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Calculus of Variations

Estimates for the first eigenvalue of Jacobi operator on hypersurfaces with constant mean curvature in spheres

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Abstract In this paper, we study the first eigenvalue of Jacobi operator on an n-dimensional non-totally umbilical compact hypersurface with constant mean curvature H in the unit sphere $S^{n+1}(1)$. We give an optimal upper bound for the first eigenvalue of Jacobi operator, which only depends on the mean curvature H and the dimension n. This bound is attained if and only if, $\varphi: M \to S^{n+1}(1)$ is isometric to $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$ when $H \neq 0$ or $\varphi: M \to S^{n+1}(1)$ is isometric to a Clifford torus $S^{n-k}\left(\sqrt{\frac{n-k}{n}}\right) \times S^k\left(\sqrt{\frac{k}{n}}\right)$, for $k = 1, 2, \ldots, n-1$ when $H = 0$.

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1 Introduction

Let $\varphi: M \to S^{n+1}(1)$ be an n-dimensional compact hypersurface in the unit sphere $S^{n+1}(1)$ of dimension $n+1$. We consider a variation of the hypersurface $\varphi: M \to S^{n+1}(1)$, for any $t \in (-\varepsilon, \varepsilon)$,

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$$\varphi_t : M \to S^{n+1}(1)$$

is an immersion with $\varphi_0 = \varphi$. The area of φ_t is given by

$$A(t) = \int_M dA_t$$

and the volume of φ_t is defined by

$$V(t) = \frac{1}{n+1} \int_M \langle \varphi_t, N(t) \rangle dA_t,$$

where $N(t)$ denotes the unit normal of φ_t. For any t, if $V(t) = V(0)$, then the variation φ_t is called volume-preserving. If the variational vector $\frac{\partial \varphi_t}{\partial t}|_{t=0} = fN$ for a smooth function f, then the variation is called a normal variation, where N is the unit normal of φ. Let H denote the mean curvature of φ. The first variation formula of the area functional $A(t)$ is given by

$$\frac{dA(t)}{dt}|_{t=0} = -\int_M nHf dA,$$

where $f = \langle \frac{\partial \varphi_t}{\partial t}|_{t=0}, N \rangle$. Thus, we know that a compact hypersurface is minimal, that is, $H \equiv 0$ if and only if

$$\frac{dA(t)}{dt}|_{t=0} = 0.$$

Hence, compact minimal hypersurfaces are critical points of the area functional $A(t)$. The second variation formula of $A(t)$ is given by

$$\frac{d^2A(t)}{dt^2}|_{t=0} = -\int_M fJf dA$$

and

$$Jf = \Delta f + (S + n) f,$$

where S denotes the squared norm of the second fundamental form of φ and Δ stands for the Laplace–Beltrami operator. The J is called a Jacobi operator or a stability operator on the minimal hypersurface φ (cf. [2,9]).

Let λ_1^J denote the first eigenvalue of the Jacobi operator J. Then

$$Ju = -\lambda_1^J u$$

and the λ_1^J is given by

$$\lambda_1^J = \inf_{f \not\equiv 0} \frac{-\int_M fJf dA}{\int_M f^2 dA}.$$

For a compact minimal hypersurface in $S^{n+1}(1)$, Simons [10] proved

$$\lambda_1^J \le -n$$

and $\lambda_1^J = -n$ if and only if $\varphi : M \to S^{n+1}(1)$ is totally geodesic. Furthermore, Wu [11] proved that for an n-dimensional compact non-totally geodesic minimal hypersurface $\varphi : M \to S^{n+1}(1)$ in $S^{n+1}(1)$, then $\lambda_1^J \le -2n$ and $\lambda_1^J = -2n$ if and only if $\varphi : M \to S^{n+1}(1)$ is a Clifford torus $S^{n-k}\left(\sqrt{\frac{n-k}{n}}\right) \times S^k\left(\sqrt{\frac{k}{n}}\right)$, for $k = 1, 2, \ldots, n-1$. Thus, we know that the upper bound for the first eigenvalue λ_1^J due to Wu is optimal and it only depends on the dimension n, does not depends on the immersion.

