Proceedings of the 6th International Congress of Chinese Mathematicians
ALM 37, pp. 37–52

Universal Estimates for Eigenvalues and Applications

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Abstract

In this survey, we discuss eigenvalues of the eigenvalue problem of Laplacian. First of all, we consider universal estimates for eigenvalues of the eigenvalue problem of Laplacian. Secondly, as applications of universal estimates for eigenvalues, we discuss the lower bound growth and the upper bound growth for eigenvalues, which are sharper in some sense. Furthermore, an obstruction for minimal immersions of complete Riemannian manifolds into Euclidean spaces is given by eigenvalues of the eigenvalue problem of Laplacian.

Mathematics Subject Classification: 35P15, 53C42, 58G25.

Keywords and Phrases: The eigenvalue problem of Laplacian, universal inequality for eigenvalues, Riemannian manifold, minimal immersion.

1 Introduction

Let M be an n-dimensional compact Riemannian manifold.

$$\Delta u = -\lambda u, \quad \text{on} \quad M$$

is called *a closed eigenvalue problem of Laplacian* on M, where Δ denotes the Laplacian on M. For the closed eigenvalue problem of the Laplacian, the spectrum of it is given by

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty,$$

where each λ_i has finite multiplicity which is repeated according to its multiplicity. Let M be an n-dimensional complete Riemannian manifold, Ω a bounded domain with piecewise smooth boundary $\partial\Omega$ in M.

The author was partially supported by JSPS Grant-in-Aid for Scientific Research (B): No. 24340013 and Challenging Exploratory Research No. 25610016.

$$\begin{cases} \Delta u = -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

is called *a Dirichlet eigenvalue problem of Laplacian*, which is also called *a fixed membrane problem.* It is well known that the spectrum of the Dirichlet eigenvalue problem of the Laplacian satisfies

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty,$$

where each λ_i has finite multiplicity which is repeated according to its multiplicity.

Remark 1.1. *In this paper, we only deal with eigenvalues of the Dirichlet eigenvalue problem of Laplacian. For the closed eigenvalue problem of Laplacian, we can obtain the similar results by using the same assertions.*

For eigenvalues of the Dirichlet eigenvalue problem of Laplacian, the following Weyl's asymptotic formula (see [23]) holds:

$$\lambda_k \sim \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to \infty,$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$. From the formula, it is not difficult to infer

$$\frac{1}{k}\sum_{i=1}^{k} \lambda_i \sim \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to \infty.$$

2 Universal inequalities for eigenvalues

2.1 The case of a Euclidean space

First of all, we consider universal inequalities for eigenvalues of the Dirichlet eigenvalue problem of Laplacian on a bounded domain Ω in an n-dimensional Euclidean space:

$$\begin{cases} \Delta u = -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

The investigation of universal inequalities for eigenvalues was initiated by Payne, Pólya and Weinberger [19], [20]. They proved

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{nk} \sum_{i=1}^{k} \lambda_i,$$

which is called a universal inequality for eigenvalues because this inequality does not depend on the domain Ω. Although this result of Payne, Pólya and Weinberger has been extended by many mathematicians in several way (cf. [1, 2, 13, 18, 24] and so on), there are two main contributions due to Hile and Protter [13] and

Yang [24]. In fact, in 1980, Hile and Protter improved this result of Payne, Pólya and Weinberger to

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{nk}{4}.$$

Yang, in 1991, has obtained a very sharp inequality (see Cheng and Yang [10] also), that is, he proved

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\lambda_i.$$

Remark 2.1. *The inequality of Yang is optimal in the sense of the order of k.*

2.2 The case of a unit sphere

For a domain Ω in an n-dimensional unit sphere, universal inequalities for eigenvalues of the Dirichlet eigenvalue problem of Laplacian:

$$\begin{cases} \Delta u = -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

has been studied by Cheng and Yang [8]. We have proved

Theorem 2.1. *Let Ω be a domain in an n-dimensional unit sphere. Eigenvalue λ_i's of the Dirichlet eigenvalue problem of the Laplacian satisfy*

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\lambda_i + \frac{n^2}{4} \right).$$

