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ESTIMATES FOR EIGENVALUES OF $\mathfrak{L}$ OPERATOR ON SELF-SHRINKERS

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In this paper, we study eigenvalues of the closed eigenvalue problem of the differential operator $\mathfrak{L}$, which is introduced by Colding and Minicozzi in [Generic mean curvature flow I; generic singularities, *Ann. Math.* **175** (2012) 755–833], on an n-dimensional compact self-shrinker in $\mathbf{R}^{n+p}$. Estimates for eigenvalues of the differential operator $\mathfrak{L}$ are obtained. Our estimates for eigenvalues of the differential operator $\mathfrak{L}$ are sharp. Furthermore, we also study the Dirichlet eigenvalue problem of the differential operator $\mathfrak{L}$ on a bounded domain with a piecewise smooth boundary in an n-dimensional complete self-shrinker in $\mathbf{R}^{n+p}$. For Euclidean space $\mathbf{R}^n$, the differential operator $\mathfrak{L}$ becomes the Ornstein–Uhlenbeck operator in stochastic analysis. Hence, we also give estimates for eigenvalues of the Ornstein–Uhlenbeck operator.

Keywords: Mean curvature flows; self-shrinkers; spheres; the differential operator $\mathfrak{L}$ and eigenvalues.

Mathematics Subject Classification 2010: 58G25, 53C40

1. Introduction

Let $X : M^n \to \mathbf{R}^{n+p}$ be an isometric immersion from an n-dimensional Riemannian manifold M^n into a Euclidean space $\mathbf{R}^{n+p}$. One considers a smooth one-parameter family of immersions:

$$F(\cdot, t) : M^n \to \mathbf{R}^{n+p}$$

satisfying $F(\cdot, 0) = X(\cdot)$ and

$$\left(\frac{\partial F(p,t)}{\partial t}\right)^N = H(p,t), \quad (p,t) \in M \times [0,T), \tag{1.1}$$

where $H(p,t)$ denotes the mean curvature vector of submanifold $M_t = F(M^n, t)$ at point $F(p,t)$. Equation (1.1) is called the mean curvature flow equation. A submanifold $X : M^n \to \mathbf{R}^{n+p}$ is said to be a self-shrinker in $\mathbf{R}^{n+p}$ if it satisfies

$$H = -X^N, \tag{1.2}$$

where X^N denotes the orthogonal projection into the normal bundle of M^n (cf. [10]). Self-shrinkers play an important role in the study of the mean curvature flow since they are not only solutions of the mean curvature flow equation, but they also describe all possible blow ups at a given singularity of a mean curvature flow. Huisken [11] proved that the sphere of radius $\sqrt{n}$ is the only closed embedded self-shrinker hypersurfaces with non-zero mean curvature. For classifications of complete non-compact embedded self-shrinker hypersurfaces, Huisken [12] and Colding and Minicozzi [6] proved that an n-dimensional complete embedded self-shrinker hypersurface with non-negative mean curvature and polynomial volume growth in $\mathbf{R}^{n+1}$ is a Riemannian product $S^k \times \mathbf{R}^{n-k}$, $0 \le k < n$. Smoczyk [14] has obtained several results for complete self-shrinkers with higher codimensions.

For study of the rigidity problem for self-shrinkers, Le and Sesum [13] and Cao and Li [1] have classified n-dimensional complete embedded self-shrinkers in $\mathbf{R}^{n+p}$ with polynomial volume growth if the squared norm $|A|^2$ of the second fundamental form satisfies $|A|^2 \le 1$. For a further study, see [4, 5, 7–9, 15] and so on.

In [6], Colding and Minicozzi introduced a differential operator $\mathfrak{L}$ and used it to study self-shrinkers. The differential operator $\mathfrak{L}$ is defined by

$$\mathfrak{L}f = \Delta f - \langle X, \nabla f \rangle \tag{1.3}$$

for a smooth function f, where Δ and ∇ denote the Laplacian and the gradient operator on the self-shrinker, respectively and $\langle \cdot, \cdot \rangle$ denotes the standard inner product of $\mathbf{R}^{n+p}$. We should notice that the differential operator $\mathfrak{L}$ plays a very important role in studying of n-dimensional complete embedded self-shrinkers in $\mathbf{R}^{n+p}$ with polynomial volume growth in order to guarantee integration by part holds as in [6].

The purpose of this paper is to study eigenvalues of the closed eigenvalue problem for the differential operator $\mathfrak{L}$ on compact self-shrinkers in $\mathbf{R}^{n+p}$ in Secs. 3 and 4 and eigenvalues of the Dirichlet eigenvalue problem of the differential operator $\mathfrak{L}$ on a bounded domain with a piecewise smooth boundary in complete self-shrinkers in $\mathbf{R}^{n+p}$ in Sec. 5. We shall adapt the idea of Cheng and Yang in [2] for studying eigenvalues of the Dirichlet eigenvalue problem of the Laplacian Δ to the differential operator $\mathfrak{L}$ by constructing appropriated trial functions for the differential operator $\mathfrak{L}$. Since the differential operator $\mathfrak{L}$ is self-adjoint with respect to measure

$e^{-\frac{|X|^2}{2}}dv$, where dv is the volume element of M^n and $|X|^2 = \langle X, X\rangle$, we know that the closed eigenvalue problem:

$$\mathfrak{L}u = -\lambda u \quad \text{on } M^n \tag{1.4}$$

for the differential operator $\mathfrak{L}$ on compact self-shrinkers in $\mathbf{R}^{n+p}$ has a real and discrete spectrum:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty,$$

where each eigenvalue is repeated according to its multiplicity. We shall prove the following theorem.