On the other hand, if one considers the volume-preserving variation of φ, then we have

$$\int_M f dA = 0.$$

From the first variation formula:

$$\frac{dA(t)}{dt}|_{t=0} = -\int_M nHf dA,$$

we know that compact hypersurfaces with constant mean curvature are critical points of the area functional $A(t)$ for the volume-preserving variation and the second variation formula of $A(t)$ is given by

$$\frac{d^2A(t)}{dt^2}|_{t=0} = -\int_M fJf dA,$$

where the Jacobi operator J on compact hypersurfaces with constant mean curvature is the same as one of compact minimal hypersurfaces ([2,4]).

Alias et al. [3] studied the first eigenvalue of the Jacobi operator J on compact hypersurfaces with constant mean curvature. They proved the following:

Theorem ABB *If $\varphi : M \to S^{n+1}(1)$ is an n-dimensional compact hypersurface with non-zero constant mean curvature H in the unit sphere $S^{n+1}(1)$, then either $\lambda_1^J = -n(1+H^2)$ and $\varphi : M \to S^{n+1}(1)$ is totally umbilical or*

$$\lambda_1^J \leq -2n(1+H^2) + \frac{n(n-2)|H|}{\sqrt{n(n-1)}} \max \sqrt{S - nH^2}$$

and the equality holds if and only if $\varphi : M \to S^{n+1}(1)$ is $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$, with $r^2 > \frac{1}{n}$ for $n \geq 2$.

According to this theorem, we know that, for $n = 2$, the upper bund of the first eigenvalue λ_1^J of the Jacobi operator of non-totally umbilical compact hypersurfaces with constant mean curvature only depends on the mean curvature H and the dimension. But for $n \geq 3$, the upper bound of the first eigenvalue λ_1^J of the Jacobi operator on non-totally umbilical compact hypersurfaces with constant mean curvature includes the term $\max \sqrt{S - nH^2}$. Hence, the upper bound of the first eigenvalue λ_1^J does not only depend on the mean curvature H and the dimension n, but also depends on the immersion φ.

It is natural and important to propose the following:

Problem 1.1 To find an optimal upper bound for the first eigenvalue λ_1^J of the Jacobi operator on non-totally umbilical compact hypersurfaces with constant mean curvature, which only depends on the mean curvature H and the dimension n.

In this paper, we give an affirmative answer for the above Problem 1.1.

Theorem 1.1 *Let $\varphi : M \to S^{n+1}(1)$ be an n-dimensional non-totally umbilical compact hypersurface with constant mean curvature H in the unit sphere $S^{n+1}(1)$.*

1. *If* $2 \leq n \leq 4$ *or* $n \geq 5$ *and* $n^2H^2 < \frac{16(n-1)}{n(n-4)}$, *then the first eigenvalue* λ_1^J *of the Jacobi operator* J *satisfies*

$$\lambda_1^J \leq -n(1+H^2) - \frac{n\left(\sqrt{4(n-1)+n^2H^2}-(n-2)|H|\right)^2}{4(n-1)}$$

and the equality holds if and only if $\varphi : M \to S^{n+1}(1)$ *is isometric to* $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$ *with* $r > 0$ *satisfying*

$$\begin{cases} 1 > r^2 > \dfrac{1}{n} & \text{for } 2 \leq n \leq 4, \\ \dfrac{n}{(n-2)^2} > r^2 > \dfrac{1}{n}, & \text{for } n \geq 5 \text{ and } n^2H^2 < \dfrac{16(n-1)}{n(n-4)} \end{cases}$$

or $\varphi : M \to S^{n+1}(1)$ *is isometric to a Clifford torus* $S^{n-k}\left(\sqrt{\frac{n-k}{n}}\right) \times S^k\left(\sqrt{\frac{k}{n}}\right)$, *for* $k = 1, 2, \ldots, n-1$ *with* $H = 0$.

