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MAXIMAL SPACE-LIKE HYPERSURFACES IN $H_1^4(-1)$ WITH ZERO GAUSS-KRONECKER CURVATURE

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Abstract. In this paper, we study complete maximal space-like hypersurfaces with constant Gauss-Kronecker curvature in an anti-de Sitter space $\mathbf{H}_1^4(-1)$. It is proved that complete maximal space-like hypersurfaces with constant Gauss-Kronecker curvature in an anti-de Sitter space $\mathbf{H}_1^4(-1)$ are isometric to the hyperbolic cylinder $\mathbf{H}^2(c_1)\times\mathbf{H}^1(c_2)$ with $S=3$ or they satisfy $S\leq 2$, where S denotes the squared norm of the second fundamental form.

1. Introduction

Let $M_s^n(c)$ be an n-dimensional connected semi-Riemannian manifold of index $s(\geq 0)$ and of constant curvature c. It is called a semi-definite space form of index s. When $s=1$, $M_1^n(c)$ is said to be a *Lorentz space form.* Such Lorentz space forms $M_1^n(c)$ can be divided into three kinds of semi-definite space forms: the de Sitter space $S_1^n(c)$, the Minkowski space R_1^n, or the anti-de Sitter space $H_1^n(c)$, according to the sign of its sectional curvature $c>0$, $c=0$, or $c<0$ respectively.

In connection with the negative settlement of the Bernstein problem due to Calabi [4] and Cheng-Yau [8], Chouque-Bruhat et al. [9] proved the following theorem independently.

Theorem A. *Let M be a complete space-like hypersurface in an $(n+1)$-dimensional Lorentz space form $M_1^{n+1}(c)$, $c\geq 0$. If M is maximal, then it is totally geodesic.*

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As a generalization of this result, complete space-like hypersurfaces with *constant mean curvature* in a Lorentz manifold have been investigated by Akutagawa [1], Li [11], Montiel [12], Nishikawa [13], Baek and the present authors [3], and Choi, Yang and the second author [16].

On the other hand, some generalizations of Theorem A for submanifolds with codimension $p \geq 1$ were given by Ishihara [10], Nakagawa and the first author [7], and the first author [5]. Among them Ishihara [10] proved that an n-dimensional complete maximal space-like submanifolds with codimension p in an $(n+p)$-dimensional semi-definite space form $M_p^{n+p}(c)$, $c \geq 0$ is totally geodesic.

Now let us consider a complete maximal space-like hypersurface in an anti-de Sitter space $\mathbf{H}_1^{n+1}(-1)$ and denote by S the squared norm of the second fundamental form of this hypersurface. Then Ishihara [10] has also proved that the squared norm S satisfies $0 \leq S \leq n$ and the hyperbolic cylinders $\mathbf{H}^{n-k}(c_1) \times \mathbf{H}^k(c_2)$, $k = 1, 2, \ldots, n-1$ are the only complete maximal space-like hypersurfaces in an anti-de Sitter space $\mathbf{H}_1^{n+1}(-1)$ satisfying $S \equiv n$.

Then it could be natural to investigate complete maximal space-like hypersurfaces in $\mathbf{H}_1^{n+1}(-1)$, which do not satisfy $S \equiv n$. When $n = 3$, the first author [6] gave several characterizations for such hypersurfaces and it was proved that hyperbolic cylinders $\mathbf{H}^2(c_1) \times \mathbf{H}^1(c_2)$ are the only complete maximal space-like hypersurfaces in $\mathbf{H}_1^4(-1)$ with nonzero constant Gauss-Kronecker curvature.

For the case that Gauss-Kronecker curvature is zero we have no result until now. Since totally geodesic maximal space-like hypersurfaces were known to have zero Gauss-Kronecker curvature, the following problem was proposed by the first author [5].

PROBLEM. [6] *Is it true that every complete maximal space-like hypersurface in $\mathbf{H}_1^4(-1)$ with zero Gauss-Kronecker curvature is totally geodesic?*

In this paper, we shall give two characterizations of such hypersurfaces, which imply the above problem may be solved affirmatively.