Theorem 1.1. *Let M^n be an n-dimensional compact self-shrinker in $\mathbf{R}^{n+p}$. Then, eigenvalues of the closed eigenvalue problem* (1.4) *satisfy*

$$\sum_{i=0}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{n}\sum_{i=0}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i + \frac{2n - \min_{M^n}|X|^2}{4}\right). \tag{1.5}$$

Remark 1.1. The sphere $S^n(\sqrt{n})$ of radius $\sqrt{n}$ is a compact self-shrinker in $\mathbf{R}^{n+p}$. For $S^n(\sqrt{n})$ and for any k, the inequality (1.5) for eigenvalues of the closed eigenvalue problem (1.4) becomes equality. Hence our results in Theorem 1.1 are sharp.

Furthermore, from the recursion formula of Cheng and Yang [3], we can obtain an upper bound for eigenvalue λ_k.

Theorem 1.2. *Let M^n be an n-dimensional compact self-shrinker in $\mathbf{R}^{n+p}$. Then, eigenvalues of the closed eigenvalue problem* (1.4) *satisfy, for any $k \geq 1$,*

$$\lambda_k + \frac{2n - \min_{M^n}|X|^2}{4} \leq \left(1 + \frac{a(\min\{n, k-1\})}{n}\right)\left(\frac{2n - \min_{M^n}|X|^2}{4}\right)k^{2/n},$$

where the bound of $a(m)$ can be formulated as:

$$\begin{cases} a(0) \leq 4, \\ a(1) \leq 2.64, \\ a(m) \leq 2.2 - 4\log\left(1 + \frac{1}{50}(m-3)\right), \quad \text{for } m \geq 2. \end{cases}$$

In particular, for $n \geq 41$ and $k \geq 41$, we have

$$\lambda_k + \frac{2n - \min_{M^n}|X|^2}{4} \leq \left(\frac{2n - \min_{M^n}|X|^2}{4}\right)k^{2/n}.$$

Results for eigenvalues of the Dirichlet eigenvalue problem of the differential operator $\mathfrak{L}$ are given in Sec. 5.

1 ## 2. Preliminaries

Suppose $X : M^n \to \mathbf{R}^{n+p}$ is an isometric immersion from Riemannian manifold M^n into the $(n+p)$-dimensional Euclidean space $\mathbf{R}^{n+p}$. Let $\{E_A\}_{A=1}^{n+p}$ be the standard basis of $\mathbf{R}^{n+p}$. The position vector can be written by $X = (x_1, x_2, \ldots, x_{n+p})$. We choose a local orthonormal frame field $\{e_1, e_2, \ldots, e_n, e_{n+1}, \ldots, e_{n+p}\}$ and the dual coframe field $\{\omega_1, \omega_2, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_{n+p}\}$ along M^n of $\mathbf{R}^{n+p}$ such that $\{e_1, e_2, \ldots, e_n\}$ is a local orthonormal basis on M^n. Thus, we have

$$\omega_\alpha = 0, \quad n+1 \le \alpha \le n+p$$

on M^n. From the Cartan's lemma, we have

$$\omega_{i\alpha} = \sum_{j=1}^{n} h_{ij}^{\alpha}\omega_j, \quad h_{ij}^{\alpha} = h_{ji}^{\alpha}.$$

The second fundamental form $\mathbf{h}$ of M^n and the mean curvature vector H are defined, respectively, by

$$\mathbf{h} = \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} h_{ij}^{\alpha}\omega_i \otimes \omega_j e_\alpha,$$

$$H = \sum_{\alpha=n+1}^{n+p} \sum_{i=1}^{n} h_{ii}^{\alpha} e_\alpha.$$

One considers the mean curvature flow for a submanifold $X : M^n \to \mathbf{R}^{n+p}$. Namely, we consider a one-parameter family of immersions:

$$F(\cdot, t) : M^n \to \mathbf{R}^{n+p}$$

satisfying $F(\cdot, 0) = X(\cdot)$ and

$$\left(\frac{\partial F(p,t)}{\partial t}\right)^N = H(p,t), \quad (p,t) \in M \times [0,T), \tag{2.1}$$

AQ: The sentence "An important ..." seems unclear. Please check and correct if necessary.

where $H(p,t)$ denotes the mean curvature vector of submanifold $M_t = F(M^n, t)$ at point $F(p,t)$. An important class of solutions to the mean curvature flow equation (2.1) are self-similar shrinkers, which profiles, self-shrinkers, satisfy

$$H = -X^N,$$

2 which is a system of quasi-linear elliptic partial differential equations of the second
3 order. Here X^N denotes the orthogonal projection of X into the normal bundle
4 of M^n.

In [6], Colding and Minicozzi introduced a differential operator $\mathfrak{L}$ and used it to study self-shrinkers. The differential operator $\mathfrak{L}$ is defined by

$$\mathfrak{L}f = \Delta f - \langle X, \nabla f \rangle \tag{2.2}$$

for a smooth function f, where Δ and ∇ denote the Laplacian and the gradient operator on the self-shrinker, respectively. For a compact self-shrinker M^n without boundary, we have

$$\begin{aligned}\int_{M^n} f\mathfrak{L}ue^{-\frac{|X|^2}{2}}dv &= \int_{M^n} f(\Delta u - \langle X, \nabla u\rangle)e^{-\frac{|X|^2}{2}}dv \\ &= \int_{M^n} f\mathrm{div}\left(e^{-\frac{|X|^2}{2}}\nabla u\right)dv = \int_{M^n} u\mathfrak{L}fe^{-\frac{|X|^2}{2}}dv,\end{aligned}$$

that is,

$$\int_{M^n} f\mathfrak{L}ue^{-\frac{|X|^2}{2}}dv = \int_{M^n} u\mathfrak{L}fe^{-\frac{|X|^2}{2}}dv, \tag{2.3}$$

for any smooth functions u, f. Hence, the differential operator $\mathfrak{L}$ is self-adjoint with respect to the measure $e^{-\frac{|X|^2}{2}}dv$. Therefore, we know that the closed eigenvalue problem:

$$\mathfrak{L}u = -\lambda u \quad \text{on } M^n \tag{2.4}$$

has a real and discrete spectrum:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty.$$

