i \delta_{ij},$$

where h_{ij} 's are components of the second fundamental form of M and λ_i 's denote principal curvatures of M . Thus, for any j , we have

$$(2.12) \quad h_{jj}nH - \sum_{i=1}^n h_{ij}h_{ji} = nH\lambda_j - \lambda_j^2.$$

From $\sum_{j=1}^n (\lambda_j - H) = 0$ and $\sum_{j=1}^n (\lambda_j - H)^2 = S - nH^2$, we have, for any j ,

$$(2.13) \quad (\lambda_j - H)^2 \leq \frac{n-1}{n} (S - nH^2).$$

Thus, we infer

$$(2.14) \quad \begin{aligned} & \lambda_j^2 - nH\lambda_j \\ &= (\lambda_j - H)^2 - (n-2)H(\lambda_j - H) - (n-1)H^2 \\ &\leq -2(n-1)H^2 + \frac{n-1}{n}S + \frac{n-2}{n}\sqrt{\frac{n-1}{n}}\sqrt{n^2H^2(S - nH^2)}. \end{aligned}$$

Hence, we have

$$(2.15) \quad \begin{aligned} & h_{jj}nH - \sum_{i=1}^n h_{ij}h_{ji} \\ &\leq -2(n-1)H^2 + \frac{n-1}{n}S \\ &\quad + \frac{n-2}{n}\sqrt{\frac{n-1}{n}}\sqrt{n^2H^2(S - nH^2)}. \end{aligned}$$

For any unit vector $v \in T_pM$, choosing an orthonormal frame $\{e_1 = v, e_2, \dots, e_n\}$ of T_pM , we have, from Gauss equation,

$$(2.16) \quad \text{Ric}_M(v, v) = \sum_{\alpha=2}^n K_N(v, e_\alpha) + h_{11}nH - \sum_{i=1}^n h_{i1}h_{1i}.$$

From (2.15) and (2.16), we obtain

$$\begin{aligned} \text{Ric}_M(v, v) &\geq \sum_{\alpha=2}^n K_N(v, e_\alpha) \\ &\quad + 2(n-1)H^2 - \frac{n-1}{n}S - \frac{n-2}{n}\sqrt{\frac{n-1}{n}}\sqrt{n^2H^2(S - nH^2)}. \end{aligned}$$

This completes the proof of Lemma 1. \square

3. HARMONIC STABLE MINIMAL SURFACES

In this section, we want to generalize a theorem due to do Carmo and Peng [10] and Fischer-Colbrie and Schoen [12] to a complete hypersurface M^2 in a Riemannian manifold N^3 . Namely, we will give a classification of complete harmonic stable minimal surfaces in a complete Riemannian manifold with non-negative Ricci curvature as follows:

Theorem 3.1. *Let M^2 be a complete harmonic stable minimal surface in a complete Riemannian manifold N^3 with non-negative Ricci curvature. Then M must be conformally equivalent to either a plane \mathbf{R}^2 or a cylinder $\mathbf{R} \times S^1$*

Proof. Let \tilde{M} be the universal covering of M . Then, if M is harmonic stable, then \tilde{M} is also harmonic stable.

In fact, by lifting function S and R_N to \tilde{M} , we define, for any compactly supported vector field X on \tilde{M} , a vector field $\bar{X}(x)$ by $\bar{X}(x) = \sum_j \bar{X}^j(x) \bar{e}_j$ where $\{\bar{e}_j\}$ is an orthonormal frame on $T_x M$ such that $\bar{X}^j(x)$ satisfy

$$(3.1) \quad |\bar{X}^j|^2(x) = \sum_{\tilde{x} \in \pi^{-1}(x)} |X^j|^2(\tilde{x}),$$

where $X(\tilde{x}) = X^j e_j(\tilde{x})$ and $\{e_j\}$ is an orthonormal frame on $T_{\tilde{x}} \tilde{M}$. Then \bar{X} is a compactly supported vector field on M and $|\bar{X}|^2 = \sum_{\tilde{x} \in \pi^{-1}(x)} |X|^2(\tilde{x})$.

