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Critical Points of the Weighted Area Functional

Qing-Ming Cheng

Abstract In this survey, we discuss critical points of functionals by various aspects. We review properties of critical points of weighted area functional, that is, self-shrinkers of mean curvature flow in Euclidean spaces and examples of compact self-shrinkers are discussed. We also review properties of critical points for weighted area functional for weighted volume-preserving variations, that is, λ-hypersurfaces of weighted volume-preserving mean curvature flow in Euclidean spaces.

Keywords Weighted area functional · Self-shrinkers · λ-hypersurfaces · $\mathscr{F}$-stability

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1 Introduction

By making use of flows (for examples, Ricci flow, mean curvature flow and so on), one obtains many important achievements among study on the differential geometry. In particular, study of mean curvature flow becomes one of the most important objects in the differential geometry of submanifolds. In this paper, we will focus on the case of self-shrinkers of mean curvature flow and λ-hypersurfaces of the weighted volume-preserving mean curvature flow in the $(n+1)$- dimensional Euclidean space $\mathbf{R}^{n+1}$ from the view of the variation principles.

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1.1 Mean Curvature Flow

Let M^n be an n-dimensional manifold and assume that

$$X : M^n \to \mathbf{R}^{n+1}$$

is an n-dimensional hypersurface in the $(n+1)$- dimensional Euclidean space $\mathbf{R}^{n+1}$. A family $X(t) = X(\cdot, t)$ of smooth immersions:

$$X(t) : M^n \to \mathbf{R}^{n+1}$$

with $X(0) = X$ is called mean curvature flow if they satisfy

$$\frac{\partial X(p,t)}{\partial t} = H(p,t),$$

where $H(p,t)$ denotes the mean curvature vector of hypersurface $M_t = X(M^n, t)$ at point $X(p,t)$. The simplest mean curvature flow is given by the one-parameter family of the shrinking spheres $M_t \subset \mathbf{R}^{n+1}$ centered at the origin and with radius $\sqrt{-n(t-T)}$ for $t \le T$. This is a smooth flow except at the origin at time $t = T$ when the flow becomes extinct.

For an n-dimensional compact convex hypersurface $M_0 = X(M^n)$ in $\mathbf{R}^{n+1}$, Huisken [15] proved that the mean curvature flow $M_t = X(M^n, t)$ remains smooth and convex until it becomes extinct at a point in the finite time. If we rescale the flow about the point, the resulting converges to the round sphere. When M_0 is non-convex, the other singularities of the mean curvature flow can occur. Grayson [13] constructed a rotationally symmetric dumbbell with a sufficiently long and narrow bar, where the neck pinches off before the two bells become extinct. For the rescaling of the singularity at the neck, the resulting blows up, can not extinctions. Hence, the resulting is not spheres, certainly. In fact, the resulting of the singularity converges to shrinking cylinders. The singularities of mean curvature flow for curves are studied very well in the work of Abresch and Langer [1], Calabi et al. [12], Hamilton [19] and so on. For higher dimensions, Huisken [15] found a key for studying singularities of mean curvature flow, that is, Huisken gave the so-called *Huisken's monotonicity formula*. Huisken [15] and Ilmanen and White proved that the monotonicity implies that the flow is asymptotically self-similar near given singularity. Recently, in the landmark paper [9] of Colding and Minicozzi, they have solved a long-standing conjecture on singularity of a generic flow.

1.2 Mean Curvature Type Flow

As one knows, for a family of immersions $X(t) : M \to \mathbf{R}^{n+1}$ with $X(0) = X$, the volume of M is defined by

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$$\frac{1}{n+1}\int_M \langle X(t), N(t)\rangle d\mu$$

Huisken [18] studied the mean curvature type flow:

$$\frac{\partial X(t)}{\partial t} = \big(-h(t)N(t) + \mathbf{H}(t)\big),$$

where $X(t) = X(\cdot, t)$, $h(t) = \frac{\int_M H(t)d\mu_t}{\int_M d\mu_t}$ and $N(t)$ is the unit normal vector of $X(t) : M \to \mathbf{R}^{n+1}$. It can be proved the above flow preserves the volume of M. Hence, one calls this flow the volume-preserving mean curvature flow. Huisken [16] proved that if the initial hypersurface is uniformly convex, then the above volume-preserving mean curvature flow has a smooth solution and it converges to a round sphere.

By making use of the Minkowski formulas, Guan and Li [14] have studied the following type of mean curvature flow

$$\frac{\partial X(t)}{\partial t} = \big(-nN(t) + \mathbf{H}(t)\big),$$

which is also a volume-preserving mean curvature flow. They have gotten that the flow converges to a solution of the isoperimetric problem if the initial hypersurface is a smooth compact, star-shaped hypersurface.