Furthermore, we have

$$\mathfrak{L}x_A = -x_A. \tag{2.5}$$

In fact,

$$\begin{aligned}\mathfrak{L}x_A &= \Delta\langle X, E_A\rangle - \langle X, \nabla x_A\rangle \\ &= \langle \Delta X, E_A\rangle - \langle X, E_A^T\rangle \\ &= \langle H, E_A\rangle - \langle X, E_A^T\rangle \\ &= -\langle X^N, E_A\rangle - \langle X, E_A^T\rangle = -x_A.\end{aligned}$$

Denote the induced metric by g and define $\nabla u \cdot \nabla v = g(\nabla u, \nabla v)$ for functions u, v. We get, from (2.5),

$$\mathfrak{L}|X|^2 = \sum_{A=1}^{n+p}(2x_A\mathfrak{L}x_A + 2\nabla x_A \cdot \nabla x_A) = 2(n - |X|^2). \tag{2.6}$$

Here we have used

$$\sum_{A=1}^{n+p} \nabla x_A \cdot \nabla x_A = n.$$

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Proposition 2.1. *For an n-dimensional compact self-shrinker M^n without boundary in $\mathbf{R}^{n+p}$, we have*

$$\min_{M^n}|X|^2 \leq n = \frac{\displaystyle\int_{M^n}|X|^2e^{-\frac{|X|^2}{2}}dv}{\displaystyle\int_{M^n}e^{-\frac{|X|^2}{2}}dv} \leq \max_{M^n}|X^N|^2.$$

Proof. Since $\mathfrak{L}$ is self-adjoint with respect to the measure $e^{-\frac{|X|^2}{2}}dv$, from (2.6), we have

$$n\int_{M^n} e^{-\frac{|X|^2}{2}}dv = \int_{M^n} |X|^2 e^{-\frac{|X|^2}{2}}dv \geq \min_{M^n} |X|^2 \int_{M^n} e^{-\frac{|X|^2}{2}}dv.$$

Furthermore, since

$$\Delta|X|^2 = 2(n + \langle X, H\rangle) = 2(n - |X^N|^2), \tag{2.7}$$

we have

$$n \leq \max_{M^n} |X^N|^2.$$

It completes the proof of this proposition. □

3. Universal Estimates for Eigenvalues

In this section, we give proof of Theorem 1.1. In order to prove our Theorem 1.1, we need to construct trial functions. Thanks to $\mathfrak{L}X = -X$, we can use coordinate functions of the position vector X of the self-shrinker M^n to construct trial functions.

Proof of Theorem 1.1. For an n-dimensional compact self-shrinker M^n in $\mathbf{R}^{n+p}$, the closed eigenvalue problem:

$$\mathfrak{L}u = -\lambda u \quad \text{on } M^n \tag{3.1}$$

for the differential operator $\mathfrak{L}$ has a discrete spectrum. For any integer $j \geq 0$, let u_j be an eigenfunction corresponding to the eigenvalue λ_j such that

$$\begin{cases} \mathfrak{L}u_j = -\lambda_j u_j, & \text{on } M^n \\ \displaystyle\int_{M^n} u_i u_j e^{-\frac{|X|^2}{2}}dv = \delta_{ij}, & \text{for any } i, j. \end{cases} \tag{3.2}$$

From the Rayleigh–Ritz inequality, we have

$$\lambda_{k+1} \leq \frac{-\displaystyle\int_{M^n} \varphi\mathfrak{L}\varphi e^{-\frac{|X|^2}{2}}dv}{\displaystyle\int_{M^n} \varphi^2 e^{-\frac{|X|^2}{2}}dv}, \tag{3.3}$$

for any function φ satisfies $\int_{M^n} \varphi u_j e^{-\frac{|X|^2}{2}}dv$, $0 \leq j \leq k$. Since $X : M^n \to \mathbf{R}^{n+p}$ is a self-shrinker in $\mathbf{R}^{n+p}$, we have

$$H = -X^N. \tag{3.4}$$

Letting x_A, $A = 1, 2, \ldots, n+p$, denote components of the position vector X, we define, for $0 \leq i \leq k$,

$$\varphi_i^A := x_A u_i - \sum_{j=0}^{k} a_{ij}^A u_j, \quad a_{ij}^A = \int_{M^n} x_A u_i u_j e^{-\frac{|X|^2}{2}}dv. \tag{3.5}$$

By a simple calculation, we obtain

$$\int_{M^n} u_j \varphi_i^A e^{-\frac{|X|^2}{2}} dv = 0, \quad i, j = 0, 1, \ldots, k. \tag{3.6}$$

From the Rayleigh-Ritz inequality, we have

$$\lambda_{k+1} \leq \frac{-\int_{M^n} \varphi_i^A \mathfrak{L} \varphi_i^A e^{-\frac{|X|^2}{2}} dv}{\int_{M^n} (\varphi_i^A)^2 e^{-\frac{|X|^2}{2}} dv}. \tag{3.7}$$