THEOREM 1. *Let M^3 be a complete maximal space-like hypersurface in an anti-de Sitter space $\mathbf{H}_1^4(-1)$ with zero Gauss-Kronecker curvature. Then, M^3 satisfies $S \leq 2$, where S denotes the squared norm of the second fundamental form.*

From Theorem 1 and the result due to the first author [6], we obtain

Corollary. *Let M^3 be a complete maximal space-like hypersurface in an anti-de Sitter space $\mathbf{H}_1^4(-1)$ with constant Gauss-Kronecker curvature. Then, M^3 is isometric to the hyperbolic cylinder $\mathbf{H}^2(c_1) \times \mathbf{H}^1(c_2)$ with $S = 3$ or M^3satisfies $S \leq 2$.*

If a maximal space-like hypersurface in $H_1^4(-1)$ is not assumed to be complete, we can assert the following:

Theorem 2. *Let M^3 be a maximal space-like hypersurface in an anti-de Sitter space $\mathbf{H}_1^4(-1)$ with zero Gauss-Kronecker curvature. If the principal curvature functions are constant along the curvature line corresponding to the zero principal curvature, then M^3 is totally geodesic.*

2. Preliminaries

We consider Minkowski space $\mathbf{R}_2^{n+2}$ as the real vector space $\mathbf{R}^{n+2}$ endowed with the Lorentzian metric $\langle .,. \rangle$ given by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1} - x_{n+2} y_{n+2} \tag{2.1}$$

for x, $y \,\epsilon\, \mathbf{R}^{n+2}$. Then, for $c > 0$, the anti-de Sitter space $\mathbf{H}_1^{n+1}(-c)$ can be defined as the following hyperquadric of $\mathbf{R}_2^{n+2}$

$$\mathbf{H}_1^{n+1}(-c) = \left\{ x \epsilon \mathbf{R}_2^{n+2} : |x|^2 = -\frac{1}{c} \right\}.$$

In this way, the anti-de Sitter space $\mathbf{H}_1^{n+1}(-c)$ inherits from $\langle , \rangle$ a metric which makes it an indefinite Riemannian manifold of constant sectional curvature $-c$. For indefinite Riemannian manifolds, refer to B. O′Neill [15].

Moreover, if $x \in \mathbf{H}_1^{n+1}(-c)$, we can put

$$T_x \mathbf{H}_1^{n+1}(-c) = \{ v \in \mathbf{R}_2^{n+2} | \langle v, x \rangle = 0 \}.$$

If ∇^L and $\bar{\nabla}$ denote the metric connections of $\mathbf{R}_2^{n+2}$ and $\mathbf{H}_1^{n+1}(-c)$ respectively, we have

$$\nabla_v^L w - \bar{\nabla}_v w = c \langle v, w \rangle x \tag{2.2}$$

for all vector fields v, w which are tangent to $\mathbf{H}_1^{n+1}(-c)$. Let

$$\phi : M^n \to \mathbf{H}_1^{n+1}(-c) \tag{2.3}$$

be a connected space-like hypersurface immersed in $\mathbf{H}_1^{n+1}(-c)$ and let ∇ be the Levi-Civita connection corresponding to the Riemannian metric g induced on M^n from $\langle , \rangle$. Then the second fundamental form $\vec{h}$ and the Weingarten endomorphism A of ϕ are given by

$$\bar{\nabla}_v w - \nabla_v w = \vec{h}(v, w), \tag{2.4}$$

$$\bar{\nabla}_v N = -Av \quad \text{and} \quad \vec{h}(v, w) = -g(Av, w)N, \tag{2.5}$$

where v, w are vector fields tangent to M^n and N is a unit timelike vector field normal to M^n. So, the mean curvature H of the immersion ϕ is given by $nH = \text{trace}A$.

Let us denote by R the curvature tensor field of M. The Gauss equation is given by

$$\begin{aligned} &R(v, w)u \\ =& -c\{g(w, u)v - g(v, u)w\} - \{g(Aw, u)Av - g(Av, u)Aw\}, \end{aligned} \tag{2.6}$$

where v, w and u are vector fields tangent to M^n. The Codazzi equation is expressed by

$$(\nabla_v A)w = (\nabla_w A)v. \tag{2.7}$$

From (2.6), we have

$$n(n-1)(r + c) = S - (nH)^2, \tag{2.8}$$

where $S = |\vec{h}|^2$ and $n(n-1)r$ denotes the squared norm of the second fundamental form and the scalar curvature of M^n, respectively.

We take a local field of orthonormal differentiable frames $e_1, \dots, e_n$ on M^n such that

$$Ae_i = \lambda_i e_i, \text{ for } i = 1, 2, \dots, n. \tag{2.9}$$

These λ_i's are called principal curvatures of M^n.