Cheng and Wei [6] introduce a definition of the weighted volume of M. For a family of immersions $X(t) : M \to \mathbf{R}^{n+1}$ with $X(0) = X$, we define *a weighted volume* of M by

$$V(t) = \int_M \langle X(t), N\rangle e^{-\frac{|X|^2}{2}} d\mu.$$

Furthermore, Cheng and Wei [7] consider a new type of mean curvature flow:

$$\frac{\partial X(t)}{\partial t} = \big(-\alpha(t)N(t) + \mathbf{H}(t)\big)$$

with

$$\alpha(t) = \frac{\int_M H(t)\langle N(t), N\rangle e^{-\frac{|X|^2}{2}} d\mu}{\int_M \langle N(t), N\rangle e^{-\frac{|X|^2}{2}} d\mu},$$

where N is the unit normal vector of $X : M \to \mathbf{R}^{n+1}$. We can prove that the flow:

$$\frac{\partial X(t)}{\partial t} = \big(-\alpha(t)N(t) + \mathbf{H}(t)\big)$$

preserves the weighted volume $V(t)$. Hence, we call this flow a weighted volume-preserving mean curvature flow.

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2 Complete Self-shrinkers of Mean Curvature Flow

2.1 Definition of Self-shrinkers

One calls a hypersurface $X : M^n \to \mathbf{R}^{n+1}$ *a self-shrinker* of mean curvature flow if

$$H + \langle X, N \rangle = 0,$$

where H denotes the mean curvature of the hypersurface $X : M^n \to \mathbf{R}^{n+1}$. One can prove that if $X : M^n \to \mathbf{R}^{n+1}$ is a self-shrinker of mean curvature flow, then

$$X(t) = \sqrt{-t}X$$

is a solution of the mean curvature flow equation, which is called *a self-similar solution of mean curvature flow*.

Letting $X : M^n \to \mathbf{R}^{n+1}$ be a hypersurface,

$$X(s) : M^n \to \mathbf{R}^{n+1}$$

is called *a variation* of $X : M^n \to \mathbf{R}^{n+1}$ if $X(s)$ is a one parameter family of immersions with $X(0) = X$. Define a functional

$$\mathscr{F}(s) = \frac{1}{(2\pi)^{n/2}} \int_M e^{-\frac{|X(s)|^2}{2}} d\mu_s.$$

By computing the first variation formula, we obtain that $X : M^n \to \mathbf{R}^{n+1}$ is a critical point of $\mathscr{F}(s)$ if and only if $X : M^n \to \mathbf{R}^{n+1}$ is a self-shrinker, that is,

$$H + \langle X, N \rangle = 0.$$

Furthermore, we know that $X : M^n \to \mathbf{R}^{n+1}$ is a minimal hypersurface in $\mathbf{R}^{n+1}$ equipped with the metric $g_{AB} = e^{-\frac{|X|^2}{n}}\delta_{AB}$ if and only if $X : M^n \to \mathbf{R}^{n+1}$ is a self-shrinker.

2.2 Examples of Complete Self-shrinkers

As a standard examples of self-shrinkers of mean curvature flow, we know the n-dimensional Euclidean space $\mathbf{R}^n$ is a complete self-shrinker in $\mathbf{R}^{n+1}$.

The n-dimensional sphere $S^n(\sqrt{n})$ with radius $\sqrt{n}$ is a compact self-shrinker in $\mathbf{R}^{n+1}$.

For a positive integer k, $\mathbf{S}^k(\sqrt{k}) \times \mathbf{R}^{n-k}$ is an n-dimensional complete noncompact self-shrinker in $\mathbf{R}^{n+1}$. Besides the standard examples of self-shrinkers of mean curvature flow, shrinking doughnuts of Angenent in [2] are only known examples of self-shrinkers of mean curvature flow are known before 2010.

Theorem 1 *For $n \geq 2$, there exists embedding revolution self-shrinkers $X : S^1 \times S^{n-1} \to \mathbf{R}^{n+1}$ in $\mathbf{R}^{n+1}$.*

Sketch of proof of theorem 1. Let $(x(s), r(s))$, $s \in (a, b)$ be a curve in the xr-plane with $r > 0$ and $S^{n-1}(1)$ denote the standard unit sphere of dimension $n-1$. Then we consider $X : (a, b) \times S^{n-1}(1) \to \mathbf{R}^{n+1}$ defined by $X(s, \alpha) = (x(s), r(s)\alpha)$, $s \in (a, b)$, $\alpha \in S^{n-1}(1)$. Namely, X is obtained by rotating the plane curve $(x(s), r(s))$ around x axis. Thus, by shooting method, $X : (a, b) \times S^{n-1}(1) \to \mathbf{R}^{n+1}$ is a self-shrinker if and only if (x, r) satisfies

$$\frac{dx}{ds} = \cos\theta$$
$$\frac{dr}{ds} = \sin\theta$$
$$\frac{d\theta}{ds} = \frac{x}{2}\sin\theta + \left(\frac{n-1}{r} - \frac{r}{2}\right)\cos\theta.$$

Let (x_R, r_R, θ_R) be the maximal solution of the above equations with initial value $(0, R, 0)$. Then for large enough R, there is a simple closed curve (x_R, r_R) in xr-plane. It can be proved that it is a graph of $x = f_R(r)$. Hence, there exists an embedding revolution self-shrinker $X : S^1 \times S^{n-1} \to \mathbf{R}^{n+1}$ in $\mathbf{R}^{n+1}$.