Since

$$\begin{aligned} \mathfrak{L}\varphi_i^A &= \Delta \varphi_i^A - \langle X, \nabla \varphi_i^A \rangle \\ &= \Delta \left(x_A u_i - \sum_{j=0}^k a_{ij}^A u_j \right) - \left\langle X, \nabla \left(x_A u_i - \sum_{j=0}^k a_{ij}^A u_j \right) \right\rangle \\ &= x_A \Delta u_i + u_i \Delta x_A + 2\nabla x_A \cdot \nabla u_i - \langle X, x_A \nabla u_i + u_i \nabla x_A \rangle \\ &\quad - \sum_{j=0}^k a_{ij}^A \Delta u_j + \left\langle X, \sum_{j=0}^k a_{ij}^A \nabla u_j \right\rangle \\ &= -\lambda_i x_A u_i + u_i \mathfrak{L} x_A + 2\nabla x_A \cdot \nabla u_i + \sum_{j=0}^k a_{ij}^A \lambda_j u_j, \end{aligned} \tag{3.8}$$

AQ: Please check the edit of equation (3.8).

we have, from (3.7) and (3.8),

$$(\lambda_{k+1} - \lambda_i) \|\varphi_i^A\|^2 \leq -\int_{M^n} \varphi_i^A (u_i \mathfrak{L} x_A + 2\nabla x_A \cdot \nabla u_i) e^{-\frac{|X|^2}{2}} dv := W_i^A, \tag{3.9}$$

where

$$\|\varphi_i^A\|^2 = \int_{M^n} (\varphi_i^A)^2 e^{-\frac{|X|^2}{2}} dv.$$

On the other hand, defining

$$b_{ij}^A = -\int_{M^n} (u_j \mathfrak{L} x_A + 2\nabla x_A \cdot \nabla u_j) u_i e^{-\frac{|X|^2}{2}} dv$$

we obtain

$$b_{ij}^A = (\lambda_i - \lambda_j) a_{ij}^A. \tag{3.10}$$

In fact,

$$\begin{aligned} \lambda_i a_{ij}^A &= \int_{M^n} \lambda_i u_i u_j x_A e^{-\frac{|X|^2}{2}} dv \\ &= -\int_{M^n} u_j x_A \mathfrak{L} u_i e^{-\frac{|X|^2}{2}} dv \\ &= -\int_{M^n} u_i \mathfrak{L}(u_j x_A) e^{-\frac{|X|^2}{2}} dv \end{aligned}$$

$$= -\int_{M^n} u_i(x_A \mathfrak{L} u_j + u_j \mathfrak{L} x_A + 2\nabla x_A \cdot \nabla u_j) e^{-\frac{|X|^2}{2}} dv$$
$$= \lambda_j a_{ij}^A + b_{ij}^A,$$

that is,

$$b_{ij}^A = (\lambda_i - \lambda_j) a_{ij}^A.$$

Hence, we have

$$b_{ij}^A = -b_{ji}^A. \tag{3.11}$$

From (3.6), (3.9) and the Cauchy–Schwarz inequality, we infer

$$\begin{aligned} W_i^A &= -\int_{M^n} \varphi_i^A (u_i \mathfrak{L} x_A + 2\nabla x_A \cdot \nabla u_i) e^{-\frac{|X|^2}{2}} dv \\ &= -\int_{M^n} \varphi_i^A \left(u_i \mathfrak{L} x_A + 2\nabla x_A \cdot \nabla u_i - \sum_{j=0}^k b_{ij}^A u_j \right) e^{-\frac{|X|^2}{2}} dv \\ &\le \|\varphi_i^A\| \left\| u_i \mathfrak{L} x_A + 2\nabla x_A \cdot \nabla u_i - \sum_{j=0}^k b_{ij}^A u_j \right\|. \end{aligned} \tag{3.12}$$

Hence, we have, from (3.9) and (3.12),

$$\begin{aligned} (\lambda_{k+1} - \lambda_i)(W_i^A)^2 &= (\lambda_{k+1} - \lambda_i)\|\varphi_i^A\|^2 \left\| u_i \mathfrak{L} x_A + 2\nabla x_A \cdot \nabla u_i - \sum_{j=0}^k b_{ij}^A u_j \right\|^2 \\ &\le W_i^A \left\| u_i \mathfrak{L} x_A + 2\nabla x_A \cdot \nabla u_i - \sum_{j=0}^k b_{ij}^A u_j \right\|^2. \end{aligned}$$

Therefore, we obtain

$$(\lambda_{k+1} - \lambda_i)^2 W_i^A \le (\lambda_{k+1} - \lambda_i) \left\| u_i \mathfrak{L} x_A + 2\nabla x_A \cdot \nabla u_i - \sum_{j=0}^k b_{ij}^A u_j \right\|^2. \tag{3.13}$$

Summing on i from 0 to k for (3.13), we have

$$\begin{aligned} &\sum_{i=0}^k (\lambda_{k+1} - \lambda_i)^2 W_i^A \\ &\quad \le \sum_{i=0}^k (\lambda_{k+1} - \lambda_i) \left\| u_i \mathfrak{L} x_A + 2\nabla x_A \cdot \nabla u_i - \sum_{j=0}^k b_{ij}^A u_j \right\|^2. \end{aligned} \tag{3.14}$$

By the definition of b_{ij}^A and (3.10), we have

$$\begin{aligned}
&\left\|u_i\mathfrak{L}x_A + 2\nabla x_A\cdot\nabla u_i - \sum_{j=0}^k b_{ij}^A u_j\right\|^2\\
&\quad= \|u_i\mathfrak{L}x_A + 2\nabla x_A\cdot\nabla u_i\|^2\\
&\qquad - 2\sum_{j=0}^k b_{ij}^A\int_{M^n}(u_i\mathfrak{L}x_A + 2\nabla x_A\cdot\nabla u_i)u_j e^{-\frac{|X|^2}{2}}dv + \sum_{j=0}^k (b_{ij}^A)^2\\
&\quad= \|u_i\mathfrak{L}x_A + 2\nabla x_A\cdot\nabla u_i\|^2 - \sum_{j=0}^k (b_{ij}^A)^2\\
&\quad= \|u_i\mathfrak{L}x_A + 2\nabla x_A\cdot\nabla u_i\|^2 - \sum_{j=0}^k (\lambda_i-\lambda_j)^2(a_{ij}^A)^2. \qquad (3.15)
\end{aligned}$$