2. *If* $n \geq 5$ *and* $n^2H^2 \geq \frac{16(n-1)}{n(n-4)}$, *the first eigenvalue* λ_1^J *of the Jacobi operator* J *satisfies*

$$\lambda_1^J \leq -2(n-1)(1+H^2) + \frac{(n-2)^4}{8(n-1)}H^2$$

and the equality holds if and only if $\varphi : M \to S^{n+1}(1)$ *is isometric to* $S^1\left(\frac{\sqrt{n}}{n-2}\right) \times S^{n-1}\left(\frac{\sqrt{(n-1)(n-4)}}{n-2}\right)$.

Remark 1.1 Since the first eigenvalue of Jacobi operator J on totally umbilical hypersurfaces satisfies $\lambda_1^J = -n(1+H^2)$, according to our theorem, one knows that for $2 \leq n \leq 4$, there are no n-dimensional compact hypersurfaces in the unit sphere with constant mean curvature H so that the first eigenvalue λ_1^J of Jacobi operator J takes a value in the internal

$$\left(-n(1+H^2) - \frac{n(\sqrt{4(n-1)+n^2H^2}-(n-2)|H|)^2}{4(n-1)},\ -n(1+H^2)\right).$$

For any $n \geq 2$, there are no n-dimensional compact hypersurfaces in the unit sphere with constant mean curvature H satisfying $n^2H^2 < \frac{16(n-1)}{n(n-4)}$ so that the first eigenvalue λ_1^J of Jacobi operator J takes a value in the internal

$$\left(-n(1+H^2) - \frac{n(\sqrt{4(n-1)+n^2H^2}-(n-2)|H|)^2}{4(n-1)},\ -n(1+H^2)\right).$$

One should compare the bound

$$-n(1+H^2) - \frac{n(\sqrt{4(n-1)+n^2H^2}-(n-2)|H|)^2}{4(n-1)}$$

with the pinching constant in the rigidity theorem of Cheng and Nakagawa [7] or Alencar and do Carmo [1].

2 Preliminaries

Throughout this paper, all manifolds are assumed to be smooth and connected without boundary. Let $\varphi : M \to S^{n+1}(1)$ be an n-dimensional hypersurface in a unit sphere $S^{n+1}(1)$. We choose a local orthonormal frame $\{\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{e}_{n+1}\}$ and the dual coframe $\{\omega_1, \ldots, \omega_n, \omega_{n+1}\}$ in such a way that $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a local orthonormal frame on M. Hence, we have

$$\omega_{n+1} = 0$$

on M. From Cartan's lemma, we have

$$\omega_{in+1} = \sum_{j=1}^{n} h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2.1}$$

The mean curvature H and the second fundamental form II of $\varphi : M \to S^{n+1}(1)$ are defined, respectively, by

$$H = \frac{1}{n}\sum_{i=1}^{n} h_{ii}, \ II = \sum_{i,j=1}^{n} h_{ij}\omega_i \otimes \omega_j \mathbf{e}_{n+1}.$$

When the mean curvature H of $\varphi : M \to S^{n+1}(1)$ is identically zero, we recall that $\varphi : M \to S^{n+1}(1)$ is by definition *a minimal hypersurface*. From the structure equations of $\varphi : M \to S^{n+1}(1)$, Gauss equation is given by

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.2}$$

From (2.2), we have

$$n(n-1)r = n(n-1) + n^2H^2 - S,$$

where $n(n-1)r$ and S denote the scalar curvature and the squared norm of the second fundamental form of $\varphi : M \to S^{n+1}(1)$, respectively. Defining the covariant derivative of h_{ij} by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{ik}\omega_{kj} + \sum_k h_{kj}\omega_{ki}, \tag{2.3}$$

we obtain the Codazzi equations

$$h_{ijk} = h_{ikj}. \tag{2.4}$$

By taking exterior differentiation of (2.3), and defining

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_l h_{ljk}\omega_{li} + \sum_l h_{ilk}\omega_{lj} + \sum_l h_{ijl}\omega_{lk}, \tag{2.5}$$

we have the following Ricci identities:

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}. \tag{2.6}$$

For any C^2-function f on M, we define its gradient and Hessian by

$$df = \sum_{i=1}^{n} f_i\omega_i,$$

$$\sum_{j=1}^{n} f_{ij}\omega_j = df_i + \sum_{j=1}^{n} f_j\omega_{ji}.$$

Thus, the Laplace–Beltrami operator Δ is given by

$$\Delta f = \sum_{i=1}^{n} f_{ii}.$$

Example 2.1 For totally umbilical sphere $S^n(r)$ of radius $r > 0$, the first eigenvalue $\lambda_1^J = -n(1 + H^2)$ with $H = \frac{1}{r}$.

Example 2.2 For Clifford torus $S^{n-k}\left(\sqrt{\frac{n-k}{n}}\right) \times S^k\left(\sqrt{\frac{k}{n}}\right)$, $k = 1, 2, \ldots, n$, the first eigenvalue $\lambda_1^J = -2n$ with $H = 0$.

Example 2.3 For hypersurfaces $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$ with $0 < r < 1$, the principal curvatures are given by

$$k_1 = -\frac{\sqrt{1-r^2}}{r}, \quad k_2 = \cdots = k_n = \frac{r}{\sqrt{1-r^2}}.$$

Hence, we know that

$$nH = \frac{nr^2 - 1}{r\sqrt{1-r^2}}, \quad S = \frac{1 - 2r^2 + nr^4}{r^2(1-r^2)}.$$

For $r^2 \geq \frac{1}{n}$, by a direct computation, we know that the first eigenvalue λ_1^J of the Jacobi operator J on $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$ satisfies

$$\lambda_1^J = -n(1 + H^2) - \frac{n\left(\sqrt{4(n-1) + n^2H^2} - (n-2)|H|\right)^2}{4(n-1)}.$$

For $n \geq 5$ and $\frac{1}{n} \leq r^2 < \frac{n}{(n-2)^2}$, we know the hypersurface $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$ satisfies

$$n^2H^2 < \frac{16(n-1)}{n(n-4)}$$

and

$$\lambda_1^J = -n(1 + H^2) - \frac{n\left(\sqrt{4(n-1) + n^2H^2} - (n-2)|H|\right)^2}{4(n-1)}.$$

The hypersurface $S^1\left(\frac{\sqrt{n}}{n-2}\right) \times S^{n-1}\left(\frac{\sqrt{(n-1)(n-4)}}{n-2}\right)$ satisfies

$$\lambda_1^J = -2(n-1)(1 + H^2) + \frac{(n-2)^4}{8(n-1)}H^2$$

with $n^2H^2 = \frac{16(n-1)}{n(n-4)}$.

3 Proof of Theorem 1.1

In this section, we give a proof of the Theorem 1.1.

Proof of Theorem 1.1 When $H \equiv 0$, according to the result of Wu [11], we have $\lambda_1^J \leq -2n$ and $\lambda_1^J = -2n$ if and only if $\varphi : M \to S^{n+1}(1)$ is isometric to a Clifford torus $S^{n-k}\left(\sqrt{\frac{n-k}{n}}\right) \times S^k\left(\sqrt{\frac{k}{n}}\right)$, for $k = 1, 2, \ldots, n-1$.

From now we assume $H \neq 0$. By making use of the Codazzi equations, Ricci identities and a standard computation of Simons' type formula (cf. [5–8,10]), we have

$$\frac{1}{2}\Delta S = \sum_{i,j,k=1}^{n} h_{ijk}^2 + nS - n^2H^2 + nHf_3 - S^2, \tag{3.1}$$

where $f_3 = \sum_{i=1}^n k_i^3$ and k_i, $i = 1, 2, \ldots, n$ denote the principal curvatures.