In [21], Kleene and Møller have proved the following:

Theorem 2 *For $n \geq 2$, an n-dimensional complete embedding revolution self-shrinker $X : M^n \to \mathbf{R}^{n+1}$ in $\mathbf{R}^{n+1}$ is one of the following:*

1. *$S^n(\sqrt{n})$,*
2. *$\mathbf{R}^n$,*
3. *$S^m(\sqrt{m}) \times \mathbf{R}^{n-m} \subset \mathbf{R}^{n+1}$, $1 \leq m \leq n-1$,*
4. *a smooth embedded self-shrinker $X : S^1 \times S^{n-1} \to \mathbf{R}^{n+1}$.*

Except the standard examples and embedded self-shrinker $X : S^1 \times S^{n-1} \to \mathbf{R}^{n+1}$ of Angenent, Drugan [11] has constructed an example of the self-shrinker of genus 0, which is not embedding. Moreover, Møller [23] have constructed new examples of closed embedding self-shrinkers with higher genus. Recently, Kapouleas et al. [20] have constructed new examples of non-compact self-shrinkers. According to the examples, one would like to propose the following:

Conjecture *An embedding self-shrinker $X : M^n \to \mathbf{R}^{n+1}$, which is homeomorphic to a sphere, is the standard round sphere $S^n(\sqrt{n})$.*

Remark 1 According to examples of Angenent and Drugan, we know that conditions of both embedding and a topological sphere are necessary. For $n = 1$, Abresch and

Langer [1] solved this conjecture affirmatively. For $n = 2$, Brendle [3] has given affirmative answer for this conjecture very recently. For $n \geq 3$, this conjecture remains open.

Sketch of proof of conjecture for $n = 2$. For an embedded self-shrinker $X : M \to \mathbf{R}^3$ of genus 0, its intersection with any plane, which passes through the origin, consists of a simple Jordan curve which is piecewise C^1. Further, one can prove that $X : M \to \mathbf{R}^3$ is star-shaped. Thus, the mean curvature of $X : M \to \mathbf{R}^3$ does not change sign. Hence, from the result of Huisken, Theorem 3 in the next section, $X : M \to \mathbf{R}^3$ is a sphere $S^2(\sqrt{2})$.

2.3 Self-shrinkers with Non-negative Mean Curvature

In this subsection, we focus on complete self-shrinkers with non-negative mean curvature. The following notation is necessary.

Polynomial area growth. We say that a complete hypersurface M^n in $\mathbf{R}^{n+1}$ has polynomial area growth if there exist constants C and d such that, for all $r \geq 1$,

$$\text{Area}(B_r(O) \cap M) \leq Cr^d$$

holds, where $B_r(O)$ is the Euclidean ball with radius r and centered at the origin.

Remark 2 The standard examples of self-shrinkers have polynomial area growth and non-negative mean curvature.

For $n = 1$, Abresch and Langer [1] classified all smooth closed self-shrinker curves in $\mathbf{R}^2$ and showed that the round circle is the only embedded self-shrinker. For $n \geq 2$, Huisken [17] studied compact self-shrinkers. He gave a complete classification of self-shrinkers with non-negative mean curvature.

Theorem 3 *If $X : M^n \to \mathbf{R}^{n+1} (n \geq 2)$ is an n-dimensional compact self-shrinker with non-negative mean curvature H in $\mathbf{R}^{n+1}$, then $X(M^n) = S^n(\sqrt{n})$.*

Remark 3 The condition of non-negative mean curvature is essential. In fact, let Δ and ∇ denote the Laplacian and the gradient operator on self-shrinker, respectively and $\langle \cdot, \cdot \rangle$ denotes the standard inner product of $\mathbf{R}^{n+1}$. Because

$$\Delta H - \langle X, \nabla H \rangle + SH - H = 0,$$

we obtain $H > 0$ from the maximum principle if the mean curvature is non-negative. From this theorem of Huisken, we know that the mean curvature of compact self-shrinker $X : S^1 \times S^{n-1} \to \mathbf{R}^{n+1}$ of Angenent changes sign.

Huisken [18] studied complete and non-compact self-shrinkers in $\mathbf{R}^{n+1}$ and proved

Theorem 4 *Let $X : M^n \to \mathbf{R}^{n+1}$ be an n-dimensional complete non-compact self-shrinker in $\mathbf{R}^{n+1}$ with $H \geq 0$ and polynomial area growth. If the squared norm S of the second fundamental form is bounded, then M^n is isometric to one of the following:*

1. $\mathbf{R}^n$,
2. $S^m(\sqrt{m}) \times \mathbf{R}^{n-m} \subset \mathbf{R}^{n+1}$, $1 \leq m \leq n-1$,
3. $\Gamma \times \mathbf{R}^{n-1}$,

where Γ is one of curves of Abresch and Langer.