Furthermore, according to the definitions of W_i^A and φ_i^A, we have from (3.10)

$$\begin{aligned}
W_i^A &= -\int_{M^n}\varphi_i^A(u_i\mathfrak{L}x_A + 2\nabla x_A\cdot\nabla u_i)e^{-\frac{|X|^2}{2}}dv\\
&= -\int_{M^n}\left(x_A u_i - \sum_{j=0}^k a_{ij}^A u_j\right)(u_i\mathfrak{L}x_A + 2\nabla x_A\cdot\nabla u_i)e^{-\frac{|X|^2}{2}}dv\\
&= -\int_{M^n}(x_A u_i^2\mathfrak{L}x_A + 2x_A u_i\nabla x_A\cdot\nabla u_i)e^{-\frac{|X|^2}{2}}dv\\
&\quad + \sum_{j=0}^k a_{ij}^A\int_{M^n} u_j(u_i\mathfrak{L}x_A + 2\nabla x_A\cdot\nabla u_i)e^{-\frac{|X|^2}{2}}dv\\
&= -\int_{M^n}\left(x_A\mathfrak{L}x_A - \frac{1}{2}\mathfrak{L}(x_A)^2\right)u_i^2 e^{-\frac{|X|^2}{2}}dv + \sum_{j=0}^k a_{ij}^A b_{ij}^A\\
&= \int_{M^n}\nabla x_A\cdot\nabla x_A u_i^2 e^{-\frac{|X|^2}{2}}dv + \sum_{j=0}^k(\lambda_i-\lambda_j)(a_{ij}^A)^2. \qquad (3.16)
\end{aligned}$$

Since

$$\begin{aligned}
2\sum_{i,j=0}^k(\lambda_{k+1}-\lambda_i)^2(\lambda_i-\lambda_j)(a_{ij}^A)^2 &= \sum_{i,j=0}^k(\lambda_{k+1}-\lambda_i)^2(\lambda_i-\lambda_j)(a_{ij}^A)^2\\
&\quad - \sum_{i,j=0}^k(\lambda_{k+1}-\lambda_j)^2(\lambda_i-\lambda_j)(a_{ij}^A)^2
\end{aligned}$$

$$= -\sum_{i,j=0}^{k} (\lambda_{k+1} - \lambda_i + \lambda_{k+1} - \lambda_j)(\lambda_i - \lambda_j)^2 (a_{ij}^A)^2$$

$$= -2 \sum_{i,j=0}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i - \lambda_j)^2 (a_{ij}^A)^2, \tag{3.17}$$

from (3.14)–(3.17), we obtain, for any A, $A = 1, 2, \ldots, n+p$,

$$\sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)^2 \int_{M^n} \nabla x_A \cdot \nabla x_A u_i^2 e^{-\frac{|X|^2}{2}} dv$$
$$\leq \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i) \| u_i \mathfrak{L} x_A + 2 \nabla x_A \cdot \nabla u_i \|^2. \tag{3.18}$$

On the other hand, since

$$\mathfrak{L} x_A = -x_A, \quad \sum_{A=1}^{n+p} (\nabla x_A \cdot \nabla u_i)^2 = \nabla u_i \cdot \nabla u_i,$$

we infer, from (2.6),

$$\sum_{A=1}^{n+p} \| u_i \mathfrak{L} x_A + 2 \nabla x_A \cdot \nabla u_i \|^2 = \sum_{A=1}^{n+p} \int_{M^n} (u_i \mathfrak{L} x_A + 2 \nabla x_A \cdot \nabla u_i)^2 e^{-\frac{|X|^2}{2}} dv$$
$$= \sum_{A=1}^{n+p} \int_{M^n} (u_i^2 (x_A)^2 - 4 u_i x_A \nabla x_A \cdot \nabla u_i$$
$$+ 4 (\nabla x_A \cdot \nabla u_i)^2) e^{-\frac{|X|^2}{2}} dv$$
$$= \sum_{A=1}^{n+p} \int_{M^n} (u_i^2 (x_A)^2 - \nabla (x_A)^2 \cdot \nabla u_i^2) e^{-\frac{|X|^2}{2}} dv$$
$$+ 4 \int_{M^n} \nabla u_i \cdot \nabla u_i e^{-\frac{|X|^2}{2}} dv$$
$$= \int_{M^n} (\mathfrak{L} |X|^2 + |X|^2) u_i^2 e^{-\frac{|X|^2}{2}} dv + 4 \lambda_i$$
$$= \int_{M^n} (2n - |X|^2) u_i^2 e^{-\frac{|X|^2}{2}} dv + 4 \lambda_i$$
$$\leq \left(2n - \min_{M^n} |X|^2 \right) + 4 \lambda_i. \tag{3.19}$$

Furthermore, because of

$$\sum_{A=1}^{n+p} \nabla x_A \cdot \nabla x_A = n, \tag{3.20}$$

taking summation on A from 1 to $n+p$ for (3.18) and using (3.19) and (3.20), we get

$$\sum_{i=0}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{n}\sum_{i=0}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i + \frac{2n-\min_{M^n}|X|^2}{4}\right).$$

It finished the proof of Theorem 1.1. □

4. Upper Bounds for Eigenvalues

The following recursion formula of Cheng and Yang [3] plays a very important role in order to prove Theorem 1.2.