Remark 2.2. *The above inequality is best possible since this inequality does not depend on the domain Ω and when Ω tends to the unit sphere, this inequality becomes an equality for all of k.*

2.3 The general case

For a general n-dimensional complete Riemannian manifold M, we want to obtain universal inequalities for eigenvalues of the Dirichlet eigenvalue problem of Laplacian:

$$\begin{cases} \Delta u = -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in M. In order to get universal inequalities for eigenvalues, the key is to construct appropriate trial functions. For the cases of a Euclidean space and a unit sphere, one can make use of the coordinate functions to construct the trial functions. Thus, one can derive universal inequalities for eigenvalues according to the Rayleigh-Ritz inequality.

For a general complete Riemannian manifold, it is very difficult to construct an appropriate trial function since it is hard to find globally defined functions. Fortunately, we have Nash's theorem. By making use of Nash's theorem, we successfully construct trial functions which satisfy good properties.

Nash's Theorem. *Each complete Riemannian manifold can be isometrically immersed in a Euclidean space.*

In my joint work [6] with Chen, we have proved

Theorem 2.2. *Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Then, there exists a constant H_0^2, which only depends on M and Ω such that eigenvalues λ_j of the Dirichlet eigenvalue problem of Laplacian satisfy, for any k,*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i+\frac{n^2}{4}H_0^2\right).$$

We should remark that El Soufi, Harrell and Ilias [12] have also proved a similar result for submanifolds, independently.

In particular, when M is a complete minimal submanifold in $\mathbf{R}^N$, we have

Corollary 2.1. *Let Ω be a bounded domain in an n-dimensional complete minimal submanifold M^n in $\mathbf{R}^N$. Then, we have*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i.$$

Remark 2.3. *We would like to notice the following:*

1. *Our universal inequality for complete minimal submanifolds in $\mathbf{R}^N$ is the same as one of Yang for the case of $\mathbf{R}^n$.*

2. *There exist many complete minimal submanifolds in $\mathbf{R}^N$.*

3. *The universal inequality for eigenvalues of Yang does not only holds for bounded domains in $\mathbf{R}^n$, but also for bounded domains in any complete minimal submanifold in $\mathbf{R}^N$.*

When M is the unit sphere $S^n(1)$, since $S^n(1)$ is a hypersurface in $\mathbf{R}^{n+1}$ with the mean curvature $H = 1$, we have obtained the theorem of Cheng and Yang [8]:

Corollary 2.2. *Let Ω be a domain in an n-dimensional unit sphere. Eigenvalue λ_i's of the Dirichlet eigenvalue problem of Laplacian satisfy*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i+\frac{n^2}{4}\right).$$

In order to prove Theorem 2.2, the following lemma plays a key role.

Lemma 2.1. *Let M be an n-dimensional complete Riemannian manifold with metric g isometrically immersed in a Euclidean space $\mathbf{R}^N$. For any point P in M, assuming that y with components y^α defined by $y^\alpha = y^\alpha(x^1, x^2, \cdots, x^n)$ is the position vector of P in $\mathbf{R}^N$, we have,*

$$\sum_{\alpha=1}^{N} g(\nabla y^\alpha, \nabla y^\alpha) = n, \quad \sum_{\alpha=1}^{N} (\Delta y^\alpha)^2 = n^2 |H|^2,$$

$$\sum_{\alpha=1}^{N} \Delta y^\alpha \nabla y^\alpha = 0, \quad \sum_{\alpha=1}^{N} g(\nabla y^\alpha, \nabla u)^2 = |\nabla u|^2,$$

for any function $u \in C^\infty(M)$, where H is the mean curvature vector of M.