In the landmark paper [9] of Colding and Minicozzi, they have removed the assumption on the second fundamental form. They have proved

Theorem 5 *Let $X : M^n \to \mathbf{R}^{n+1}$ be an n-dimensional complete embedded self-shrinker in $\mathbf{R}^{n+1}$ with $H \geq 0$ and polynomial area growth. Then M^n is isometric to one of the following:*

1. $S^n(\sqrt{n})$,
2. $\mathbf{R}^n$,
3. $S^m(\sqrt{m}) \times \mathbf{R}^{n-m} \subset \mathbf{R}^{n+1}$, $1 \leq m \leq n-1$.

3 The Weighted Volume-Preserving Variations

In this section, we will give a survey of results on λ-hypersurfaces in the recent paper of Cheng and Wei [6].

3.1 *Definition of λ-hypersurfaces*

Let $X : M^n \to \mathbf{R}^{n+1}$ be an n-dimensional hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbf{R}^{n+1}$. We denote a variation of X by $X(t) : M \to \mathbf{R}^{n+1}$, $t \in (-\varepsilon, \varepsilon)$ with $X(0) = X$. We define *a weighted area functional* $A : (-\varepsilon, \varepsilon) \to \mathbf{R}$ by

$$A(t) = \int_M e^{-\frac{|X(t)|^2}{2}} d\mu_t,$$

where $d\mu_t$ is the area element of M in the metric induced by $X(t)$. The *weighted volume function* $V : (-\varepsilon, \varepsilon) \to \mathbf{R}$ of M is defined by

$$V(t) = \int_M \langle X(t), N\rangle e^{-\frac{|X|^2}{2}} d\mu.$$

The weighted volume $V(t)$ is preserved by the weighted volume-preserving mean curvature flow

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$$\frac{\partial X(t)}{\partial t} = \big(-\alpha(t)N(t) + \mathbf{H}(t)\big).$$

We say that a variation $X(t)$ of X is *a weighted volume-preserving normal variation* if $V(t) = V(0)$ for all t and $\frac{\partial X(t)}{\partial t}|_{t=0} = fN$. By computing the first variation formula, we have

Proposition 1 *Let $X : M \to \mathbf{R}^{n+1}$ be an immersion. The following statements are equivalent:*

1. *$\langle X, N\rangle + H = \lambda$, which is constant.*
2. *For all weighted volume-preserving variations, $A^{'}(0) = 0$.*
3. *For all variations, $J^{'}(0) = 0$, where $J(t) = A(t) + \lambda V(t)$.*

Definition 1 Let $X : M \to \mathbf{R}^{n+1}$ be an n-dimensional hypersurface in $\mathbf{R}^{n+1}$. If $\langle X, N\rangle + H = \lambda$, we call $X : M \to \mathbf{R}^{n+1}$ *a λ-hypersurface of the weighted volume-preserving mean curvature flow*.

Remark 4 The λ-hypersurface for $\lambda = 0$ is a self-shrinker of mean curvature flow.

According to the proposition 1 and the first variation formula, we have

Theorem 6 *Let $X : M \to \mathbf{R}^{n+1}$ be a hypersurface. The following statements are equivalent:*

1. *$X : M \to \mathbf{R}^{n+1}$ is a λ-hypersurface.*
2. *$X : M \to \mathbf{R}^{n+1}$ is a critical point of the weighted area functional $A(t)$ for all weighted volume- preserving variations.*
3. *$X : M \to \mathbf{R}^{n+1}$ is a hypersurface with constant mean curvature λ in $\mathbf{R}^{n+1}$ equipped with the metric $g_{AB} = e^{-\frac{|X|^2}{n}}\delta_{AB}$.*

As standard examples of λ-hypersurfaces, we know that all of self-shrinkers of mean curvature flow are λ-hypersurfaces. Spheres $X : S^n(r) \to \mathbf{R}^{n+1}$ with radius $r > 0$ are compact λ-hypersurfaces in $\mathbf{R}^{n+1}$ with $\lambda = \frac{n}{r} - r$. For a positive integer k, $X : \mathbf{S}^k(r) \times \mathbf{R}^{n-k}$ is an n-dimensional complete non-compact λ-hypersurface in $\mathbf{R}^{n+1}$ with $\lambda = \frac{k}{r} - r$. Hence, there are many λ-hypersurfaces, which are not self-shrinkers of mean curvature flow.