A recursion formula of Cheng and Yang. Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{k+1}$ be any positive real numbers satisfying

$$\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}\mu_i(\mu_{k+1}-\mu_i).$$

Define

$$\Lambda_k = \frac{1}{k}\sum_{i=1}^{k}\mu_i, \quad T_k = \frac{1}{k}\sum_{i=1}^{k}\mu_i^2, \quad F_k = \left(1+\frac{2}{n}\right)\Lambda_k^2 - T_k.$$

Then, we have

$$F_{k+1} \leq C(n,k)\left(\frac{k+1}{k}\right)^{\frac{4}{n}} F_k, \tag{4.1}$$

where

$$C(n,k) = 1 - \frac{1}{3n}\left(\frac{k}{k+1}\right)^{\frac{4}{n}} \frac{\left(1+\frac{2}{n}\right)\left(1+\frac{4}{n}\right)}{(k+1)^3} < 1.$$

Proof of Theorem 1.2. From Proposition 2.1, we know

$$\mu_{i+1} = \lambda_i + \frac{2n-\min_{M^n}|X|^2}{4} > 0,$$

for any $i = 0, 1, 2, \ldots$. Then, we obtain from (1.5)

$$\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)\mu_i. \tag{4.2}$$

Thus, we know that μ_i's satisfy the condition of the above recursion formula of Cheng and Yang [3]. Furthermore, since

$$\mathfrak{L}x_A = -x_A \quad \text{and} \quad \int_{M^n} x_A e^{-\frac{|X|^2}{2}} dv = 0, \quad \text{for } A = 1, 2, \ldots, n+p,$$

$\lambda = 1$ is an eigenvalue of $\mathfrak{L}$ with multiplicity at least $n+p$. Thus,

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n+1} \leq 1.$$

Hence, we have

$$\sum_{j=1}^{n}(\mu_{j+1}-\mu_1)=\sum_{j=1}^{n}\lambda_j\le n\le 2n-\min_{M^n}|X|^2=4\mu_1 \tag{4.3}$$

because of $\min_{M^n}|X|^2\le n$ according to Proposition 2.1. Hence, we can prove Theorem 1.2 as in [3] almost word by word. For the convenience of readers, we shall give a self-contained proof. First of all, according to the above recursion formula of Cheng and Yang, we have

$$F_k\le C(n,k-1)\left(\frac{k}{k-1}\right)^{\frac{4}{n}}F_{k-1}\le k^{\frac{4}{n}}F_1=\frac{2}{n}k^{\frac{4}{n}}\mu_1^2.$$

Furthermore, we infer, from (4.2)

$$\left[\mu_{k+1}-\left(1+\frac{2}{n}\right)\Lambda_k\right]^2\le\left(1+\frac{4}{n}\right)F_k-\frac{2}{n}\left(1+\frac{2}{n}\right)\Lambda_k^2.$$

Hence, we have

$$\frac{\frac{2}{n}}{\left(1+\frac{4}{n}\right)}\mu_{k+1}^2+\frac{1+\frac{2}{n}}{1+\frac{4}{n}}\left(\mu_{k+1}-\left(1+\frac{4}{n}\right)\Lambda_k\right)^2\le\left(1+\frac{4}{n}\right)F_k.$$

Thus, we derive

$$\mu_{k+1}\le\left(1+\frac{4}{n}\right)\sqrt{\frac{n}{2}F_k}\le\left(1+\frac{4}{n}\right)k^{\frac{2}{n}}\mu_1. \tag{4.4}$$

Define

$$\begin{aligned}
a_1(n)&=\frac{n\left(1+\frac{4}{n}\right)\left(1+\frac{8}{n+1}+\frac{8}{(n+1)^2}\right)^{\frac{1}{2}}}{(n+1)^{\frac{2}{n}}}-n,\\
a_2(k,n)&=\frac{n}{k^{\frac{2}{n}}}\left(1+\frac{4(n+k+4)}{n^2+5n-4(k-1)}\right)-n,\\
a_2(k)&=\max\{a(n,k),k\le n\le 400\},\\
a_3(k)&=\frac{4}{1-\frac{k}{400}}-2\log k,\\
a(k)&=\max\{a_1(k),a_2(k+1)),a_3(k+1)\}.
\end{aligned}$$

Case 1. For $k\ge n+1$, we have

$$\begin{aligned}
\mu_{k+1}&\le\frac{\left(1+\frac{4}{n}\right)\left(1+\frac{8}{n+1}+\frac{8}{(n+1)^2}\right)^{\frac{1}{2}}}{(n+1)^{\frac{2}{n}}}k^{\frac{2}{n}}\mu_1\\
&=\left(1+\frac{a_1(n)}{n}\right)k^{\frac{2}{n}}\mu_1,
\end{aligned} \tag{4.5}$$

where $a_1(n) \leq 2.31$. In fact, since μ_{k+1} satisfies (4.2), we have, from (4.1),

$$\mu_{k+1}^2 \leq \frac{n}{2}\left(1+\frac{4}{n}\right)^2 F_k \leq \frac{n}{2}\left(1+\frac{4}{n}\right)^2\left(\frac{k}{n+1}\right)^{\frac{4}{n}} F_{n+1}. \tag{4.6}$$

On the other hand,

$$\begin{aligned} F_{n+1} &= \frac{2}{n}\Lambda_{n+1}^2 - \sum_{i=1}^{n+1}\frac{(\mu_i-\Lambda_{n+1})^2}{n+1} \\ &\leq \frac{2}{n}\Lambda_{n+1}^2 - \frac{(\mu_1-\Lambda_{n+1})^2+\frac{1}{n}(\mu_1-\Lambda_{n+1})^2}{n+1} \\ &= \frac{2}{n}\left(\Lambda_{n+1}^2-\frac{(\mu_1-\Lambda_{n+1})^2}{2}\right). \end{aligned} \tag{4.7}$$