$$(3.2) \quad \begin{aligned} & \int_{\tilde{M}} \left\{ \frac{1}{2} S |X|^2 + R_N |X|^2 - K_M |X|^2 \right\} d\tilde{M} \\ &= \int_M \left\{ \frac{1}{2} S |\bar{X}|^2 + R_N |\bar{X}|^2 - K_M |\bar{X}|^2 \right\} dM \\ &\leq \int_M |\nabla \bar{X}|^2 dM. \end{aligned}$$

On the other hand, from Schwarz's inequality, we infer

$$(3.3) \quad \begin{aligned} & |\bar{X}^j|^2 |\nabla \bar{X}^j|^2(x) \\ &= \left| \sum_{\tilde{x} \in \pi^{-1}(x)} X^j \nabla X^j(\tilde{x}) \right|^2 \\ &\leq \sum_{\tilde{x} \in \pi^{-1}(x)} |X^j|^2(\tilde{x}) \sum_{\tilde{x} \in \pi^{-1}(x)} |\nabla X^j|^2(\tilde{x}) \\ &= |\bar{X}^j|^2 \sum_{\tilde{x} \in \pi^{-1}(x)} |\nabla X^j|^2(\tilde{x}). \end{aligned}$$

Hence,

$$(3.4) \quad |\nabla \bar{X}^j|^2(x) \leq \sum_{\tilde{x} \in \pi^{-1}(x)} |\nabla X^j|^2(\tilde{x}).$$

We choose \bar{e}_j and e_j such that $\nabla \bar{e}_j = 0$ at x and $\nabla e_j = 0$ at \tilde{x} , respectively. Then, we have, at x ,

$$(3.5) \quad |\nabla \bar{X}|^2 = \sum |\nabla \bar{X}^j|^2$$

and, at \tilde{x} ,

$$(3.6) \quad |\nabla X|^2 = \sum |\nabla X^j|^2.$$

Since x and \tilde{x} are arbitrary, we have

$$(3.7) \quad \int_M |\nabla \bar{X}|^2 dM \leq \int_{\tilde{M}} |\nabla X|^2 d\tilde{M}.$$

Thus, we know that \tilde{M} is harmonic stable if M is harmonic stable.

By the uniformization theorem, \tilde{M} must be conformally equivalent to either the unit disk D^2 or the plane \mathbf{R}^2 . If \tilde{M} is conformally equivalent to the unit disk, we know that there exist non-constant bounded functions with finite Dirichlet integral on \tilde{M} since the harmonic property in the dimension 2 is invariant under conformal transformation. Let u be such a harmonic function. We know that du is a harmonic one form. From the Bochner-Weitzenbock formula, we have

$$(3.8) \quad \frac{1}{2} \Delta |\nabla u|^2 = \text{Ric}_{\tilde{M}}(\nabla u, \nabla u) + |\nabla^2 u|^2.$$

Fix a point $p \in \tilde{M}$, let $B_p(r)$ denote the geodesic ball with radius p and centered at p . We choose a cut off function φ_r with compact support such that

$$(3.9) \quad \begin{cases} \varphi_r = 1, & \text{in } B_p(r), \\ \varphi_r = 0, & \text{in } \tilde{M} \setminus B_p(r+1), \\ |\nabla \varphi_r| \leq 1, & \text{on } \tilde{M}. \end{cases}$$

We consider vector field $X_r = \varphi_r \nabla u$. Since \tilde{M} is harmonic stable, we know

$$(3.10) \quad \int_{\tilde{M}} \{S|X_r|^2 + \text{Ric}_N(\nu, \nu)|X_r|^2\} d\tilde{M} \leq \int_{\tilde{M}} |\nabla X_r|^2 d\tilde{M}.$$

