EXTRINSIC ESTIMATES FOR EIGENVALUES OF THE LAPLACE OPERATOR

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Abstract. For a bounded domain in a complete Riemannian manifold M^n isometrically immersed in a Euclidean space, we derive extrinsic estimates for eigenvalues of the Dirichlet eigenvalue problem of the Laplace operator, which depend on the mean curvature of the immersion. Further, we also obtain an upper bound for the $(k+1)^{\text{th}}$ eigenvalue, which is best possible in the meaning of order on k.

1. introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$. The Dirichlet eigenvalue problem of the Laplacian is given by

$$\begin{cases} \Delta u = -\lambda u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

It is well known that the spectrum of this problem is real and purely discrete

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \nearrow \infty,$$

where each λ_i has finite multiplicity which is repeated according to its multiplicity.

The investigation of universal inequalities for eigenvalues of (1.1) was initiated by Payne, Pólya and Weinberger [13] and [14]. They proved

$$\lambda_{k+1} - \lambda_k \le \frac{4}{nk} \sum_{i=1}^{k} \lambda_i. \tag{1.2}$$

Although the result of Payne, Pólya and Weinberger has been extended by many mathematicians in several way, there are two main contributions due to Hile and Protter [9] and Yang [15]. In 1980, Hile and Protter improved the result of Payne, Pólya and Weinberger to

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \ge \frac{nk}{4}. \tag{1.3}$$

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Further, Yang [15] (cf. [6]) has obtained very sharp inequality, that is, he has derived

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\big(\lambda_{k+1}-(1+\frac{4}{n})\lambda_i\big)\leq 0. \tag{1.4}$$

From (1.4), one can infer

$$\lambda_{k+1}\leq \frac{1}{k}(1+\frac{4}{n})\sum_{i=1}^{k}\lambda_i. \tag{1.5}$$

The inequalities (1.4) and (1.5) are called Yang's first inequality and second inequality, respectively (cf. [1], [2]). By making use of the Chebyshev's inequality, it is not difficult to prove the following relation

$$(1.4)\Longrightarrow(1.5)\Longrightarrow(1.3)\Longrightarrow(1.2).$$

In [1] and [2], Ashbaugh has also given a different proof.

On the other hand, from the Weyl's asymptotic formula, one has

$$\lambda_k\sim\frac{4\pi^2}{(\omega_n\mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}},\quad k\to\infty,$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$. Further, Pólya conjectured eigenvalue λ_k should satisfy

$$\lambda_k\geq\frac{4\pi^2}{(\omega_n\mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}},$$

for $k=1,2,\cdots$. On the conjecture of Pólya, Li and Yau [12] attacked it and obtained

$$\lambda_k\geq\frac{n}{n+2}\frac{4\pi^2}{(\omega_n\mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}},$$

for $k=1,2,\cdots$. Recently, Cheng and Yang [6] have obtained a very sharp upper bound of λ_{k+1}, that is, they have proved

$$\lambda_{k+1}\leq C_0(n,k)k^{2/n}\lambda_1,$$

where

$$C_0(n,k)=\begin{cases}\dfrac{j^2_{n/2,1}}{j^2_{n/2-1,1}}, & \text{for}\ \ k=1\\ 1+\dfrac{a(\min\{n,k-1\})}{n}, & \text{for}\ \ k\geq 2\end{cases}$$

and $a(1)\leq 2.64$ and $a(m)\leq 2.2-4\log(1+\frac{m-3}{50})$ for $m\geq 2$ is a constant depending only on m, and $j_{p,k}$ denotes the k^{th} positive zero of the standard Bessel function $J_p(x)$ of the first kind of order p. From the Weyl's asymptotic formula, we know that the upper bound of Cheng and Yang is best possible in the meaning of order on k. It is natural and important to obtain universal inequalities for eigenvalues of the Dirichlet eigenvalue problem on a bounded domain in a complete Riemannian manifold. Since the Weyl's asymptotic formula also holds in this case, it is also important to obtain the lower bound and upper bound of λ_k.