It is obvious that $\Lambda_{n+1}^2 - \frac{(\mu_1-\Lambda_{n+1})^2}{2}$ is an increasing function of Λ_{n+1}. From (4.3), we have

$$\mu_{n+1}+\cdots+\mu_2 \leq (n+4)\mu_1. \tag{4.8}$$

Thus, we derive

$$\Lambda_{n+1} \leq \left(1+\frac{4}{n+1}\right)\mu_1. \tag{4.9}$$

Hence, we have

$$\frac{n}{2}F_{n+1} \leq \left(1+\frac{8}{n+1}+\frac{8}{(n+1)^2}\right)\mu_1^2. \tag{4.10}$$

1 From (4.6) and (4.10), we complete the proof of (4.5).

Case 2. For $k \geq 55$ and $n \geq 54$, we have

$$\mu_{k+1} \leq k^{\frac{2}{n}}\mu_1. \tag{4.11}$$

If $k \geq n+1$, from Case 1, we have

$$\mu_{k+1} \leq \frac{1}{(n+1)^{\frac{2}{n}}}\left(1+\frac{4}{n}\right)^2 k^{\frac{2}{n}}\mu_1.$$

Since

$$\begin{aligned} (n+1)^{\frac{2}{n}} &= \exp\left(\frac{2}{n}\log(n+1)\right) \\ &\geq 1+\frac{2}{n}\log(n+1)+\frac{2}{n^2}(\log(n+1))^2 \\ &\geq \left(1+\frac{1}{n}\log(n+1)\right)^2, \end{aligned} \tag{4.12}$$

we have

$$\mu_{k+1} \le \left(\frac{1 + \dfrac{4}{n}}{1 + \dfrac{1}{n}\log(n+1)} \right)^2 k^{\frac{2}{n}} \mu_1. \tag{4.13}$$

Then, when $n \ge 54$, $\log(n+1) \ge 4$, we have

$$\mu_{k+1} \le k^{\frac{2}{n}} \mu_1.$$

On the other hand, if $k \le n$, then $\Lambda_k \le \Lambda_{n+1}$. Since

$$\begin{aligned}
\frac{n}{2} F_k &= \Lambda_k^2 - \frac{n}{2} \frac{\sum_{i=1}^{k} (\mu_i - \Lambda_k)^2}{k} \\
&\le \Lambda_k^2 - \frac{n}{2} \frac{(\mu_1 - \Lambda_k)^2 + \dfrac{\left\{ \sum_{i=2}^{k} (\mu_i - \Lambda_k) \right\}^2}{k-1}}{k} \\
&\le \Lambda_k^2 - \frac{(\mu_1 - \Lambda_k)^2}{2} \\
&\le \Lambda_{n+1}^2 - \frac{(\mu_1 - \Lambda_{n+1})^2}{2} \le \left(1 + \frac{4}{n}\right)^2 \mu_1^2,
\end{aligned}$$

we have

$$\mu_{k+1} \le \left(1 + \frac{4}{n}\right) \sqrt{\frac{n}{2} F_k} \le \frac{1}{k^{\frac{2}{n}}} \left(1 + \frac{4}{n}\right)^2 k^{\frac{2}{n}} \mu_1 \le \left(\frac{1 + \dfrac{4}{n}}{1 + \dfrac{\log k}{n}} \right)^2 k^{\frac{2}{n}} \mu_1.$$

Here we used $k^{\frac{2}{n}} \ge (1 + \frac{\log k}{n})^2$. By the same assertion as above, when $k \ge 55$, we also have

$$\mu_{k+1} \le k^{\frac{2}{n}} \mu_1.$$

Case 3. For $k \le 54$ and $k \le n$, we have

$$\mu_{k+1} \le \left(1 + \frac{\max\{a_2(k), a_3(k)\}}{n}\right) k^{\frac{2}{n}} \mu_1.$$

Because of $k \le n$ and $k \le 54$, from (4.3), we derive

$$\mu_{k+1} \le \frac{1}{n - k + 1} \{(n+5)\mu_1 - k\Lambda_k\}. \tag{4.14}$$

Since formula (4.2) is a quadratic inequality for μ_{k+1}, we have

$$\mu_{k+1} \le \left(1 + \frac{4}{n}\right) \Lambda_k. \tag{4.15}$$

Since the right-hand side of (4.14) is a decreasing function of Λ_k and the right-hand side of (4.15) is an increasing function of Λ_k, for $\frac{1}{n-k+1}\{(n+5)\mu_1-k\Lambda_k\} = (1+\frac{4}{n})\Lambda_k$, we infer

$$\begin{aligned}\mu_{k+1} &\leq \frac{1}{k^{\frac{2}{n}}}\left(1+\frac{4(n+k+4)}{n^2+5n-4(k-1)}\right)k^{\frac{2}{n}}\mu_1\\ &= \left(1+\frac{a_2(k,n)}{n}\right)k^{\frac{2}{n}}\mu_1. \end{aligned}\tag{4.16}$$

From the definition of $a_2(k) = \max\{a(n,k), k \leq n \leq 400\}$, when $n \leq 400$, we obtain

$$\mu_{k+1} \leq \left(1+\frac{a_2(k)}{n}\right)k^{\frac{2}{n}}\mu_1. \tag{4.17}$$