$$\begin{aligned}
& \int_{\tilde{M}} |\nabla X_r|^2 d\tilde{M} \\
&= \int_{\tilde{M}} \{ |\nabla \varphi_r|^2 |\nabla u|^2 + 2 \langle \varphi_r d\varphi_r \otimes \nabla u, \nabla(\nabla u) \rangle + \varphi_r^2 |\nabla \nabla u|^2 \} d\tilde{M} \\
&= \int_{\tilde{M}} \{ |\nabla \varphi_r|^2 |\nabla u|^2 + 2 \langle \varphi_r d\varphi_r \otimes \nabla u, \nabla(\nabla u) \rangle \\
&\quad + \varphi_r^2 (\frac{1}{2} \Delta |\nabla u|^2 - \text{Ric}_{\tilde{M}}(\nabla u, \nabla u)) \} d\tilde{M} \\
&= \int_{\tilde{M}} \{ |\nabla \varphi_r|^2 |\nabla u|^2 - \varphi_r^2 \text{Ric}_{\tilde{M}}(\nabla u, \nabla u) \} d\tilde{M} \\
&\leq \int_{\tilde{M}} \{ |\nabla \varphi_r|^2 |\nabla u|^2 + \frac{1}{2} S \varphi_r^2 |\nabla u|^2 \} d\tilde{M}.
\end{aligned}$$

Here we used Lemma 1 and M is minimal. Therefore, we infer

$$\begin{aligned}
(3.11) \quad & \int_{\tilde{M}} \{ \frac{1}{2} S \varphi_r^2 |\nabla u|^2 + \text{Ric}_N(\nu, \nu) \varphi_r^2 |\nabla u|^2 \} d\tilde{M} \\
& \leq \int_{\tilde{M}} |\nabla \varphi_r|^2 |\nabla u|^2 d\tilde{M}.
\end{aligned}$$

From the definition of φ_r , we have

$$\int_{B_p(r)} \{ \frac{1}{2} S |\nabla u|^2 + \text{Ric}_N(\nu, \nu) |\nabla u|^2 \} d\tilde{M} \leq \int_{B_p(r+1) \setminus B_p(r)} |\nabla u|^2 d\tilde{M}.$$

Since ∇u is a L^2 -harmonic vector field and r is arbitrary, we have

$$(3.12) \quad \int_{\tilde{M}} \{ \frac{1}{2} S |\nabla u|^2 + \text{Ric}_N(\nu, \nu) |\nabla u|^2 \} d\tilde{M} = 0.$$

Since N has non-negative Ricci curvature, we have $S \equiv 0$ and $\text{Ric}_N(\nu, \nu) = 0$ on \tilde{M} because u is a non-constant harmonic function. Hence \tilde{M} is totally geodesic and $\text{Ric}_N(\nu, \nu) = 0$ on \tilde{M} . Thus, from Gauss equation, we know that \tilde{M} has non-negative Gauss curvature. From Blanc-Fiala-Huber's theorem in [13], we infer that there are no non-trivial bounded harmonic functions on \tilde{M} . This is a contradiction. Hence, \tilde{M} is conformally equivalent to the plane \mathbf{R}^2 . Making use of the uniformization theorem again, we conclude that M must be conformally equivalent to the plane \mathbf{R}^2 or the cylinder $\mathbf{R} \times S^1$. \square

4. L^2 -HARMONIC ONE FORMS ON MINIMAL HYPERSURFACES

It is well known that in order to study structures of topology and curvature of non-compact Riemannian manifolds, harmonic function theory plays an important role. Furthermore, in L^2 -Hodge theory, we know that harmonic differential forms also play an important role in the investigation of the topology of non-compact Riemannian manifolds. Palmer [21] proved that there are no non-trivial L^2 -harmonic 1-forms on complete stable minimal hypersurfaces in a Euclidean space. Moreover, Miyaoka [20] extended his result to complete stable minimal hypersurfaces in a complete Riemannian manifold with non-negative sectional curvature. In this section, we obtain the following:

Theorem 4.1. *Let M be a complete harmonic stable minimal hypersurface in a complete Riemannian manifold N^{n+1} with non-negative sectional curvature. Then, there exist no nontrivial L^2 -harmonic 1-forms on M .*

Proof. Since M is harmonic stable, then for any vector field $X \in \Gamma_c(TM)$, we have

$$(4.1) \quad \int_M \{S|X|^2 + \text{Ric}_N(\nu, \nu)|X|^2\} dM \leq \int_M |\nabla X|^2 dM.$$