Putting $\mu_i = k_i - H$, we have

$$B := \sum_{i=1}^{n} \mu_i^2 = S - nH^2 \geq 0, \quad f_3 = B_3 + 3HB + nH^3, \tag{3.2}$$

where $B_3 = \sum_{i=1}^n \mu_i^3$. The following inequality is known (cf. [7,8]):

$$|B_3| \leq \frac{n-2}{\sqrt{n(n-1)}} B^{\frac{3}{2}}, \tag{3.3}$$

and the equality holds if and only if at least $n-1$ of k_i, for $i = 1, 2, \ldots, n$, are equal with each other. Since H is constant, we can assume $H > 0$. Thus, from (3.1), (3.2) and (3.3), we have

$$\frac{1}{2}\Delta B = \frac{1}{2}\Delta S \geq \sum_{i,j,k=1}^{n} h_{ijk}^2 + B(n + nH^2 - B) - nH\frac{n-2}{\sqrt{n(n-1)}} B^{\frac{3}{2}}. \tag{3.4}$$

For any constant $\alpha > 0$ and $\varepsilon > 0$, we consider a function $f_\varepsilon = (B + \varepsilon)^\alpha > 0$. Hence, we have, from (3.4),

$$\begin{aligned} \Delta f_\varepsilon &= \alpha(\alpha-1)(B+\varepsilon)^{\alpha-2}|\nabla B|^2 + \alpha(B+\varepsilon)^{\alpha-1}\Delta B \\ &\geq \alpha(\alpha-1)(B+\varepsilon)^{\alpha-2}|\nabla B|^2 \\ &\quad + 2\alpha(B+\varepsilon)^{\alpha-1}\left(\sum_{i,j,k=1}^{n} h_{ijk}^2 + B(n + nH^2 - B) - nH\frac{n-2}{\sqrt{n(n-1)}} B^{\frac{3}{2}}\right). \end{aligned} \tag{3.5}$$

Since H is constant, we have

$$\begin{aligned} \nabla_k(nH) &= \sum_{i=1}^{n} h_{iik} = 0, \quad h_{kkk}^2 \leq (n-1)\sum_{i \neq k} h_{iik}^2 \\ |\nabla B|^2 &= \sum_{k=1}^{n}(2\sum_{i=1}^{n}\mu_i h_{iik})^2 \leq 4B\sum_{i,k=1}^{n} h_{iik}^2. \end{aligned} \tag{3.6}$$

Thus, we obtain

$$
\begin{aligned}
|\nabla B|^2 &\le 4B \sum_{i,k=1}^{n} h_{iik}^2 \\
&= 4B \left(\frac{n}{n+2} \sum_{k=1}^{n} h_{kkk}^2 + \frac{2}{n+2} \sum_{k=1}^{n} h_{kkk}^2 + \sum_{i \neq k} h_{iik}^2 \right) \\
&\le \frac{4n}{n+2} B \left(\sum_{k=1}^{n} h_{kkk}^2 + 3 \sum_{i \neq k} h_{iik}^2 \right). \qquad (3.7)
\end{aligned}
$$