For the Dirichlet eigenvalue problem of the Laplacian on a compact homogeneous Riemannian manifold or on a compact minimal submanifold in a sphere, many mathematicians have studied universal inequalities for eigenvalues (for examples [3], [5], [7], [8], [10], [11], [16] and so on). More recently, Cheng and Yang [3], [5] have derived universal inequalities for eigenvalues of the Dirichlet eigenvalue problem of the Laplacian on a domain in a sphere or in a complex projective space. The upper bound for λ_{k+1} can also be obtained by the same proof as in [6].

Unfortunately, for a general complete Riemannian manifold, it is very hard to find an appropriate trial function with "nice" properties such that one can infer universal inequalities for eigenvalues. Fortunately, we have the Nash's theorem: each complete Riemannian manifold can be isometrically immersed in a Euclidean space. In this paper, we shall make use of this theorem to construct appropriate trial functions with "nice" properties. By making use of these trial functions, we derive universal inequalities for eigenvalues and the upper bounds for eigenvalues, which is best possible in the meaning of order on k.

Theorem 1.1. *Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M^n isometrically immersed in $\mathbb{R}^N$. For the Dirichlet eigenvalue problem of the Laplacian:*

$$\begin{cases} \Delta u = \lambda u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.6}$$

we have

$$\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)\mu_i, \tag{1.7}$$

where $\mu_i = \lambda_i + \frac{n^2}{4}\|H\|^2$, λ_i denotes the i^{th} eigenvalue of (1.6) and H is the mean curvature vector field of M^n with $\|H\|^2 = \sup_{\Omega}|H|^2$.

Since the formula (1.7) is a quadratic inequality of μ_{k+1}, it is easy to infer the following:

Corollary 1.1. *Under the same assumptions as in the theorem 1.1, we have*

$$\mu_{k+1} \leq \left(1+\frac{4}{n}\right)\frac{1}{k}\sum_{i=1}^{k}\mu_i. \tag{1.8}$$

Remark 1.1. Our universal inequality (1.7) is the Yang-type first inequality and (1.8) is the Yang-type second inequality.

In particular, when M^n is isometrically minimally immersed in $\mathbb{R}^N$, we have

Corollary 1.2. *Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M^n isometrically minimally immersed in $\mathbb{R}^N$. Then, we have*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i. \tag{1.9}$$

Remark 1.2. Since $\mathbb{R}^n$ can be seen as a totally geodesic minimal hypersurface in $\mathbb{R}^{n+1}$, we know that the results of Yang is included in the corollary 1.2. Further, since there exist many complete minimal submanifolds in $\mathbb{R}^N$, we know that the Yang's inequalities for eigenvalues also hold for any bounded domain in any complete minimal submanifold in $\mathbb{R}^N$.

Since the n-dimensional unit sphere $S^n(1)$ can be seen as a totally umbilical hypersurface with the mean curvature 1 in $\mathbb{R}^{n+1}$, from our theorem 1.1, we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i+\frac{n^2}{4}).$$

which is the Yang-type inequality for eigenvalues of the Dirichlet eigenvalue problem of Laplacian on a domain in a unit sphere obtained by Cheng and Yang [3].

In order to obtain the upper bound for λ_{k+1}, the universal inequality for lower order eigenvalues of the eigenvalue problem (1.6) is necessary.

Theorem 1.2. *Under the same assumptions as in the theorem 1.1, we have*

$$\frac{\mu_2+\mu_3+\cdots+\mu_{n+1}}{\mu_1} \leq n+4. \tag{1.10}$$

Corollary 1.3. *Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M^n isometrically minimally immersed in $\mathbb{R}^N$. Then, we have*

$$\frac{\lambda_2+\lambda_3+\cdots+\lambda_{n+1}}{\lambda_1} \leq n+4. \tag{1.11}$$

Remark 1.3. According to the same arguments as in the remark 1.2, we know that the result for lower order eigenvalues of Payne, Pólya and Weinberger [14] does not only hold for a bounded domain in $\mathbb{R}^n$, but also for a bounded domain in any complete minimal submanifod in $\mathbb{R}^N$.