Proof of Theorem 2.2. According to the Rayleigh-Ritz inequality, Cheng and Yang in [9] have proved, for any function $f \in C^3(\Omega) \cap C^2(\partial\Omega)$ and any integer k, we have

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \|u_i \nabla f\|^2 \le \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \|2\nabla f \cdot \nabla u_i + u_i \Delta f\|^2,$$

where $\|f\|^2 = \int_M f^2$ and $\nabla f \cdot \nabla u_i = g(\nabla f, \nabla u_i)$ and u_i is an orthonormal eigenfunction corresponding to λ_i. Putting $f = y^\alpha$ and taking sum on α from 1 to N, we obtain our inequality by making use of the above lemma of Chen and Cheng. $\square$

2.4 A conjecture

For the hyperbolic space $H^n(-1)$, from Theorem 2.2, we can obtain universal inequalities for eigenvalues

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \le \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\lambda_i + \frac{n^2}{4} H_0^2 \right).$$

But we do not think that it is a good idea to use Nash theorem. If we do it, we shall lose many good properties of $H^n(-1)$. Thus, we can not decide the constant H_0 in universal inequalities for eigenvalues. Hence, it is a good idea to deal with this problem for the hyperbolic space $H^n(-1)$, directly.

Although many mathematicians want to derive universal inequalities for eigenvalues, directly, the sharp results on universal inequalities for eigenvalues are not obtained yet.

Conjecture. For a bounded domain Ω in $H^n(-1)$, eigenvalue λ_i's of the Dirichlet eigenvalue problem of Laplacian satisfy

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \le \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\lambda_i - \frac{(n-1)^2}{4} \right).$$

For this conjecture, Cheng and Yang in [11] have found a kind of appropriate trial functions for $M = H^n(-1)$. Hence, we can derive a universal inequality for eigenvalues, that is, we prove the following:

Theorem 2.3. *For a bounded domain Ω in $H^n(-1)$, eigenvalue λ_i's of the Dirichlet eigenvalue problem of Laplacian satisfy*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq 4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i-\frac{(n-1)^2}{4}\right).$$

As an application of our universal inequality for eigenvalues, we can obtain

Corollary 2.3. *Let Ω be a bounded domain in $H^n(-1)$. Then, eigenvalue $\lambda_k(\Omega)$, for any k, of the Dirichlet eigenvalue problem of Laplacian satisfies*

$$\lim_{\Omega\to H^n(-1)}\lambda_k(\Omega)=\frac{(n-1)^2}{4}.$$

3 Application to lower bounds for eigenvalues

For eigenvalues of the Dirichlet eigenvalue problem of the Laplacian, the Weyl's asymptotic formula holds:

$$\lambda_k \sim \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}}, \quad k\to\infty,$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$. From the formula, it is not difficult to infer

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i \sim \frac{n}{n+2}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}}, \quad k\to\infty.$$

3.1 A conjecture of Pólya

For a bounded domain in the Euclidean space $\mathbf{R}^n$, Pólya [21] proved, for $k = 1, 2, \cdots$,

$$\lambda_k \geq \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}},$$

if Ω is a tiling domain in $\mathbf{R}^n$. Furthermore, he conjectured the following:

Conjecture of Pólya. If Ω is a bounded domain in $\mathbf{R}^n$, then eigenvalue λ_k of the Dirichlet eigenvalue problem of the Laplacian satisfies, for $k = 1, 2, \cdots$,

$$\lambda_k \geq \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}}.$$

For this conjecture of Pólya, there are many mathematicians to attack it. For examples, Berezin [3], Lieb [17], Li and Yau [16] and so on. The following is the result of Li and Yau [16].

Theorem 3.1. *If Ω is a bounded domain in $\mathbf{R}^n$, then eigenvalue λ_k of the Dirichlet eigenvalue problem of Laplacian satisfies, for $k = 1, 2, \cdots$,*

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i \geq \frac{n}{n+2}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}}.$$

Remark 3.1. *According to the Weyl's asymptotic formula*

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i \sim \frac{n}{n+2}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{\mathrm{n}}}}k^{\frac{2}{n}}, \quad k \to \infty,$$

we know that the result of Li and Yau is optimal in the sense of average.

From this formula, we have, for $k = 1, 2, \cdots$,

$$\lambda_k \geq \frac{n}{n+2}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{\mathrm{n}}}}k^{\frac{2}{n}},$$

which gives a partial solution for the conjecture of Pólya with a factor $\dfrac{n}{n+2}$.

In order to prove this theorem, main methods, which are used by Li and Yau, are the following:

1. the Fourier transform,
2. a lemma of Hörmander.