If ω is a nontrivial L^2 -harmonic 1-form, we know that its dual vector field X is a L^2 -harmonic vector field. We choose a cut off function as follows:

$$(4.2) \quad \begin{cases} \varphi_r = 1, & \text{in } B_p(r), \\ \varphi_r = 0, & \text{in } M \setminus B_p(r+1), \\ |\nabla \varphi_r| \leq 1 \text{ and } 0 \leq \varphi_r \leq 1, & \text{on } M. \end{cases}$$

Hence, $\varphi_r X \in \Gamma_c(TM)$. Thus,

$$(4.3) \quad \int_M \{S|\varphi_r X|^2 + \text{Ric}_N(\nu, \nu)|\varphi_r X|^2\} dM \leq \int_M |\nabla(\varphi_r X)|^2 dM.$$

Since the sectional curvature of N is non-negative, we have

$$(4.4) \quad \text{Ric}_N(\nu, \nu) \geq 0.$$

Therefore,

$$(4.5) \quad \int_M S|\varphi_r X|^2 dM \leq \int_M |\nabla(\varphi_r X)|^2 dM.$$

By a direct computation, we infer

$$(4.6) \quad |\nabla(\varphi_r X)|^2 = |\nabla\varphi_r|^2 |X|^2 + \varphi_r^2 |\nabla X|^2 + \varphi_r \nabla\varphi_r \cdot \nabla |X|^2.$$

Since X is harmonic vector field, from Bochner-Weitzenböck formula, we have

$$(4.7) \quad \frac{1}{2} \Delta |X|^2 = |\nabla X|^2 + \text{Ric}_M(X, X).$$

From Lemma 1 in section 2, we have

$$(4.8) \quad \text{Ric}_M(X, X) \geq -\frac{n-1}{n} S |X|^2,$$

because M is minimal and N has non-negative sectional curvature. Thus, we infer

$$(4.9) \quad \begin{aligned} & \int_M S |\varphi_r X|^2 dM \\ & \leq \int_M \{ |\nabla\varphi_r|^2 |X|^2 + \varphi_r^2 |\nabla X|^2 + \varphi_r \nabla\varphi_r \cdot \nabla |X|^2 \} dM \\ & = \int_M [|\nabla\varphi_r|^2 |X|^2 + \varphi_r^2 \{ \frac{1}{2} \Delta |X|^2 - \text{Ric}_M(X, X) \} \\ & \quad + \varphi_r \nabla\varphi_r \cdot \nabla |X|^2] dM \\ & \leq \int_M \{ |\nabla\varphi_r|^2 |X|^2 + \frac{n-1}{n} S \varphi_r^2 |X|^2 \} dM. \end{aligned}$$

Here we used the Stokes' formula, (4.7) and (4.8). Hence, we obtain

$$(4.10) \quad \frac{1}{n} \int_M S |\varphi_r X|^2 dM \leq \int_M |\nabla\varphi_r|^2 |X|^2 dM.$$

From the definition of φ_r , we have

$$(4.11) \quad \frac{1}{n} \int_{B_p(r)} S |X|^2 dM \leq \int_{B_p(r+1) \setminus B_p(r)} |\nabla\varphi_r|^2 |X|^2 dM.$$

Since $\int_M |X|^2 dM < \infty$, we have

$$(4.12) \quad \int_M S |X|^2 dM = 0.$$

Hence, $S \equiv 0$ on M . Namely, M is totally geodesic. From Gauss equation, we know that M has non-negative sectional curvature since N has nonnegative sectional curvature. Hence, the Ricci curvature of M is non-negative. From (4.7), we have

$$(4.13) \quad \frac{1}{2} \Delta |X|^2 = |\nabla X|^2 + \text{Ric}_M(X, X) \geq |\nabla X|^2.$$

Then from Sato's inequality $|\nabla X| \geq |\nabla|X||$, we have

$$(4.14) \quad |X|\Delta|X| = \frac{1}{2}\Delta|X|^2 - |\nabla|X||^2 \geq |\nabla X|^2 - |\nabla|X||^2 \geq 0.$$

Thus, $|X|$ is subharmonic function. A theorem of Yau in [28] yields $|X| = 0$. This is a contradiction. Hence, there exist no nontrivial L^2 -harmonic 1-forms on M . This finishes the proof of Theorem 4.1. \square

Since if M is stable, from Proposition 1, M is harmonic stable, we have the following Corollary, which is a generalization of the theorem due to Palmer [21] and was proved by Miyaoka in [20].