For any constant β, we have

$$
\begin{aligned}
\lambda_1^J \int_M f_\varepsilon^2 dA &\le -\int_M f_\varepsilon J f_\varepsilon dA \\
&= -\beta \int_M f_\varepsilon \Delta f_\varepsilon dA - \int_M \left((1-\beta) f_\varepsilon \Delta f_\varepsilon + (S+n) f_\varepsilon^2 \right) dA \\
&= \beta \int_M |\nabla f_\varepsilon|^2 dA - \int_M f_\varepsilon \left\{ (1-\beta) \left(\alpha(\alpha-1)(B+\varepsilon)^{\alpha-2} |\nabla B|^2 \right.\right. \\
&\quad \left.\left. + \alpha (B+\varepsilon)^{\alpha-1} \Delta B \right) + (B + nH^2 + n) f_\varepsilon \right\} dA \\
&= \alpha \int_M f_\varepsilon \{1 + 2\alpha\beta - \beta - \alpha\} (B+\varepsilon)^{\alpha-2} |\nabla B|^2 dA \\
&\quad - \int_M f_\varepsilon^2 \left\{ \frac{\alpha(1-\beta)}{B+\varepsilon} \Delta B + B + nH^2 + n \right\} dA.
\end{aligned}
$$

By taking α and β satisfying

$$
\alpha > \frac{n-2}{4n}, \quad 1 - \beta = \frac{2n\alpha}{4n\alpha + 2 - n}, \qquad (3.8)
$$

we have

$$
(n-2)(1-\beta) - 4n\alpha(1-\beta) + 2n\alpha = 0.
$$

Since

$$
\sum_{i,j,k=1}^{n} h_{ijk}^2 = \sum_{k=1}^{n} h_{kkk}^2 + 3 \sum_{i \neq k} h_{iik}^2 + \sum_{i \neq j \neq k \neq i}^{n} h_{ijk}^2,
$$

from (3.7), we obtain

$$
\begin{aligned}
&(1 + 2\alpha\beta - \beta - \alpha) |\nabla B|^2 - 2(1-\beta)(B+\varepsilon) \sum_{i,j,k=1}^{n} h_{ijk}^2 \\
&\le \frac{2}{n+2} B \{(n-2)(1-\beta) - 4n\alpha(1-\beta) + 2n\alpha\} \left(\sum_{k=1}^{n} h_{kkk}^2 + 3 \sum_{i \neq k} h_{iik}^2 \right) = 0. \qquad (3.9)
\end{aligned}
$$

Thus, we infer

$$\begin{aligned}
&\lambda_1^J \int_M f_\varepsilon^2 dA \\
&\quad \le \alpha \int_M f_\varepsilon (B+\varepsilon)^{\alpha-2} \left\{ (1+2\alpha\beta-\beta-\alpha)\,|\nabla B|^2 - 2(1-\beta)(B+\varepsilon) \sum_{i,j,k=1}^n h_{ijk}^2 \right\} dA \\
&\qquad - \int_M f_\varepsilon^2 \left\{ \frac{2\alpha(1-\beta)B}{B+\varepsilon} \left((n+nH^2-B) - nH \frac{(n-2)}{\sqrt{n(n-1)}} B^{\frac{1}{2}} \right) + B + nH^2 + n \right\} dA \\
&\quad \le -\int_M f_\varepsilon^2 \frac{B}{B+\varepsilon} \left(\{1-2\alpha(1-\beta)\}\, B - \frac{2\alpha(1-\beta)(n-2)}{\sqrt{n(n-1)}} nHB^{\frac{1}{2}} + \varepsilon \right) dA \\
&\qquad - 2\alpha(1-\beta)(n+nH^2) \int_M f_\varepsilon^2 \frac{B}{B+\varepsilon} dA - (n+nH^2) \int_M f_\varepsilon^2 dA.
\end{aligned}$$

For $1-2\alpha(1-\beta) > 0$, we obtain

$$\begin{aligned}
&\lambda_1^J \int_M f_\varepsilon^2 dA \\
&\quad \le \int_M f_\varepsilon^2 \frac{B}{B+\varepsilon} \left(\frac{\alpha^2(1-\beta)^2(n-2)^2}{(1-2\alpha(1-\beta))n(n-1)} (nH)^2 - \varepsilon \right) dA \\
&\qquad - 2\alpha(1-\beta)(n+nH^2) \int_M f_\varepsilon^2 \frac{B}{B+\varepsilon} dA - (n+nH^2) \int_M f_\varepsilon^2 dA.
\end{aligned}$$