Lemma of Hömander. *If f is a function defined on $\mathbf{R}^n$ satisfying*

$$0 \leq f \leq a_1, \quad \int_{\mathbf{R}^n}|z|^2 f(z)dz \leq a_2,$$

then, we have

$$\int_{\mathbf{R}^n} f(z)dz \leq \Big(a_1\omega_n\Big)^{\frac{2}{n+2}} a_2^{\frac{n}{n+2}}\left(\frac{n+2}{n}\right)^{\frac{n}{n+2}},$$

where a_1 and a_2 are constant.

Proof of Theorem 3.1. Let u_i be an eigenfunction corresponding to eigenvalue λ_i such that $\{u_i\}$ becomes an orthonormal basis of $L^2(\Omega)$. By defining a function

$$\varphi(x,y) = \begin{cases} \sum_{i=1}^{k} u_i(x)u_i(y), & (x,y) \in \Omega \times \Omega, \\ 0, & \text{the other}, \end{cases}$$

and

$$f(z) = \int_{\mathbf{R}^n} |\hat{\varphi}(z,y)|^2 dy,$$

where $\hat{\varphi}(z,y)$ is the Fourier transform of $\varphi(x,y)$ in x, we have

$$0 \le f \le (2\pi)^{-n} \mathrm{vol}\Omega,$$

$$\int_{\mathbf{R}^n} f(z)dz = k$$

and

$$\int_{\mathbf{R}^n} |z|^2 f(z)dz = \sum_{i=1}^{k} \lambda_i.$$

Thus, from Lemma of Hörmander, we have

$$\frac{1}{k}\sum_{i=1}^{k} \lambda_i \ge \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}.$$

It completes the proof of Theorem 3.1.

3.2 A problem of Chavel

For a complete Riemannian manifold M, eigenvalues of the Dirichlet eigenvalue problem of the Laplacian also satisfy the Weyl's asymptotic formula. Hence, it is natural to try to obtain a lower bound for eigenvalues.

In fact, for a complete Riemannian manifold, by making use of the Sobolev constant s, Li [15], Chavel and Feldman [5] proved

$$\lambda_k \ge \begin{cases} c(n,s)\dfrac{k^{\frac{1}{n-1}}}{(\mathrm{vol}\Omega)^{\frac{2}{n}}}, & n > 2, \\ c(n,s)\dfrac{k^{\frac{1}{3}}}{\mathrm{vol}\Omega}, & n = 2, \end{cases}$$

where $c(n,s)$ is a constant depending only on n and the Sobolev constant s. They have applied this result to prove the uniform convergences of the Heat Kernel. Since the order of k is not optimal, one wants to ask whether it is possible to get lower bounds with the optimal order of k.

In fact, Chavel proposed the following in his famous book: Eigenvalues in Riemannian Geometry, 1984 (p. 330):

The problem of Chavel. For a complete Riemannian manifold, it is desirable to prove that eigenvalues of the Dirichlet eigenvalue problem of Laplacian satisfy

$$\lambda_k \ge c\left(\frac{k}{\mathrm{vol}\Omega}\right)^{\frac{2}{n}},$$

where c is a constant.

More general, one would like to ask *whether is it possible for one to consider the same problem as the conjecture of Pólya for a complete Riemannian manifold other than* $\mathbf{R}^n$*?*

First of all, a difficulty which we will encounter is that there is no the Fourier transform for a complete Riemannian manifold. In order to derive the result of Li and Yau, Lemma of Hörmander plays an important role. But there is no this kind of lemma for a complete Riemannian manifold. In order to consider the same problem as the conjecture of Pólya, we need to come over the following problems:

- What method will we use to replace the Fourier transform?
- What method will we use to replace Lemma of Hörmander?

Therefore, our purpose is to study the lower bounds with the optimal order of k for eigenvalues of the Dirichlet eigenvalue problem of Laplacian on a bounded domain in complete Riemannian manifolds.

3.3 A generalized conjecture of Pólya

First of all, we will propose a version of the conjecture of Pólya for complete Riemannian manifolds.