Since the upper bound for λ_{k+1} of Cheng and Yang [6] does hold not only for eigenvalues, but also for any positive real numbers which satisfy Yang's first inequality and the inequality of Payne, Pólya and Weinberger which is same as (1.11), we infer, from the theorem 1.1 and the therorem 1.2,

Theorem 1.3. *Under the same assumptions as in theorem 1.1, we have*

$$\mu_{k+1} \leq (1+\frac{a(min\{n,k-1\})}{n})k^{2/n}\mu_1,$$

where the bound of $a(m)$ can be formulated as:

$$\begin{cases} a(0) \leq 4, \\ a(1) \leq 2.64, \\ a(m) \leq 2.2-4\log(1+\frac{1}{50}(m-3)), \qquad \textit{for} \quad m \geq 2. \end{cases}$$

In particular, for $n \geq 41$ and $k \geq 41$, we have

$$\mu_{k+1} \leq k^{2/n}\mu_1.$$

Especially, when M^n is a complete minimal submanifold in $\mathbb{R}^N$, we have

Corollary 1.4. *Under the same assumptions as in the corollary 1.2, we have*

$$\lambda_{k+1} \leq (1 + \frac{a(min\{n, k-1\})}{n})k^{2/n}\lambda_1$$

and when $n \geq 41$ *and* $k \geq 41$*,*

$$\lambda_{k+1} \leq k^{2/n}\lambda_1.$$

2. Proof of Theorem 1.1

Throughout this paper we will agree the following convention on ranges of indices:

$$1 \leq i, j, \cdots, \leq n; \quad 1 \leq \alpha, \beta, \cdots, \leq N; \quad n+1 \leq A, B, \cdots, \leq N.$$

Let M^n be an n-dimensional complete Riemannian manifold isometrically immersed in $\mathbb{R}^N$. Let $\Omega \subset M^n$ be a bounded domain of M^n and $P \in \Omega$ be an arbitrary point of Ω. Let $(x^1, \cdots, x^n)$ be an arbitrary coordinate system in a neighborhood U of P in M^n. Assume that y with components y^α defined by

$$y^\alpha = y^\alpha(x^1, \cdots, x^n), \qquad 1 \leq \alpha \leq N,$$

is the position vector of P in $\mathbb{R}^N$. Since M^n is isometrically immersed in $\mathbb{R}^N$, then

$$g_{ij} = g\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) = \langle \sum_{\alpha=1}^{N} \frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha}, \sum_{\beta=1}^{N} \frac{\partial y^\beta}{\partial x^j}\frac{\partial}{\partial y^\beta}\rangle = \sum_{\alpha=1}^{N} \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\alpha}{\partial x^j}, \tag{2.1}$$

where g denotes the induced metric of M^n from $\mathbb{R}^N$, $\langle , \rangle$ is the standard inner product in $\mathbb{R}^N$. At the point P,

$$\sum_{\alpha=1}^{N} g(\nabla y^\alpha, \nabla y^\alpha) = \sum_{\alpha=1}^{N}\sum_{i,j=1}^{n} \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\alpha}{\partial x^j} g^{ij} = \sum_{i,j=1}^{n} g^{ij} g_{ij} = n, \tag{2.2}$$

where ∇ is the gradient operator on M^n.