Corollary 4.1. *Let M be a complete stable minimal hypersurface in a complete Riemannian manifold N^{n+1} with non-negative sectional curvature. Then, there exist no nontrivial L^2 -harmonic 1-forms on M .*

Corollary 4.2. *Let M be a complete harmonic stable minimal hypersurface in a complete Riemannian manifold N^{n+1} with non-negative sectional curvature. Then, there exist no nontrivial harmonic functions with finite energy on M .*

Proof. If u is a harmonic function with finite energy on M , then du is a L^2 -harmonic 1-form on M . From Theorem 4.1, we know that $du = 0$. Hence, u is constant. Thus, there exist no nontrivial harmonic functions with finite energy on M . \square

5. HARMONIC STABLE MINIMAL HYPERSURFACES

In this section we consider more general case than the result given in section 4. That is, we want to give a result for a harmonic stable minimal hypersurface in a complete Riemannian manifold as follows:

Theorem 5.1. *Let M be a complete harmonic stable proper minimal hypersurface in a complete Riemannian manifold N^{n+1} with non-negative sectional curvature. If M is nonparabolic, then M must have only one nonparabolic end.*

Proof. For the construction of harmonic functions, we shall use the same assertion as in [18]. Since M is nonparabolic, then M has a nonparabolic end, denoted

by E . If M has only one end E , then there is nothing to do. We suppose that M has at least two ends. Assume that E and F are two ends of M , given by unbounded connected components of $M \setminus B_p(1)$. Since M is proper, we know that the Busemann function β is unbounded in F and $|\nabla \beta| \leq 1$. Since N has non-negative sectional curvature, we know that β is a convex exhaustion function of N .

In particular, the restriction of β on M is a subharmonic function with respect to the induced metric because M is minimal. Hence, β satisfies the conditions in Proposition 2 in section 2. Since $|\nabla \beta| \leq 1$, we know that $s(r)$ is at most linear growth. Hence, there exists a sequence of harmonic functions $\{u_i\}$ defined on $F(r_i) = F \cup B_p(r_i)$ that converges to a positive harmonic function u defined on F . Moreover, they satisfies

$$(5.1) \quad \begin{cases} \Delta u_i = 0, & \text{in } F(r_i), \\ u_i = 0, & \text{on } \partial F, \\ \int_{\partial F} \frac{\partial u_i}{\partial r} = 1. \end{cases}$$

Since E is nonparabolic, there exists a sequence of harmonic functions $\{v_i\}$ defined on $E(r_i) = E \cup B_p(r_i)$ that converges to a positive harmonic function u defined on E . Moreover, they satisfies

$$(5.2) \quad \begin{cases} \Delta v_i = 0, & \text{in } E(r_i), \\ v_i = 0, & \text{on } \partial E, \\ \int_{\partial E} \frac{\partial v_i}{\partial r} = 1. \end{cases}$$

Let us define function f_i on $B_p(r_i)$ by

$$(5.3) \quad f_i = \begin{cases} v_i, & \text{in } E(r_i), \\ -u_i, & \text{in } F(r_i), \\ 0 & \text{in } B_p(r_i) \setminus (E(r_i) \cup F(r_i)). \end{cases}$$

f_i is harmonic function on $E(r_i) \cup F(r_i)$ and

$$(5.4) \quad \int_{\partial B_p(1)} \frac{\partial f_i}{\partial r} = 0.$$

We consider boundary value problem

$$(5.5) \quad \begin{cases} \Delta w_i = 0, & \text{in } B_p(r_i), \\ w_i = f_i, & \text{on } \partial B_p(r_i). \end{cases}$$