3.2 $\mathscr{F}$-Functional

We define a $\mathscr{F}$-functional by

$$\begin{aligned}\mathscr{F}(s) = \mathscr{F}_{X_s,t_s}(X(s)) &= (4\pi t_s)^{-\frac{n}{2}}\int_M e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s \\ &+ \lambda(4\pi)^{-\frac{n}{2}}\frac{1}{\sqrt{t_s}}\int_M \langle X(s) - X_s, N\rangle e^{-\frac{|X|^2}{2}}d\mu,\end{aligned}$$

where X_s and t_s denote the variations of $X_0 = O$, $t_0 = 1$, respectively and $\frac{\partial X(0)}{\partial s} = fN$. One calls that $X : M \to \mathbf{R}^{n+1}$ is *a critical point of* $\mathscr{F}(s)$ if it is critical with respect to all normal variations and all variations X_s and t_s of $X_0 = O$, $t_0 = 1$.

Theorem 7 *Let $X : M \to \mathbf{R}^{n+1}$ be a hypersurface. The following statements are equivalent:*

1. *$X : M \to \mathbf{R}^{n+1}$ is a λ-hypersurface.*
2. *$X : M \to \mathbf{R}^{n+1}$ is a hypersurface with constant mean curvature λ in $\mathbf{R}^{n+1}$ equipped with the metric $g_{AB} = e^{-\frac{|X|^2}{n}}\delta_{AB}$.*
3. *$X : M \to \mathbf{R}^{n+1}$ is a critical point of the weighted area functional $A(t)$ for all weighted volume-preserving variations.*
4. *$X : M \to \mathbf{R}^{n+1}$ is a critical point of $\mathscr{F}(s)$.*

3.3 *Stability of Compact λ-hypersurfaces*

Definition 2 One calls that a critical point $X : M \to \mathbf{R}^{n+1}$ of the $\mathscr{F}$-functional $\mathscr{F}(s)$ is *$\mathscr{F}$-stable* if, for every normal variation $X(s)$ of X, there exist variations X_s and t_s of $X_0 = O$, $t_0 = 1$ such that

$$\mathscr{F}''(0) \geq 0.$$

One calls that a critical point $X : M \to \mathbf{R}^{n+1}$ of the $\mathscr{F}$-functional $\mathscr{F}(s)$ is *$\mathscr{F}$-unstable* if there exist a normal variation $X(s)$ of X such that for all variations X_s and t_s of $X_0 = O$, $t_0 = 1$,

$$\mathscr{F}''(0) < 0.$$

For stability of λ-hypersurfaces, the following is proved in Cheng and Wei [6]:

Theorem 8 • *If $r \leq \sqrt{n}$ or $r > \sqrt{n+1}$, the n-dimensional round sphere*

$$X : S^n(r) \to \mathbf{R}^{n+1}$$

is $\mathscr{F}$-stable;

• *If $\sqrt{n} < r \leq \sqrt{n+1}$, the n-dimensional round sphere*

$$X : S^n(r) \to \mathbf{R}^{n+1}$$

is $\mathscr{F}$-unstable.

According to our Theorem 3, we would like to propose the following: AQ1

Problem 3.1 Is it possible to prove that spheres $S^n(r)$ with $r \leq \sqrt{n}$ or $r > \sqrt{n+1}$ are the only $\mathscr{F}$-stable compact λ-hypersurfaces?

Remark 5 Colding and Minicozzi [9] have proved that the sphere $S^n(\sqrt{n})$ is the only $\mathscr{F}$-stable compact self-shrinkers. In order to prove this result, the property that the mean curvature H is an eigenfunction of Jacobi operator plays a very important role. But for λ-hypersurfaces, the mean curvature H is not an eigenfunction of Jacobi operator in general.

3.4 Complete λ-hypersurfaces

Theorem 9 *Let $X : M \to \mathbf{R}^{n+1}$ be an n-dimensional complete embedded λ-hypersurface with polynomial area growth in $\mathbf{R}^{n+1}$. If $H - \lambda \geq 0$ and*

$$\lambda(f_3(H-\lambda) - S) \geq 0,$$

then $X : M \to \mathbf{R}^{n+1}$ is isometric to one of the following:

1. $\mathbf{R}^n$
2. $S^n(r)$, *for* $r > 0$,
3. $S^k(r) \times \mathbf{R}^{n-k}$, $0 < k < n$, $r > 0$,

where $S = \sum_{i,j} h_{ij}^2$ is the squared norm of the second fundamental form and $f_3 = \sum_{i,j,k} h_{ij}h_{jk}h_{ki}$.

Remark 6 For $\lambda = 0$, Huisken [19] and Colding and Minicozzi [9] proved this result. In this case, from the maximum principle, one can prove $H > 0$ if $H \geq 0$. $H \geq 0$ is essential according to examples of Angenent and so on. For $\lambda \neq 0$, we can not prove $H - \lambda > 0$ if $H - \lambda \geq 0$ from the maximum principle only. We need to use the condition $\lambda(f_3(H-\lambda) - S) \geq 0$. We think that this condition

$$\lambda(f_3(H-\lambda) - S) \geq 0$$

is essential. We are trying to construct examples in the forthcoming paper.