Lemma 2.1. *For any function* $u \in C^\infty(M^n)$*, we have*

$$\sum_{\alpha=1}^{N} \Big(g(\nabla y^\alpha, \nabla u)\Big)^2 = |\nabla u|^2, \tag{2.3}$$

$$\sum_{\alpha=1}^{N} (\Delta y^\alpha)^2 = n^2|H|^2, \tag{2.4}$$

$$\sum_{\alpha=1}^{N} \Delta y^\alpha \nabla y^\alpha = 0, \tag{2.5}$$

where $|H|$ *is the mean curvature of* M^n*.*

Proof. Let ∇' and $\mathbf{h}$ denote the connection of R^N and the second fundamental form of M^n, respectively. We choose a new coordinate system $\bar{y} = (\bar{y}^1, \cdots, \bar{y}^N)$ of $\mathbb{R}^N$ given by $y - y(P) = \bar{y}A$ such that $(\frac{\partial}{\partial \bar{y}^1})_P, \cdots, (\frac{\partial}{\partial \bar{y}^n})_P$ span $T_P M^n$ and at P, $g\Big(\frac{\partial}{\partial \bar{y}^i}, \frac{\partial}{\partial \bar{y}^j}\Big) = \delta_{ij}$, where $A = (a^\alpha_\beta) \in O(N)$ is an $N \times N$ orthogonal matrix. Then we have

$$\nabla'_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{\alpha=1}^{N} \frac{\partial^2 \bar{y}^\alpha}{\partial x^i \partial x^j} \frac{\partial}{\partial \bar{y}^\alpha}. \tag{2.6}$$

Therefore, from the formula of Gauss, we infer

$$h^A_{ij} = h^A(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \frac{\partial^2 \bar{y}^A}{\partial x^i \partial x^j}, \tag{2.7}$$

where $h^A_{ij} = \langle \nabla'_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial \bar{y}^A} \rangle$ denotes the component of the second fundamental form $\mathbf{h}$ of M^n. For $u \in C^\infty(M^n)$, at P, we have

$$\begin{aligned}
\sum_{\alpha=1}^{N} \Big(g(\nabla y^\alpha, \nabla u)\Big)^2 &= \sum_{\alpha=1}^{N} \Big[g\Big(\nabla(y^\alpha(P) + \sum_{\beta=1}^{N} a^\alpha_\beta \bar{y}^\beta), \nabla u\Big)\Big]^2 \\
&= \sum_{\alpha=1}^{N} \Big[g\Big(\sum_{\beta=1}^{N} a^\alpha_\beta \nabla \bar{y}^\beta, \nabla u\Big)\Big]^2 \\
&= \sum_{\alpha=1}^{N} \Big(\sum_{\beta=1}^{N} \sum_{i=1}^{n} a^\alpha_\beta \frac{\partial \bar{y}^\beta}{\partial \bar{y}^i} \frac{\partial u}{\partial \bar{y}^i}\Big)^2 \\
&= \sum_{i=1}^{n} \Big(\sum_{\alpha=1}^{N} a^\alpha_i a^\alpha_i\Big) \frac{\partial u}{\partial \bar{y}^i} \frac{\partial u}{\partial \bar{y}^i} \\
&= |\nabla u|^2.
\end{aligned}$$

where $|\nabla u|^2 = g(\nabla u, \nabla u)$. Since P is any point, it finishes the proof of (2.3).

Let H be the mean curvature vector of M^n. Then, from the standard calculation, we have

$$\Delta y = nH. \tag{2.8}$$

Therefore, we derive

$$\sum_{\alpha=1}^{N} (\Delta y^\alpha)^2 = n^2 |H|^2. \tag{2.9}$$

Since $\nabla_i y$ is tangent to M^n, we have

$$\sum_{\alpha=1}^{N} \Delta y^\alpha \nabla_i y^\alpha = \langle \Delta y, \nabla_i y \rangle = 0.$$

Thus, we have

$$\sum_{\alpha=1}^{N} \Delta y^\alpha \nabla y^\alpha = 0. \tag{2.10}$$

$\square$

The following theorem of Cheng and Yang [5] will play an important role to prove our theorem 1.1

Theorem CY. *Let λ_i be the i^{th} eigenvalue of the Dirichlet eigenvalue problem on an n-dimensional compact Riemannian manifold $\bar{\Omega} = \Omega \cup \partial\Omega$ with boundary $\partial\Omega$ and u_i be the orthonormal eigenfunction corresponding to λ_i. Then, for any function $f \in C^3(\Omega) \cap C^2(\partial\Omega)$ and any integer k, we have*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\|u_i\nabla f\|^2 \leq \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\|2\nabla f\cdot\nabla u_i + u_i\Delta f\|^2,$$

where $\|f\|^2 = \int_M f^2$ and $\nabla f \cdot \nabla u_i = g(\nabla f, \nabla u_i)$.