Since $\varphi : M \to S^{n+1}(1)$ is not totally umbilical, we have

$$\lim_{\varepsilon \to 0} \int_M f_\varepsilon^2 dA = \int_M B^{2\alpha} dA > 0.$$

Letting $\varepsilon \to 0$, we derive

$$\lambda_1^J \le -(1+2\alpha(1-\beta))n(1+H^2) + \frac{\alpha^2(1-\beta)^2}{(1-2\alpha(1-\beta))} \frac{(n-2)^2}{n(n-1)} n^2 H^2. \tag{3.10}$$

For $n = 2$, we have

$$\lambda_1^J \le -(1+2\alpha(1-\beta))n(1+H^2).$$

From (3.8), we have $\beta = \dfrac{1}{2}$ for any $0 < \alpha < 1$. Hence, we obtain

$$\lambda_1^J \le -2n(1+H^2).$$

For $2 < n \le 4$ or $n \ge 5$ and $n^2H^2 < \dfrac{16(n-1)}{n(n-4)}$, we have

$$\frac{1}{2} > \frac{1}{2}\left(1 - \sqrt{\frac{(n-2)^2 H^2}{4(n-1)+n^2H^2}} \right) > \frac{1}{2} - \frac{1}{n} \ge \frac{1}{2} - \frac{1}{\sqrt{2n}}. \tag{3.11}$$

Observe from (3.8) that $1-2\alpha(1-\beta) > 0$ if and only if

$$\frac{1}{2} - \frac{1}{\sqrt{2n}} < \alpha < \frac{1}{2} + \frac{1}{\sqrt{2n}}. \tag{3.12}$$

Defining

$$w(\alpha) = \alpha(1-\beta) = \frac{2n\alpha^2}{4n\alpha + 2 - n},$$

$w(\alpha)$ is an increasing function of α, for $\alpha > \frac{1}{2} - \frac{1}{n}$ and

$$w\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{2} - \frac{1}{n}, \quad w\left(\frac{1}{2} + \frac{1}{\sqrt{2n}}\right) = \frac{1}{2}.$$

According to (3.11) and (3.12), there exists a α satisfying

$$\frac{1}{2} - \frac{1}{n} < \alpha < \frac{1}{2} + \frac{1}{\sqrt{2n}}$$

such that

$$w(\alpha) = \frac{1}{2}\left(1 - \sqrt{\frac{(n-2)^2H^2}{4(n-1)+n^2H^2}}\right). \tag{3.13}$$

Therefore, we have, for this α,

$$1 - 2\alpha(1-\beta) = \sqrt{\frac{(n-2)^2H^2}{4(n-1)+n^2H^2}} > 0. \tag{3.14}$$

From (3.10), we obtain

$$\begin{aligned} \lambda_1^J \le & -n(1+H^2) \\ & - 2\alpha(1-\beta)n\frac{4(n-1)\left(1-2\alpha(1-\beta)\right)(1+H^2) - 2\alpha(1-\beta)(n-2)^2H^2}{4(n-1)\left(1-2\alpha(1-\beta)\right)}. \end{aligned} \tag{3.15}$$

From (3.14), we infer

$$\begin{aligned} & 4(n-1)\left(1-2\alpha(1-\beta)\right)(1+H^2) - 2\alpha(1-\beta)(n-2)^2H^2 \\ & = \left\{4(n-1)(1+H^2)\sqrt{\frac{(n-2)^2H^2}{4(n-1)+n^2H^2}} - \left(1 - \sqrt{\frac{(n-2)^2H^2}{4(n-1)+n^2H^2}}\right)(n-2)^2H^2\right\} \\ & = 4(n-1)\sqrt{\frac{(n-2)^2H^2}{4(n-1)+n^2H^2}} - (n-2)^2H^2 + \sqrt{\frac{(n-2)^2H^2}{4(n-1)+n^2H^2}}n^2H^2 \\ & = \sqrt{4(n-1)+n^2H^2}\sqrt{(n-2)^2H^2} - (n-2)^2H^2 \\ & = \sqrt{(n-2)^2H^2}\left(\sqrt{4(n-1)+n^2H^2} - (\sqrt{(n-2)^2H^2}\right). \end{aligned}$$