Sketch of proof of theorem 9. For λ-hypersurfaces, we define a differential operator $\mathscr{L}$ by

$$\mathscr{L}f = \Delta f - \langle X, \nabla f\rangle.$$

Since

$$\mathscr{L}H = H + S(\lambda - H)$$

and

$$H - \lambda \geq 0,$$

we have

$$\mathscr{L}H - H \leq 0.$$

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If $\lambda \le 0$, we conclude from the maximum principle that either $H \equiv \lambda$ or $H - \lambda > 0$. If $H \equiv \lambda$, we can prove $H = \lambda = 0$ and $X : M \to \mathbf{R}^{n+1}$ is the Euclidean space $\mathbf{R}^n$. If $\lambda > 0$, we have $f_3(H - \lambda) - S \ge 0$. we are able to prove $H - \lambda > 0$ also.

Since $X : M \to \mathbf{R}^{n+1}$ is an n-dimensional complete embedded λ-hypersurface, we can not use Stokes formula for non-compact case directly. But since $X : M \to \mathbf{R}^{n+1}$ is an n-dimensional complete embedded λ-hypersurface with polynomial area growth, we can make use of Stokes formula for several special functions. For $H - \lambda > 0$, we consider function $\log(H - \lambda)$. We have

$$\mathscr{L} \log(H - \lambda) = 1 - S + \frac{\lambda}{H - \lambda} - |\nabla \log(H - \lambda)|^2$$

and

$$\mathscr{L}\sqrt{S} \ge \sqrt{S} - \sqrt{S}S + \frac{\lambda f_3}{\sqrt{S}}.$$

In order to use Stokes formula for functions S, $\log(H - \lambda)$ and $\sqrt{S}$, we need to prove the following:

- $\int_M S(1 + |X|^2)e^{-\frac{|X|^2}{2}} d\mu < +\infty$
- $\int_M S^2 e^{-\frac{|X|^2}{2}} d\mu < +\infty.$
- $\int_M |\nabla\sqrt{S}|^2 e^{-\frac{|X|^2}{2}} d\mu < +\infty,$
- $\int_M \sum_{i,j,k} h_{ijk}^2 e^{-\frac{|X|^2}{2}} d\mu < +\infty.$
- $\int_M S|\nabla \log(H - \lambda)|^2 e^{-\frac{|X|^2}{2}} d\mu < +\infty.$

Thus, we can have

$$\begin{aligned} &\int_M \langle \nabla S, \nabla \log(H - \lambda)\rangle e^{-\frac{|X|^2}{2}} d\mu \\ &= -\int_M S\mathscr{L} \log(H - \lambda) e^{-\frac{|X|^2}{2}} d\mu \end{aligned}$$

and

$$\int_M |\nabla\sqrt{S}|^2 e^{-\frac{|X|^2}{2}} d\mu = -\int_M \sqrt{S}\mathscr{L}\sqrt{S} e^{-\frac{|X|^2}{2}} d\mu.$$

Putting

$$\mathscr{L} \log(H - \lambda) = 1 - S + \frac{\lambda}{H - \lambda} - |\nabla \log(H - \lambda)|^2$$

and

$$\mathscr{L}\sqrt{S} \ge \sqrt{S} - \sqrt{S}S + \frac{\lambda f_3}{\sqrt{S}}$$

into the above two formulas, we have

$$0 \geq \int_M \left|\nabla\sqrt{S} - \sqrt{S}\nabla \log(H-\lambda)\right|^2 e^{-\frac{|X|^2}{2}} d\mu + \int_M \lambda(f_3 - \frac{S}{H-\lambda}) e^{-\frac{|X|^2}{2}} d\mu.$$

From $\lambda(f_3(H-\lambda) - S) \geq 0$, we have

$$\lambda(f_3 - \frac{S}{H-\lambda}) = 0,$$

$$\frac{S}{(H-\lambda)^2} = \text{constant}$$

$$h_{ijk}(H-\lambda) = h_{ij}H_{,k},$$

for any i, j, k. By making use of local assertions, we can obtain that $X : M \to \mathbf{R}^{n+1}$ is isometric to a sphere $S^n(r)$ or $S^k(r) \times \mathbf{R}^{n-k}$ with $\lambda = \frac{k}{r} - r$.

4 Area of Complete λ-hypersurfaces

In study on Riemannian geometry, estimates of the volume of complete and non-compact Riemannian manifolds are very important. For examples, the comparison volume theorem on complete and non-compact Riemannian manifolds of Bishop and Gromov and lower bound growth on volume of complete and non-compact Riemannian manifolds due to Calabi and Yau are very important results in Riemannian geometry. In this section, we shall review several results on estimates of area of complete λ-hypersurfaces in Cheng and Wei [6].

4.1 *Upper Bound Growth of Area of Complete λ-hypersurfaces*

It is well-known that the comparison volume (area) theorem of Bishop and Gromov is a very powerful tool for studying Riemannian geomery.