Next we consider the case of $n = 3$. Since $\nabla_{e_i} e_j$ are tangent to M^3 and e_1, e_2, e_3 is a local field of orthonormal differentiable frames, we know that there are 9 functions $a_1, a_2, \dots, a_9$ such that

(2.10)
$$\nabla_{e_1} e_1 = a_1 e_2 + a_2 e_3,\ \nabla_{e_1} e_2 = -a_1 e_1 + a_3 e_3,\ \nabla_{e_1} e_3 = -a_2 e_1 - a_3 e_2,$$

(2.11)
$$\nabla_{e_2} e_1 = -a_4 e_2 + a_6 e_3,\ \nabla_{e_2} e_2 = a_4 e_1 + a_5 e_3,\ \nabla_{e_2} e_3 = -a_6 e_1 - a_5 e_2,$$

(2.12)
$$\nabla_{e_3} e_1 = a_9 e_2 - a_7 e_3,\ \nabla_{e_3} e_2 = -a_9 e_1 - a_8 e_3,\ \nabla_{e_3} e_3 = a_7 e_1 + a_8 e_2.$$

The following Generalized Maximum Principle due to Omori and Yau will be used in order to prove our theorems.

Generalized Maximum Principle. (Omori [14] and Yau [17]) *Let M^n be a complete Riemannian manifold whose Ricci curvature is bounded from below and $f \in \mathcal{C}^2(M)$ a function bounded from above on M^n. Then for any $\epsilon > 0$, there exists a point $p \in M^n$ such that*

$$f(p) \geq \sup f - \epsilon, \quad \|\mathrm{grad} f\|(p) < \epsilon, \quad \nabla_i \nabla_i f(p) < \epsilon,$$

for $i = 1, 2, \dots, n$.

3. Proofs of Theorems

In order to prove our theorems, we shall prepare two lemmas, firstly.

Lemma 1. *Let M^3 be a space-like hypersurface in an anti-de Sitter space $\mathbf{H}^4_1(-1)$. If the principal curvatures λ_i's are different from each others on an open subset $\mathfrak{U}$ of M^3, then on $\mathfrak{U}$, we have the following:*

$$\begin{aligned}
&e_1(\lambda_2) = a_4(\lambda_2 - \lambda_1),\ e_1(\lambda_3) = a_7(\lambda_3 - \lambda_1),\\
&e_2(\lambda_1) = a_1(\lambda_1 - \lambda_2),\ e_2(\lambda_3) = a_8(\lambda_3 - \lambda_2),\\
&e_3(\lambda_1) = a_2(\lambda_1 - \lambda_3),\ e_3(\lambda_2) = a_5(\lambda_2 - \lambda_3),\\
&a_9(\lambda_1 - \lambda_2) = a_3(\lambda_2 - \lambda_3) = a_6(\lambda_1 - \lambda_3),
\end{aligned}$$

where the above functions a_i, $i = 1, \dots, 9$ are given in section 2.

Proof. Since these principal curvatures λ_i's are different from each other on the open subset $\mathfrak{U}$ of M, then on $\mathfrak{U}$, λ_i's are differentiable functions. From Codazzi equation (2.7), we have

$$(\nabla_{e_1}A)e_2 = (\nabla_{e_2}A)e_1.$$

From (2.9), we obtain

$$\nabla_{e_1}(\lambda_2 e_2) - A\nabla_{e_1}e_2 = \nabla_{e_2}(\lambda_1 e_1) - A\nabla_{e_2}e_1,$$

$$e_1(\lambda_2)e_2 + \lambda_2\nabla_{e_1}e_2 - A\nabla_{e_1}e_2 = e_2(\lambda_1)e_1 + \lambda_1\nabla_{e_2}e_1 - A\nabla_{e_2}e_1.$$

From (2.10) and (2.11), we infer

$$\begin{aligned} &e_1(\lambda_2)e_2 + \lambda_2(-a_1e_1 + a_3e_3) + a_1\lambda_1e_1 - a_3\lambda_3e_3 \\ &= e_2(\lambda_1)e_1 + \lambda_1(-a_4e_2 + a_6e_3) + a_4\lambda_2e_2 - a_6\lambda_3e_3. \end{aligned}$$

Hence, we have

$$e_1(\lambda_2) = a_4(\lambda_2 - \lambda_1),\ e_2(\lambda_1) = a_1(\lambda_1 - \lambda_2),\ a_3(\lambda_2 - \lambda_3) = a_6(\lambda_1 - \lambda_3).$$

Similarly, we can prove the other also holds. Now we complete the proof of Lemma 1. □

Since M^3 is maximal and the Gauss-Kronecker curvature is zero, we can assume $\lambda_1 = \lambda = -\lambda_2, \lambda_3 = 0$. Then we are able to state the following:

LEMMA 2. *Let M^3 be a maximal space-like hypersurface with zero Guass-Kronecker curvature in an anti-de Sitter space $\mathbf{H}_1^4(-1)$. If S is not zero on an open subset $\mathfrak{U}$ of M^3, then on $\mathfrak{U}$, we have*

$$e_1(a_4) + e_2(a_1) = \lambda^2 - 1 + a_1^2 + a_2^2 + 2a_3^2 + a_4^2, \tag{3.1}$$

$$e_3(a_1) + \frac{1}{2}e_1(a_3) = a_1a_2 - \frac{1}{2}a_3a_4, \tag{3.2}$$

$$e_3(a_4) - \frac{1}{2}e_2(a_3) = a_2a_4 + \frac{1}{2}a_1a_3, \tag{3.3}$$

$$e_3(a_2) = -1 + a_2^2 - a_3^2, \tag{3.4}$$

$$e_1(a_2) = e_2(a_3),\ e_1(a_3) = -e_2(a_2),\ e_3(a_3) = 2a_2a_3, \tag{3.5}$$

where $\lambda = \lambda_1 \neq 0$.

Proof. Since M^3 is maximal and the Gauss-Kronecker curvature is zero, we may assume $\lambda_1 = \lambda = -\lambda_2 \neq 0$, $\lambda_3 = 0$. According to Lemma 1, we have

$$e_1(\lambda) = 2a_4\lambda,\ e_2(\lambda) = 2a_1\lambda,\ e_3(\lambda) = a_2\lambda, \tag{3.6}$$

and

$$a_5 = a_2, \quad 2a_9 = -a_3 = a_6, \quad a_7 = a_8 = 0. \tag{3.7}$$

From (2.10), (2.11), and (2.12), we can obtain the following formulas

$$[e_1, e_2] = -a_1e_1 + a_4e_2 + 2a_3e_3, \tag{3.8}$$

$$[e_1, e_3] = -a_2e_1 - \frac{1}{2}a_3e_2, \tag{3.9}$$

$$[e_2, e_3] = \frac{1}{2}a_3e_1 - a_2e_2. \tag{3.10}$$

From the definition of the curvature tensor and the Gauss equation (2.6), we have

$$\nabla_{e_1}\nabla_{e_2}e_2 - \nabla_{e_2}\nabla_{e_1}e_2 - \nabla_{[e_1,e_2]}e_2 = R(e_1, e_2)e_2 = (\lambda^2 - 1)e_1. \tag{3.11}$$

From (2.10) and (2.11), we have

(3.12)
$$\begin{aligned}
&\nabla_{e_1}\nabla_{e_2}e_2 - \nabla_{e_2}\nabla_{e_1}e_2 - \nabla_{[e_1,e_2]}e_2 \\
&= \nabla_{e_1}(a_4e_1 + a_2e_3) - \nabla_{e_2}(-a_1e_1 + a_3e_3) - \nabla_{(-a_1e_1+a_4e_2+2a_3e_3)}e_2 \\
&= \big\{e_1(a_4) + e_2(a_1) - a_1^2 - a_2^2 - 2a_3^2 - a_4^2\big\}e_1 + \big\{e_1(a_2) - e_2(a_3)\big\}e_3.
\end{aligned}$$

From (3.11) and (3.12), we infer

$$e_1(a_4) + e_2(a_1) = \lambda^2 - 1 + a_1^2 + a_2^2 + 2a_3^2 + a_4^2,\ e_1(a_2) = e_2(a_3).$$

Making use of a similar proof, we can obtain

$$e_1(a_3) = -e_2(a_2),$$

$$e_3(a_1) + \frac{1}{2}e_1(a_3) = a_1a_2 - \frac{1}{2}a_3a_4,$$

$$e_3(a_2) = -1 + a_2^2 - a_3^2,$$

$$e_3(a_4) - \frac{1}{2}e_2(a_3) = a_2a_4 + \frac{1}{2}a_1a_3,$$
$$e_3(a_3) = 2a_2a_3,$$

from

$$\nabla_{e_1}\nabla_{e_2}e_1 - \nabla_{e_2}\nabla_{e_1}e_1 - \nabla_{[e_1,e_2]}e_1 = R(e_1,e_2)e_1 = (1-\lambda^2)e_2,$$
$$\nabla_{e_1}\nabla_{e_3}e_1 - \nabla_{e_3}\nabla_{e_1}e_1 - \nabla_{[e_1,e_3]}e_1 = R(e_1,e_3)e_1 = e_3,$$
$$\nabla_{e_2}\nabla_{e_3}e_2 - \nabla_{e_3}\nabla_{e_2}e_2 - \nabla_{[e_2,e_3]}e_2 = R(e_2,e_3)e_2 = e_3$$

and

$$\nabla_{e_3}\nabla_{e_1}e_3 - \nabla_{e_1}\nabla_{e_3}e_3 - \nabla_{[e_1,e_3]}e_1 = R(e_3,e_1)e_3 = e_1.$$