The solution w_i minimizes Dirichlet integral. Hence,

$$(5.6) \quad \begin{aligned} & \int_{B_p(r_i)} |\nabla w_i|^2 dM \\ & \leq \int_{B_p(r_i)} |\nabla f_i|^2 dM \\ & = \int_{E(r_i)} |\nabla v_i|^2 dM + \int_{F(r_i)} |\nabla u_i|^2 dM \\ & \leq 2Cr_i. \end{aligned}$$

On the other hand, f_i converges to the function

$$(5.7) \quad f = \begin{cases} v, & \text{in } E, \\ -u, & \text{in } F, \\ 0 & \text{in } M \setminus (E \cup F). \end{cases}$$

There exists a constant $C_1 > 0$ independent of i such that the sequence $\{w_i\}$ converges to a harmonic function w on M satisfying

$$(5.8) \quad |w - f| \leq C_1.$$

We consider a function

$$(5.9) \quad \phi(x) = \begin{cases} 1, & \text{in } B_p(r), \\ \frac{r_i - d(p, x)}{r_i - r}, & \text{in } B_p(r_i) \setminus B_p(r), \\ 0 & \text{in } M \setminus B_p(r), \end{cases}$$

where $d(p, x)$ is the distance function from p and $0 < r < r_i$. Thus,

$$(5.10) \quad \phi(x) \nabla w_i \in \Gamma_c(TM).$$

Since M is harmonic stable, we have

$$(5.11) \quad \int_M \{S|\phi \nabla w_i|^2 + \text{Ric}_N(\nu, \nu)|\phi \nabla w_i|^2\} dM \leq \int_M |\nabla(\phi \nabla w_i)|^2 dM.$$

Since w_i is harmonic in $B_p(r_i)$, we have, in $B_p(r_i)$,

$$(5.12) \quad \frac{1}{2}\Delta|\nabla w_i|^2 = |\nabla\nabla w_i|^2 + \text{Ric}_M(\nabla w_i, \nabla w_i).$$

Therefore, we infer

$$\begin{aligned} & |\nabla(\phi\nabla w_i)|^2 \\ &= |\nabla\phi|^2|\nabla w_i|^2 + \phi\nabla\phi \cdot \nabla|\nabla w_i|^2 + \phi^2|\nabla\nabla w_i|^2 \\ &= |\nabla\phi|^2|\nabla w_i|^2 + \phi\nabla\phi \cdot \nabla|\nabla w_i|^2 + \phi^2\left\{\frac{1}{2}\Delta|\nabla w_i|^2 - \text{Ric}_M(\nabla w_i, \nabla w_i)\right\} \end{aligned}$$

Hence, we have, from Lemma 1,

$$\begin{aligned} (5.13) \quad & \int_M |\nabla(\phi\nabla w_i)|^2 dM \\ &= \int_M \{|\nabla\phi|^2|\nabla w_i|^2 - \phi^2\text{Ric}_M(\nabla w_i, \nabla w_i)\} dM \\ &\leq \int_M \{|\nabla\phi|^2|\nabla w_i|^2 + \frac{n-1}{n}S\phi^2|\nabla w_i|^2\} dM. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} (5.14) \quad & \int_M S|\phi\nabla w_i|^2 dM \leq n \int_M |\nabla\phi|^2|\nabla w_i|^2 dM \\ &= \int_{B_p(r_i) \setminus B_p(r)} \frac{1}{(r_i - r)^2} |\nabla w_i|^2 dM \\ &= \frac{1}{(r_i - r)^2} 2Cr_i. \end{aligned}$$

Letting $i \rightarrow \infty$, we have

$$(5.15) \quad \int_{B_p(r)} S|\nabla w_i|^2 dM = 0.$$

Since r is arbitrary, we conclude $|\nabla w|^2 S = 0$. Since w is a non-constant harmonic function, we know that $S = 0$ on M . Hence, M is totally geodesic. From Gauss equation, we infer that M has non-negative sectional curvature, because N has non-negative sectional curvature. From the splitting theorem of Cheeger and Gromoll [6], we know $M = \mathbf{R} \times P$, which contradicts the assumption that M is nonparabolic. Hence, M must have only one end. \square