Proof of Theorem 1.1 Let u_i be the eigenfunction corresponding to the eigenvalue λ_i such that $\{u_i\}_{i\in\mathbb{N}}$ becomes an orthonormal basis of $L^2(\Omega)$. Put $f^\alpha = y^\alpha, 1 \leq \alpha \leq N$. Since M^n is complete and Ω is a bounded domain, we know that $\bar{\Omega}$ is a compact Riemannian manifold with boundary. From the theorem CY of Cheng and Yang, we infer

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\|u_i\nabla f^\alpha\|^2 \leq \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\|2\nabla f^\alpha\cdot\nabla u_i + u_i\Delta f^\alpha\|^2. \tag{2.11}$$

Taking sum on α from 1 to N, we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\sum_{\alpha=1}^{N}\|u_i\nabla f^\alpha\|^2 \leq \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\sum_{\alpha=1}^{N}\|2\nabla f^\alpha\cdot\nabla u_i + u_i\Delta f^\alpha\|^2.$$

From (2.2) and the lemma 2.1, we infer

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\sum_{\alpha=1}^{N}\|u_i\nabla f^\alpha\|^2 = \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i^2\sum_{\alpha=1}^{N}|\nabla y^\alpha|^2 = n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2,$$

$$\begin{aligned}
&\sum_{\alpha=1}^{N}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\|2\nabla y^\alpha\cdot\nabla u_i + u_i\Delta y^\alpha\|^2\\
&= \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_\Omega\Big\{4\sum_{\alpha=1}^{N}|\nabla y^\alpha\cdot\nabla u_i|^2 + u_i^2\sum_{\alpha=1}^{N}(\Delta y^\alpha)^2 + 2\sum_{\alpha=1}^{N}(\Delta y^\alpha\nabla y^\alpha)\cdot\nabla u_i^2\Big\}\\
&\leq 4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_\Omega|\nabla u_i|^2 + n^2\|H\|^2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\\
&= 4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i + n^2\|H\|^2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i),
\end{aligned}$$

where $\|H\|^2 = \sup_\Omega |H|^2$. Therefore, we derive

$$n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq 4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i + n^2\|H\|^2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i).$$

Putting $\mu_i = \lambda_i + \frac{n^2}{4}\|H\|$, we obtain the inequality (1.7).

$\square$

3. Proof of Theorem 1.2

In this section we shall give a proof of the theorem 1.2

Proof. Let u_i be the eigenfunction corresponding to the eigenvalue λ_i such that $\{u_i\}_{i\in\mathbb{N}}$ becomes an orthonormal basis of $L^2(\Omega)$. Hence, $\int_\Omega u_i u_j = \delta_{ij}$ for $\forall\ i, j \in \mathbb{N}$. We consider the $N \times N$-matrix $B = (b_{\alpha\beta})$ defined by $b_{\alpha\beta} = \int_\Omega y^\alpha u_1 u_{\beta+1}$, where $y = (y^\alpha)$ is the position vector of the immersion in $\mathbb{R}^N$. From the orthogonalization of Gram and Schmidt, there exist an upper triangle matrix $R = (R_{\alpha\beta})$ and an orthogonal matrix $Q = (q_{\alpha\beta})$ such that $R = QB$. Thus,

$$R_{\alpha\beta} = \sum_{\gamma=1}^{N} q_{\alpha\gamma} b_{\gamma\beta} = \int_\Omega \sum_{\gamma=1}^{N} q_{\alpha\gamma} y^\gamma u_1 u_{\beta+1} = 0, \text{ for } 1 \le \beta < \alpha \le N. \tag{3.1}$$