The generalized conjecture of Pólya. Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Then, there exists a constant $c(M,\Omega)$, which only depends on M and Ω such that eigenvalues of the Dirichlet eigenvalue problem of Laplacian satisfy, for $k = 1, 2, \cdots$,

$$\lambda_k + c(M,\Omega) \geq \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}},$$

$$\frac{1}{k}\sum_{i=1}^{k} \lambda_i + c(M,\Omega) \geq \frac{n}{n+2}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}.$$

Remark 3.2. *About the constant $c(M,\Omega)$, we propose the following:*

1. *When M is a complete minimal submanifold in the Euclidean space $\mathbf{R}^N$,*
$$c(M,\Omega) = 0.$$
2. *When M is the unit sphere $S^n(1)$,*
$$c(M,\Omega) = \frac{n^2}{4}.$$
3. *When M is the hyperbolic space $\mathbf{H}^n(-1)$,*
$$c(M,\Omega) = -\frac{(n-1)^2}{4}.$$

For this generalized conjecture of Pólya, we have proved, in Cheng and Yang [11]

Theorem 3.2. *Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Then, there exists a constant H_0^2, which only depends on M and Ω such that eigenvalues of the Dirichlet eigenvalue problem of Laplacian satisfy, for $k = 1, 2, \cdots$,*

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{n^2}{4}H_0^2 \geq \frac{n}{\sqrt{(n+2)(n+4)}}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}},$$

$$\lambda_k + \frac{n^2}{4}H_0^2 \geq \frac{n}{\sqrt{(n+2)(n+4)}}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}}.$$

Corollary 3.1. *Let Ω be a domain in the n-dimensional unit sphere $S^n(1)$. Then, eigenvalues of the Dirichlet eigenvalue problem of Laplacian satisfy, for $k = 1, 2, \cdots$,*

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{n^2}{4} \geq \frac{n}{\sqrt{(n+2)(n+4)}}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}}.$$

Corollary 3.2. *For any bounded domain Ω in an n-dimensional complete minimal submanifold M in the Euclidean space $\mathbf{R}^N$, eigenvalues of the Dirichlet eigenvalue problem of Laplacian must satisfy, for $k = 1, 2, \cdots$,*

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i \geq \frac{n}{\sqrt{(n+2)(n+4)}}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}}.$$

Remark 3.3. *From Theorem 3.2, we know that the problem of Chavel is solved.*

In order to prove our theorem, we need to come over the problems, which we have mentioned:

- What method will we use to replace the Fourier transform? The answer is universal inequalities for eigenvalues.
- What method will we use to replace Lemma of Hörmander? The answer is a recursion formula of Cheng and Yang.

3.4 A recursion formula

In [10], Cheng and Yang have proved the following:

The recursion formula of Cheng and Yang. Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{k+1}$ be any positive real numbers satisfying

$$\sum_{i=1}^{k}(\mu_{k+1} - \mu_i)^2 \leq \frac{4}{t}\sum_{i=1}^{k}\mu_i(\mu_{k+1} - \mu_i).$$

Define

$$G_k = \frac{1}{k}\sum_{i=1}^{k}\mu_i, \quad T_k = \frac{1}{k}\sum_{i=1}^{k}\mu_i^2,$$

$$F_k = \left(1 + \frac{2}{t}\right) G_k^2 - T_k.$$

Then, we have the following recursion formula

$$F_{k+1} \leq \left(\frac{k+1}{k}\right)^{\dfrac{4}{t}} F_k,$$

where $t \geq 1$ is any positive real number.

Remark 3.4. *We should notice that by making use of the recursion formula of Cheng and Yang, we can not only derive lower bounds for eigenvalues, but also derive upper bounds for eigenvalues.*

3.5 Proof of Theorem 3.2

We shall give a proof of Theorem 3.2.