The comparison volume theorem. For n-dimensional complete and non-compact Riemannian manifolds with nonnegative Ricci curvature, geodesic balls have at most polynomial volume growth:

$$\text{Area}(B_r(x_0)) \leq Cr^n.$$

Furthermore, Cao and Zhou have studied upper bound growth of volume of geodesic balls for n-dimensional complete and non-compact gradient shrinking Ricci solitons. They have proved

Theorem 10 *For n-dimensional complete and non-compact gradient shrinking Ricci solitons, geodesic balls have at most polynomial volume growth:*

$$\mathrm{Area}(B_r(x_0)) \leq Cr^k.$$

Remark 7 There exist n-dimensional complete and non-compact gradient shrinking Ricci solitons, which Ricci curvature is not nonnegative.

It is natural to ask the following:

Problem 4.1 Whether is it possible to give an upper bound growth of area for complete and noncompact λ-hypersurfaces?

Fot this problem 4.1, Cheng and Wei in [6] have proved the following:

Theorem 11 *Let $X : M \to \mathbf{R}^{n+1}$ be a complete and non-compact proper λ-hypersurface in the Euclidean space $\mathbf{R}^{n+1}$. Then, there is a positive constant C such that for $r \geq 1$,*

$$\mathrm{Area}(B_r(0) \cap X(M)) \leq Cr^{n+\frac{\lambda^2}{2}-2\beta-\frac{\inf H^2}{2}},$$

where $\beta = \frac{1}{4}\inf(\lambda - H)^2$.

Remark 8 The estimate in our theorem is best possible because the cylinders $S^k(r_0) \times \mathbf{R}^{n-k}$ satisfy the equality. For $\lambda = 0$, that is, for self-shrinkers, this result in theorem is proved by Ding and Xin [10] and Cheng and Zhou [8].

Furthermore, Cheng and Wei [6] have proved

Theorem 12 *A complete and non-compact λ-hypersurface $X : M \to \mathbf{R}^{n+1}$ in the Euclidean space $\mathbf{R}^{n+1}$ has polynomial area growth if and only if $X : M \to \mathbf{R}^{n+1}$ is proper.*

Proof According to the Theorem 11, we only need to prove that $X : M \to \mathbf{R}^{n+1}$ is proper if $X : M \to \mathbf{R}^{n+1}$ has polynomial area growth. Since $X : M \to \mathbf{R}^{n+1}$ has polynomial area growth, we can prove that the weighted area of $X : M \to \mathbf{R}^{n+1}$ is finite. If $X : M \to \mathbf{R}^{n+1}$ is not proper, then, there exists a real $r > 0$ such that $U = X^{-1}(\bar{B}_r(0))$ is not compact in M. For a constant $r_0 > 0$, there exists a sequence $\{p_k\} \subset U$ such that $d(p_k, p_l) \geq r_0$ for any $k \neq l$, where d denotes the distance of $X : M \to \mathbf{R}^{n+1}$. If we take $r_0 < 2r$, we have $B^X_{\frac{r_0}{2}}(p_k) \subset B_{2r}(0)$, where $B^X_a(p_k)$ denotes the geodesic ball in $X(M)$ with radius a centered at $X(p_k)$. Because of $\langle X, N\rangle + H = \lambda$, we have

$$|H| \leq |X| + |\lambda|.$$

For any $p \in M$ such that $X(p) \in X(M) \cap B_{2r}(0)$, we have

$$|H(p)| \leq |X(p)| + |\lambda| \leq 2r + |\lambda|.$$

For $p \in U$ such that $X(p) \in B^X_{\frac{r_0}{2}}(p_k)$, by defining $\rho(p) = d(p, p_k)$, we have

$$\frac{1}{2}\Delta\rho(p)^2 \geq n - (2r + |\lambda|)\rho(p).$$

Taking $r_0 < \min\{2r, \frac{n}{2r+|\lambda|}\}$ and $0 < a < \frac{r_0}{2}$, we have

$$\int_{B^X_a(p_k)} (n - (2r + |\lambda|)\rho) d\mu \leq \frac{1}{2}\int_{B^X_a(p_k)} \Delta\rho^2 dv \leq a\alpha(a),$$

where $\alpha(a)$ denotes the area of $\partial B^X_a(p_k)$. By co-area formula, we obtain

$$(n - r_0 a)A(a) \leq a\alpha(a),$$

where $A(a)$ denotes the area of $B^X_a(p_k)$. Hence, we have

$$A(a) \geq \omega_n a^n e^{-\frac{r_0^2}{2}}.$$

Thus, we can conclude that the weighted area is infinite. This is a contradiction.