Defining $g^\alpha = \sum_{\gamma=1}^{N} q_{\alpha\gamma} y^\gamma$, we have

$$\int_\Omega g^\alpha u_1 u_{\beta+1} = \int_\Omega \sum_{\gamma=1}^{N} q_{\alpha\gamma} y^\gamma u_1 u_{\beta+1} = 0, \text{ for } 1 \le \beta < \alpha \le N. \tag{3.2}$$

Therefore, the functions defined by

$$\Psi_\alpha = (g^\alpha - a^\alpha)u_1, \ a^\alpha = \int_\Omega g^\alpha u_1^2, \text{ for } \ 1 \le \alpha \le N \tag{3.3}$$

satisfy

$$\int_\Omega \Psi_\alpha u_{\beta+1} = 0, \qquad \text{for } 0 \le \beta < \alpha \le N.$$

From the Rayleigh-Ritz inequality, we have, for $1 \le \alpha \le N$,

$$\lambda_{\alpha+1} \le \frac{\|\nabla \Psi_\alpha\|^2}{\|\Psi_\alpha\|^2}. \tag{3.4}$$

From the definition of Ψ_α, we derive

$$\Delta\Psi_\alpha = \Delta g^\alpha u_1 + 2\nabla g^\alpha \cdot \nabla u_1 - \lambda_1 u_1 g^\alpha + \lambda_1 a^\alpha u_1. \tag{3.5}$$

Therefore, (3.4) can be written as

$$(\lambda_{\alpha+1} - \lambda_1)\|\Psi_\alpha\|^2 \le \int_\Omega (-\Delta g^\alpha u_1 - 2\nabla g^\alpha \cdot \nabla u_1)\Psi_\alpha. \tag{3.6}$$

From the Cauchy-Schwarz inequality, we obtain

$$\left(\int_\Omega (-\Delta g^\alpha u_1 - 2\nabla g^\alpha \cdot \nabla u_1)\Psi_\alpha\right)^2 \le \|\Psi_\alpha\|^2 \|(\Delta g^\alpha u_1 + 2\nabla g^\alpha \cdot \nabla u_1)\|^2. \tag{3.7}$$

Multiplying (3.7) by $(\lambda_{\alpha+1}-\lambda_1)$, we infer, from (3.6),

$$\begin{aligned}(\lambda_{\alpha+1}-\lambda_1)&\Big(\int_\Omega(-\Delta g^\alpha u_1-2\nabla g^\alpha\cdot\nabla u_1)\Psi_\alpha\Big)^2\\ &\le(\lambda_{\alpha+1}-\lambda_1)\|\Psi_\alpha\|^2\|(\Delta g^\alpha u_1+2\nabla g^\alpha\cdot\nabla u_1)\|^2\\ &\le\Big(\int_\Omega(-\Delta g^\alpha u_1-2\nabla g^\alpha\cdot\nabla u_1)\Psi_\alpha\Big)\|(\Delta g^\alpha u_1+2\nabla g^\alpha\cdot\nabla u_1)\|^2.\end{aligned}$$

Hence, we derive

$$(\lambda_{\alpha+1}-\lambda_1)\int_\Omega(-\Delta g^\alpha u_1-2\nabla g^\alpha\cdot\nabla u_1)\Psi_\alpha\le\|(\Delta g^\alpha u_1+2\nabla g^\alpha\cdot\nabla u_1)\|^2. \tag{3.8}$$

From the lemma 2.1 and the definition of g^α, taking sum on α from 1 to N, we have

$$\begin{aligned}&\sum_{\alpha=1}^N\|(\Delta g^\alpha u_1+2\nabla g^\alpha\cdot\nabla u_1)\|^2\\ &=\sum_{\alpha=1}^N\int_\Omega\{(\Delta g^\alpha)^2u_1^2+4(\nabla g^\alpha\cdot\nabla u_1)^2+2(\Delta g^\alpha\nabla g^\alpha)\cdot\nabla u_1^2\}\\ &=n^2\int_\Omega|H|^2u_1^2+4\int_\Omega|\nabla u_1|^2\\ &\le4\lambda_1+n^2\|H\|^2.\end{aligned} \tag{3.9}$$