Proof of Theorem 3.2. From Theorem 2.2, we have

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)\left(\lambda_i + \frac{n^2}{4}H_0^2\right).$$

Letting $\mu_i = \lambda_i + \frac{n^2}{4}H_0^2$, we have

$$\sum_{i=1}^{k}(\mu_{k+1} - \mu_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\mu_{k+1} - \mu_i)\mu_i.$$

From the recursion formula of Cheng and Yang with $t = n$, we infer

$$\frac{F_{k+1}}{(k+1)^{\frac{4}{n}}} \leq \frac{F_k}{k^{\frac{4}{n}}}.$$

According to the Weyl's asymptotic formula, we derive, for any positive integer k,

$$\frac{F_k}{k^{\frac{4}{n}}} \geq \frac{2n}{(n+2)(n+4)}\frac{16\pi^4}{(\omega_n \mathrm{vol}\Omega)^{\frac{4}{n}}}.$$

Since

$$F_k \leq \frac{2}{n}G_k^2,$$

we infer

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{n^2}{4}H_0^2 \geq \frac{n}{\sqrt{(n+2)(n+4)}}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}}.$$

This finishes the proof of Theorem 3.2. $\square$

3.6 Further research

From Weyl's asymptotic formula, we have

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j \sim \frac{n}{n+2}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}}, \quad k \to +\infty.$$

According to Theorem 3.1, we know

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j \geq \frac{n}{n+2}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \cdots .$$

Hence, the constant $\frac{n}{n+2}$ is optimal. The next landmark goal is to find the second term in the asymptotic formula for eigenvalues. Safarov and Vassiliev [22], under suitable assumptions on Ω, have given

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j = \frac{n}{n+2}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}} + c_n\frac{|\partial\Omega|}{(\mathrm{vol}\Omega)^{1+\frac{1}{n}}}k^{\frac{1}{n}} + o(k^{\frac{1}{n}}),$$

when $k \to +\infty$, where c_n is a positive constant dependent only on the dimension n. We shall propose the following:

Open problem. Let Ω be a bounded domain in $\mathbf{R}^n$. For eigenvalue λ_k of the Dirichlet eigenvalue problem of Laplacian, for $k = 1, 2, \cdots$, does the following hold?

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j \geq \frac{n}{n+2}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}} + c_n\frac{|\partial\Omega|}{\mathrm{vol}\Omega^{1+\frac{1}{n}}}k^{\frac{1}{n}},$$

where c_n is a positive constant depending only on the dimension n.

Remark 3.5. *For $n = 2$, Kovařík, Vugalter and Weidl* [14] *have made an important breakthrough for this open problem.*

In my joint work [7] with Qi, we study the n-dimensional case. We have proved the following:

Theorem 3.3. *Let Ω be an n-dimensional polytope in $\mathbf{R}^n$. Then, there exists a positive integer N, such that, for all $k \geq N$,*

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j \geq \frac{n}{n+2}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}} + \frac{\pi}{81\cdot 2^{n-1}(n+2)\omega_n^{\frac{1}{n}}}\frac{|\partial\Omega|}{(\mathrm{vol}\Omega)^{1+\frac{1}{n}}}k^{\frac{1}{n}-\varepsilon(k)},$$

where

$$\varepsilon(k) = 2\left[\sqrt{\frac{1}{n+12}\log_2\left(\left(\frac{\mathrm{vol}\Omega}{c_1}\right)^{n-1}\left(\frac{4n\pi^2}{n+2}\right)^{\frac{n}{2}}\frac{k}{\omega_n\mathrm{vol}\Omega}\right)}\right]^{-1},$$

$$c_1 = \sqrt{\frac{3}{\omega_n}\left(\frac{4n\pi^2}{n+2}\right)^{\frac{n}{2}}},$$

and $|\partial\Omega|$ denotes the area of the boundary of Ω.

4 Application to upper bounds for eigenvalues

4.1 Upper bounds for eigenvalues

First of all, for a bounded domain in the Euclidean space $\mathbf{R}^n$, according to the partial solution of the conjecture of Pólya due to Li and Yau, we have, for $k = 1, 2, \cdots$,

$$\lambda_k \geq \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}$$

and from the Weyl's asymptotic formula, we know

$$\lambda_k \sim \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}.$$

Hence, it is also very important to obtain upper bounds for eigenvalues with the optimal order of k. But, it is very hard to obtain an upper bound for eigenvalues with the optimal order of k. In order to obtain an upper bound for eigenvalues with the optimal order of k, we need the recursion formula of Cheng and Yang [10]

$$\frac{F_{k+1}}{(k+1)^{\frac{4}{n}}} \leq \frac{F_k}{k^{\frac{4}{n}}}.$$

By making use of the recursion formula of Cheng and Yang and universal inequalities for eigenvalues, we have proved in [10]