From (3.14), (3.15) and the above equality, we obtain

$$\begin{aligned}\lambda_1^J \leq & -n(1+H^2) - \frac{n\left(1-\sqrt{\frac{(n-2)^2H^2}{4(n-1)+n^2H^2}}\right)}{4(n-1)\sqrt{\frac{(n-2)^2H^2}{4(n-1)+n^2H^2}}} \\ & \times \sqrt{(n-2)^2H^2}\left(\sqrt{4(n-1)+n^2H^2} - \left(\sqrt{(n-2)^2H^2}\right)\right) \\ = & -n(1+H^2) - \frac{n}{4(n-1)}\left(\sqrt{4(n-1)+n^2H^2} - (n-2)|H|\right)^2.\end{aligned}$$

If the equality holds, we know that $h_{ijk} = 0$, for any $i, j, k = 1, 2, \ldots, n$. Hence, we know that the second fundamental form is parallel and S is constant. Thus, we know that $\varphi : M \to S^{n+1}(1)$ is isometric to $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$ since, from the (3.3), the $n-1$ of the principal curvatures are equal with each other. From the examples in the Sect. 2, we know that r satisfies

$$\begin{cases} r^2 > \dfrac{1}{n} & \text{for } 2 \leq n \leq 4, \\ \dfrac{1}{n} < r^2 < \dfrac{n}{(n-2)^2}, & \text{for } n \geq 5 \text{ and } n^2H^2 < \dfrac{16(n-1)}{n(n-4)}. \end{cases}$$

If $n \geq 5$ and $n^2H^2 \geq \frac{16(n-1)}{n(n-4)}$, we take

$$\alpha(1-\beta) = \frac{1}{2} - \frac{1}{n},$$

that is,

$$\beta = 0 \quad \text{and} \quad \alpha = \frac{1}{2} - \frac{1}{n},$$

Thus, the inequality (3.10) becomes

$$\lambda_1^J \leq -2(n-1)(1+H^2) + \frac{(n-2)^4}{8(n-1)}H^2.$$

If the equality holds, we know

$$(1-2\alpha)\sqrt{B} = \frac{\alpha(n-2)}{\sqrt{n(n-1)}}nH.$$

Thus, we have

$$S = B + nH^2 = nH^2 + \frac{(n-2)^4}{16n(n-1)}n^2H^2. \tag{3.16}$$

because of

$$\alpha = \frac{1}{2} - \frac{1}{n}.$$

Since S is constant, the first eigenvalue λ_1^J of the Jacobi operator is given by

$$\lambda_1^J = -S - n = -2(n-1)(1+H^2) + \frac{(n-2)^4}{8(n-1)}H^2.$$

Hence, we obtain

$$S = n - 2 + 2(n-1)H^2 - \frac{(n-2)^4}{8(n-1)}H^2. \tag{3.17}$$

From (3.16) and (3.17), we get

$$n - 2 = (2-n)H^2 + \frac{(n-2)^4(n+2)}{16(n-1)}H^2,$$
$$1 = \frac{n(n-4)}{16(n-1)}n^2H^2,$$

that is,

$$n^2H^2 = \frac{16(n-1)}{n(n-4)}.$$

Since, from the (3.3), the $n-1$ of the principal curvatures are equal with each other, From the examples in the Sect. 2, we know that $\varphi : M \to S^{n+1}(1)$ is isometric to $S^1\left(\frac{\sqrt{n}}{n-2}\right) \times S^{n-1}\left(\frac{\sqrt{(n-1)(n-4)}}{n-2}\right)$. It completes the proof of Theorem 1.1. □

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