From the Stokes' theorem and the definition of Ψ_α, we conclude

$$\begin{aligned}&\int_\Omega(-\Delta g^\alpha u_1-2\nabla g^\alpha\cdot\nabla u_1)\Psi_\alpha\\ &=\int_\Omega(-\Delta g^\alpha u_1-2\nabla g^\alpha\cdot\nabla u_1)(g^\alpha u_1-a^\alpha u_1)\\ &=\int_\Omega-\Delta g^\alpha g^\alpha u_1^2-\frac{1}{2}\nabla(g^\alpha)^2\cdot\nabla u_1^2\\ &=\int_\Omega|\nabla g^\alpha|^2u_1^2.\end{aligned} \tag{3.10}$$

By (3.8), (3.9), (3.10) and (2.2), we deduce

$$\sum_{\alpha=1}^N\lambda_{\alpha+1}\int_\Omega|\nabla g^\alpha|^2u_1^2\le(4+n)\lambda_1+n^2\|H\|^2. \tag{3.11}$$

For any point P, we use the same transformation of coordinates as in the proof of the lemma 2.1 $y-y(P)=\bar{y}A$. Since A and Q are orthogonal matrices, QA is also an orthogonal matrix. Hence, we have, for any α,

$$
\begin{aligned}
|\nabla g^\alpha|^2 &= g(\nabla g^\alpha, \nabla g^\alpha)\\
&= \sum_{\beta,\gamma=1}^{N} q_{\alpha\gamma} q_{\alpha\beta} g(\nabla y^\gamma, \nabla y^\beta)\\
&= \sum_{\beta,\gamma=1}^{N} q_{\alpha\gamma} q_{\alpha\beta} g(\sum_{\mu=1}^{N} a_\mu^\gamma \nabla \bar{y}^\mu, \sum_{\nu=1}^{N} a_\nu^\beta \nabla \bar{y}^\nu)\\
&= \sum_{\beta,\gamma,\mu,\nu=1}^{N} q_{\alpha\gamma} a_\mu^\gamma q_{\alpha\beta} a_\nu^\beta g(\nabla \bar{y}^\mu, \nabla \bar{y}^\nu)\\
&= \sum_{j=1}^{n} \big(\sum_{\beta=1}^{N} q_{\alpha\beta} a_j^\beta\big)^2 \le 1.
\end{aligned}
\tag{3.12}
$$

Therefore, from (3.12), we have

$$
\begin{aligned}
\sum_{\alpha=1}^{N} \lambda_{\alpha+1} |\nabla g^\alpha|^2 &\ge \sum_{i=1}^{n} \lambda_{i+1} |\nabla g^i|^2 + \lambda_{n+1} \sum_{A=n+1}^{N} |\nabla g^A|^2\\
&= \sum_{i=1}^{n} \lambda_{i+1} |\nabla g^i|^2 + \lambda_{n+1} (n - \sum_{i=1}^{n} |\nabla g^i|^2)\\
&= \sum_{i=1}^{n} \lambda_{i+1} |\nabla g^i|^2 + \lambda_{n+1} \sum_{i=1}^{n} (1 - |\nabla g^i|^2)\\
&\ge \sum_{i=1}^{n} \lambda_{i+1} |\nabla g^i|^2 + \sum_{i=1}^{n} \lambda_{i+1} (1 - |\nabla g^i|^2)\\
&= \sum_{i=1}^{n} \lambda_{i+1}.
\end{aligned}
\tag{3.13}
$$

From (3.11) and (3.13), we infer

$$
\frac{\mu_2 + \mu_3 + \cdots + \mu_{n+1}}{\mu_1} \le n + 4.
$$

with $\mu_i = \lambda_i + \frac{n^2}{4} \|H\|^2$. $\square$

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