Thus, the proof is completed. □

Proof of Theorem 1. From a result due to Ishihara [10], we know $S \leq 3$. If $\sup S = 0$, then our theorem is true. Next we consider the case of $\sup S > 0$. Then let us construct an open subset $\mathfrak{U}$ of M^3 in such a way that

$$\mathfrak{U} = \{p \in M^3; S(p) > 0\}.$$

Since the Gauss-Kronecker curvature is zero and M^3 is maximal, we can assume

$$\lambda_1 = \lambda, \quad \lambda_2 = -\lambda \quad \text{and} \quad \lambda_3 = 0.$$

Thus, on such an open subset $\mathfrak{U}$, these principal curvatures λ_1, λ_2 and λ_3 are different from each other. Hence, they are differentiable on $\mathfrak{U}$.

Now we are able to assume that $\lambda > 0$ on $\mathfrak{U}$. From the Gauss equation, we know that the sectional curvature is bounded from below by -1. Applying the Generalized Maximum Principle due to Omori [14] and Yau [17] in section 2 to the function S, we know that there exists a sequence $\{p_k\} \subset M^3$ such that

$$\lim_{k\to\infty} S(p_k) = \sup S, \quad \lim_{k\to\infty} \|\text{grad } S\|(p_k) = 0, \tag{3.13}$$

$$\lim_{k\to\infty} \sup \nabla_i\nabla_i S(p_k) \leq 0, \text{ for } i = 1,2,3. \tag{3.14}$$

Since $\sup S > 0$, we can assume $\{p_k\} \subset \mathfrak{U}$. On $\mathfrak{U}$, $S = \lambda_1^2+\lambda_2^2+\lambda_3^2 = 2\lambda^2$. Hence, we have

$$\sup S = \lim_{k\to\infty} S(p_k) = 2\lim_{k\to\infty} \lambda(p_k)^2. \tag{3.15}$$

From (3.13) and

$$e_1(\lambda) = 2a_4\lambda, \; e_2(\lambda) = 2a_1\lambda, \; e_3(\lambda) = a_2\lambda, \tag{3.16}$$

we have

$$\lim_{k\to\infty} a_1(p_k) = 0, \; \lim_{k\to\infty} a_2(p_k) = 0, \; \lim_{k\to\infty} a_4(p_k) = 0. \tag{3.17}$$

From (3.14) and $S = 2\lambda^2$, $\lambda > 0$, we have

$$\begin{gathered} \lim_{k\to\infty} \sup e_1e_1 \; (\lambda)(p_k) \leq 0, \; \lim_{k\to\infty} \sup e_2e_2(\lambda)(p_k) \leq 0, \\ \lim_{k\to\infty} \sup e_3e_3(\lambda)(p_k) \leq 0. \end{gathered} \tag{3.18}$$

From (3.16), we have

$$\begin{aligned} e_1e_1(\lambda) &= 2e_1(a_4)\lambda + 2a_4e_1(\lambda), \\ e_2e_2(\lambda) &= 2e_2(a_1)\lambda + 2a_1e_2(\lambda), \\ e_3e_3(\lambda) &= e_3(a_2)\lambda + a_2e_3(\lambda). \end{aligned}$$

Thus, we obtain

$$\lim_{k\to\infty} \sup e_1(a_4)(p_k) \leq 0 \text{ and } \lim_{k\to\infty} \sup e_2(a_1)(p_k) \leq 0.$$

From the formula (3.1) in Lemma 2, we have

$$\lim_{k\to\infty} \lambda(p_k)^2 \leq 1.$$

Hence, we infer $\sup S \leq 2$. Now we complete the proof of Theorem 1.□

Proof of Theorem 2. If there exists a point $p \in M^3$ such that $S(p) > 0$, then by using the similar assertion as in the proof of Theorem 1, we have that on an open subset $\mathfrak{U}$, $S(p) > 0$ and these principal curvatures are differentiable. From the assumption of Theorem 2, we have $a_2 = 0$ according to (3.6). From (3.4), we infer $-1 - a_3^2 = 0$. This is impossible. Hence, $S \equiv 0$ on M^3, that is, M^3 is totally geodesic. Thus, Theorem 2 is proved. □

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