4.2 Lower Bound Growth of Area of Complete λ-hypersurfaces

Calabi [4] and Yau [25] studied lower bound growth of volume for n-dimensional complete and non-compact Riemannian manifolds with nonnegative Ricci curvature. They proved the following:

Theorem 13 *For n-dimensional complete and non-compact Riemannian manifolds with nonnegative Ricci curvature, geodesic balls have at least linear volume growth:*

$$\mathrm{Area}(B_r(x_0)) \geq Cr.$$

For an n-dimensional complete and non-compact gradient shrinking Ricci soliton M, Cao and Zhou [5] have proved that M must have infinite area. Furthermore, Munteanu and Wang [24] have proved that volume of geodesic balls for n-dimensional complete and non-compact gradient shrinking Ricci solitons have at least linear growth:

$$\mathrm{Area}(B_r(x_0)) \geq Cr.$$

In [6], Cheng and Wei have studied lower bound growth of area for complete and noncompact λ-hypersurfaces. The following is proved:

Theorem 14 *Let $X : M \to \mathbf{R}^{n+1}$ be an n-dimensional complete proper λ-hypersurface. Then, for any $p \in M$, there exists a constant $C > 0$ such that*

$$\text{Area}(B_r(0) \cap X(M)) \geq Cr,$$

for all $r > 1$.

Remark 9 The estimate in the theorem is best possible because $S^{n-1}(r_0) \times \mathbf{R}$ satisfy the equality. For $\lambda = 0$, that is, for self-shrinkers, this result is proved by Li and Wei [22].

References

1. Abresch, U., Langer, J.: The normalized curve shortening flow and homothetic solutions. J. Diff. Geom. **23**, 175–196 (1986)
2. Angenent, S.: Shrinking doughnuts. In: Nonlinear diffusion equations and their equilibrium states, Birkhaüser, Boston-Basel-Berlin, vol. 7, pp. 21–38 (1992)
3. Brendle, S.: Embedded self-similar shrinkers of genus 0, arXiv:1411.4640
4. Calabi, E.: On manifolds with non-negative Ricci-curvature II. Notices Amer. Math. Soc. **22**, A205 (1975)
5. Cao, H.-D., Zhou, D.: On complete gradient shrinking Ricci solitons. J. Diff. Geom. **85**, 175–185 (2010)
6. Cheng, Q.-M., Wei, G.: Complete λ-hypersurfaces of the weighted volume—preserving mean curvature flow, arXiv:1403.3177
7. Cheng, Q.-M., Wei, G.: The weighted volume—preserving mean curvature flow, preprint
8. Cheng, X., Zhou, D.: Volume estimate about shrinkers. Proc. Amer. Math. Soc. **141**, 687–696 (2013)
9. Colding, T.H., Minicozzi II, W.P.: Generic mean curvature flow I. Generic Singularities. Ann. of Math. **175**, 755–833 (2012)
10. Ding, Q., Xin, Y.L.: Volume growth, eigenvalue and compactness for self-shrinkers. Asia J. Math. **17**, 443–456 (2013)
11. Drugan, G.: An immersed S^2 self-shrinker, to appear in Trans. Amer. Math. Soc
12. Grayson, M.A.: The heat equation shrinks embedded plane curves to round points. J. Differential Geom. **26**, 285–314 (1987)
13. Grayson, M.A.: Shortening Embedded Curves. Ann. of Math. **129**, 71–111 (1989)
14. Guan, P., Li, J.: A mean curvature type flow in space forms, arXiv:1309.5099
15. Huisken, G.: Flow by mean curvature of convex surfaces into spheres. J. Diff. Geom. **22**, 237–266 (1984)
16. Huisken, G.: The volume preserving mean curvature flow. J. reine angew. Math. **382**, 35–48 (1987)
17. Huisken, G.: Asymptotic behavior for singularities of the mean curvature flow. J. Diff. Geom. **31**, 285–299 (1990)
18. Huisken, G.: Local and global behaviour of hypersurfaces moving by mean curvature, Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), Proceedings of Symposia in Pure Mathematics, Part 1, American Mathematical Society, Providence, RI, vol. 54, pp. 175–191(1993)
19. Hamilton, R.S., Isoperimetric estimates for the curve shrinking flow in the plane. In: Modern Methods in Complex Analysis (Princeton, NJ, : Annals of Mathematics Studdies, Princeton University Press, Princeton, NJ 1995, vol. 137, pp. 201–222 (1992)

20. Kapouleas, N., Kleene, S.J., Møller, N.M.: Mean curvature self-shrinkers of high genus: non-compact examples, to appear in J. Reine Angew. Math
21. Kleene, S., Møller, N. M.: Self-shrinkers with a rotation symmetry, to appear in Trans. Amer. Math. Soc
22. Li, H., Wei, Y.: Lower volume growth estimates for self-shrinkers of mean curvature flow. Proc. Amer. Math. Soc. **142**, 3237–3248 (2014)
23. Møller, N. M.: Closed self-shrinking surfaces in $\mathbf{R}^3$ via the torus, arXiv:1111.318
24. Munteanu, O., Wang, J.: Analysis of the weighted Laplacian and applications to Ricci solitons. Comm. Anal. Geom. **20**, 55–94 (2012)
25. Yau, S.T.: Some function-theoretic properties of complete Riemannian manifold and their applications to geometry. Indiana Univ. Math. J. **25**, 659–670 (1976)

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Chapter 4

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