Theorem 4.1. *For a bounded domain $\Omega \subset \mathbf{R}^n$, eigenvalues of the Dirichlet eigenvalue problem of Laplacian satisfy, for any $k > 1$,*

$$\lambda_{k+1} \leq C_0(n)\lambda_1 k^{\frac{2}{n}},$$

where

$$C_0(n) \leq 1 + \frac{2.6}{n}.$$

Remark 4.1. *The upper bound of Theorem* 4.1 *is best possible in the sense of order of k and it is a universal inequality.*

For a complete Riemannian manifold M, Chen and Cheng [6] have obtained the following

Theorem 4.2. *For a bounded domain Ω in an n-dimensional complete Riemannian manifold, there exists a constant H_0^2 such that eigenvalues of the Dirichlet eigenvalue problem of Laplacian satisfy, for $k > 1$,*

$$\lambda_{k+1} + \frac{n^2}{4} H_0^2 \leq C_0(n) \left(\lambda_1 + \frac{n^2}{4} H_0^2 \right) k^{\frac{2}{n}},$$

In particular, for complete minimal submanifolds in $\mathbf{R}^N$, we have

Corollary 4.1. *Let M be an n-dimensional complete minimal submanifold in the Euclidean space. Then, for any bounded domain Ω in M, eigenvalues of the Dirichlet eigenvalue problem of Laplacian satisfy*

$$\lambda_{k+1} \leq C_0(n)\lambda_1 k^{\frac{2}{n}}.$$

5 An obstruction for complete minimal immersions

For a given n-dimensional complete Riemannian manifold M, it is a very important problem in the differential geometry, to study whether there exist an isometric minimal immersion from M into the Euclidean space $\mathbf{R}^N$.

For a compact Riemannian manifold M, it is well-known that there exist no isometric minimal immersions from M into the Euclidean space $\mathbf{R}^N$. But, for complete and non-compact Riemannian manifolds, there is no any criterion to decide whether there exists an isometric minimal immersion from a complete and non-compact Riemannian manifold into the Euclidean space $\mathbf{R}^N$.

From estimates for eigenvalues of the Dirichlet eigenvalue problem of the Laplacian, when M is an n-dimensional complete minimal submanifold in the Euclidean space $\mathbf{R}^N$, for any bounded domain Ω in M, the following holds.

- The universal inequality for eigenvalues is given by
$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i,$$
which is the same as one in the Euclidean space $\mathbf{R}^n$.

- The upper bound for eigenvalues of the Dirichlet eigenvalue problem of the Laplacian are given by
$$\lambda_{k+1} \leq C_0(n)k^{\frac{2}{n}}\lambda_1,$$
for $k>1$, which is also the same as one in the Euclidean space $\mathbf{R}^n$.

- The lower bound for eigenvalues of the Dirichlet eigenvalue problem of the Laplacian are given by, for $k=1,2,\cdots$,
$$F_k \geq \frac{2n}{(n+2)(n+4)}\frac{16\pi^4}{(\omega_n \mathrm{vol}\Omega)^{\frac{4}{n}}}k^{\frac{4}{n}},$$
with
$$G_k=\frac{1}{k}\sum_{i=1}^{k}\lambda_i, \quad T_k=\frac{1}{k}\sum_{i=1}^{k}\lambda_i^2, \quad F_k=\left(1+\frac{2}{n}\right)G_k^2-T_k,$$
which is also the same as one in the Euclidean space $\mathbf{R}^n$.

Thus, we have the following:

An obstruction for complete minimal immersions. For an n-dimensional complete Riemannian manifold M, if there exists a minimal isometric immersion from M into a Euclidean space $\mathbf{R}^N$, then, for any bounded domain Ω in M, eigenvalues of the Dirichlet eigenvalue problem of the Laplacian must have the almost same behaviors as ones in the Euclidean space $\mathbf{R}^n$.

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