REFERENCES

1. F. Almgren, *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem*, Ann. of Math. **84** (1966), 277-292.
2. S. Bernstein, *Sur un theoreme de geometrie et ses applications aux derivees partielles du type elliptique*, Comm. Soc. Math. Kharkov **15** (1915-1917), 38-45.
3. E. Bombieri, E. De Giorgi and E. Guisti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243-268.
4. H. Cao, Y. Shen and S. Zhu, *The structure of stable minimal hypersurfaces in \mathbf{R}^{n+1}* , Math. Res. Let. **4** (1997), 637-644.
5. J. Cheeger and D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. **92** (1972), 413-443.
6. J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Diff. Geom. **6** (1971), 119-128.
7. Q.-M. Cheng and H. Nakagawa, *Totally umbilical hypersurfaces*, Hiroshima Math. J. **20** (1990), 1-10.
8. Q.-M. Cheng and Q.-R. Wan, *Complete hypersurfaces of \mathbf{R}^4 with constant mean curvature*, Mh. Math. **118** (1994), 171-204.
9. E. De Giorgi, *Una estensione del teorema di Bernstein*, Ann. Scuola Nor. Sup. Pisa **19** (1965), 79-85.
10. M. do Carmo and C.K. Peng, *Stable complete minimal surfaces in \mathbf{R}^3 are planes*, Bull. Amer. Math. Soc. **1** (1979), 903-906.
11. D. Fischer-Colbrie, *On complete minimal surfaces with finite Morse index in three manifolds*, Invent. Math. **82** (1985), 121-132.
12. D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33** (1980), 199-211.
13. W. Fleming, *On the oriented Plateau problem*, Rend. Circ. Mat. Palermo **11** (1963), 69-90.
14. A. Huber, *On subharmonic functions and differential geometry in the large*, Comm. Math. Helv. **32** (1957), 13-72.
15. P. Li and L.F. Tam, *Symmetric Green's functions on complete manifolds*, Amer. J. Math. **109** (1987), 1129-1154.
16. P. Li and J. Wang, *Complete manifolds with positive spectrum*, J. Diff. Geom. **58** (2001), 501-534.
17. P. Li and J. Wang, *Minimal hypersurfaces with finite index*, Math. Res. Let. **9** (2002), 95-103.
18. P. Li and J. Wang, *Stable minimal hypersurfaces in a nonnegatively curved manifold*, J. Reine Ang. Math. (Crelles) **566** (2004), 215-230.
19. J. Mei and S. Xu, *On minimal hypersurfaces with finite harmonic indices*, Duke Math. J. **110** (2001), 195-215.
20. R. Miyaoka, *L^2 -harmonic 1-forms on a complete stable minimal hypersurfaces*, Geom. and Global Anal. (1993), 289-293.
21. B. Palmer, *Stability of minimal hypersurfaces*, Comment. Math. Helvetici **66** (1991), 185-188.
22. V. Pogorelov, *On the stability of minimal surfaces*, Sov. Math. Dokl **24** (1981), 274-276.
23. R. Schoen and S.T. Yau, *Harmonic maps and the topology of stable hypersurfaces and manifolds of nonnegative Ricci curvature*, Comm. Math. Helv. **39** (1976), 333-341.
24. Y.-B. Shen and X.-H. Zhu, *On stable complete minimal hypersurfaces in \mathbf{R}^{n+1}* , Amer. J. Math. **120** (1998), 103-116.
25. J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. **80** (1964), 1-21.
26. C.J. Sung, L.F. Tam and J. Wang, *Spaces of harmonic function*, J. London Math. Soc. **61** (2000), 789-806.
27. S.T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201-228.
28. S.T. Yau, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana Univ. Math. J. **25** (1976), 659-670.

29. S.T. Yau and R. Schoen, *Differential Geometry*, Science Press, Beijing, 1991.

QING-MING CHENG

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ENGINEERING

SAGA UNIVERSITY, SAGA 840-8502, JAPAN

CHENG@MS.SAGA-U.AC.JP

YOUNG JIN SUH

DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701,
SOUTH KOREA

YJSUH@MAIL.KNU.AC.KR