When $n > 400$ holds, from (4.4), we have

$$\mu_{k+1} \leq \left(1+\frac{4}{n-k}\right)\mu_1.$$

Since $n > 400$ and $k \leq 54$, we know $\frac{2}{n}\log k < \frac{1}{50}$. Hence, we have

$$\begin{aligned}k^{-\frac{2}{n}} = e^{-\frac{2}{n}\log k} &= 1-\frac{2}{n}\log k+\frac{1}{2}\left(\frac{2}{n}\log k\right)^2-\cdots\\ &\leq 1-\frac{2}{n}\log k+\frac{1}{2}\left(\frac{2}{n}\log k\right)^2.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\left(1+\frac{4}{n-k}\right)k^{-\frac{2}{n}} &\leq \left(1+\frac{4}{n-k}\right)\left(1-\frac{2}{n}\log k+\frac{1}{2}\left(\frac{2}{n}\log k\right)^2\right)\\ &\leq 1+\frac{\left(4\Big/\left(1-\frac{k}{400}\right)-2\log k\right)}{n}.\end{aligned}$$

Hence, we infer

$$\begin{aligned}\mu_{k+1} &\leq \left(1+\frac{4}{n-k}\right)k^{-\frac{2}{n}}k^{\frac{2}{n}}\mu_1\\ &\leq \left(1+\frac{\left(4\Big/\left(1-\frac{k}{400}\right)-2\log k\right)}{n}\right)k^{\frac{2}{n}}\mu_1\\ &= \left(1+\frac{a_3(k)}{n}\right)k^{\frac{2}{n}}\mu_1.\end{aligned}\tag{4.18}$$

AQ: Please check the edit of Table 1.

Table 1. The values of $a_1(k)$, $a_2(k+1)$ and $a_3(k+1)$.

k	$a_1(k)$	$a_2(k+1)$	$a_3(k+1)$
1	≤ 2.31	≤ 2.62	≤ 2.64
2	≤ 2.27	≤ 2.05	≤ 1.84
3	≤ 2.2	≤ 2.00	≤ 1.27
4	≤ 2.12	≤ 1.96	≤ 0.84
5	≤ 2.03	≤ 1.90	≤ 0.48
6	≤ 1.94	≤ 1.84	≤ 0.18
7	≤ 1.86	≤ 1.77	≤ -0.07
8	≤ 1.77	≤ 1.70	≤ -0.30
9	≤ 1.69	≤ 1.63	≤ -0.50
10	≤ 1.61	≤ 1.56	≤ -0.68
11	≤ 1.53	≤ 1.49	≤ -0.84
12	≤ 1.46	≤ 1.42	≤ -0.99
13	≤ 1.39	≤ 1.35	≤ -1.13
14	≤ 1.32	≤ 1.29	≤ -1.26
15	≤ 1.25	≤ 1.22	≤ -1.37
16	≤ 1.18	≤ 1.16	≤ -1.48
17	≤ 1.12	≤ 1.10	≤ -1.59
18	≤ 1.06	≤ 1.04	≤ -1.68
19	≤ 1.00	≤ 0.98	≤ -1.78
20	≤ 0.94	≤ 0.92	≤ -1.86
21	≤ 0.89	≤ 0.87	≤ -1.94
22	≤ 0.83	≤ 0.82	≤ -2.02
23	≤ 0.78	≤ 0.76	≤ -2.10
24	≤ 0.72	≤ 0.71	≤ -2.17
25	≤ 0.67	≤ 0.66	≤ -2.23
26	≤ 0.62	≤ 0.61	≤ -2.30
27	≤ 0.58	≤ 0.57	≤ -2.36
28	≤ 0.53	≤ 0.52	≤ -2.42
29	≤ 0.48	≤ 0.47	≤ -2.47
30	≤ 0.44	≤ 0.43	≤ -2.53
31	≤ 0.39	≤ 0.38	≤ -2.58
32	≤ 0.35	≤ 0.34	≤ -2.63
33	≤ 0.31	≤ 0.30	≤ -2.68
34	≤ 0.27	≤ 0.26	≤ -2.72
35	≤ 0.23	≤ 0.22	≤ -2.77
36	≤ 0.19	≤ 0.18	≤ -2.81
37	≤ 0.15	≤ 0.14	≤ -2.85
38	≤ 0.11	≤ 0.10	≤ -2.89
39	≤ 0.07	≤ 0.07	≤ -2.93
40	≤ 0.03	≤ 0.03	≤ -2.97
41	≤ -0.00	≤ -0.01	≤ -3.00

By Table 1 of the values of $a_1(k)$, $a_2(k+1)$ and $a_3(k+1)$ which are calculated by using Mathematica, we have $a_1(1) \leq a_2(2) \leq a_3(2) = a(1) \leq 2.64$ and, for $k \geq 2$,

$$a_3(k+1) \leq a_2(k+1) \leq a_1(k).$$

Hence, $a(k) = a_1(k)$ for $k \geq 2$. Further, for $k \geq 41$, we know $a(k) < 0$. Hence, for $k \geq 2$, we derive

$$\mu_{k+1} \leq \left(1 + \frac{a(\min\{n, k-1\})}{n}\right) k^{\frac{2}{n}} \mu_1$$

and for $n \geq 41$ and $k \geq 41$, we have

$$\mu_{k+1} \leq k^{\frac{2}{n}} \mu_1.$$

When $k = 1, a(0) = 4$ from (4.4). It is easy to check that, when $k \geq 3$, by a simple calculation,

$$a(k) \leq 2.2 - 4 \log\left(1 + \frac{k-3}{50}\right).$$

This completes the proof of Theorem 1.2. □

5. The Dirichlet Eigenvalue Problem

For a bounded domain Ω with a piecewise smooth boundary $\partial\Omega$ in an n-dimensional complete self-shrinker in $\mathbf{R}^{n+p}$, we consider the following Dirichlet eigenvalue problem of the differential operator $\mathfrak{L}$:

$$\begin{cases} \mathfrak{L}u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

This eigenvalue problem has a real and discrete spectrum:

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty,$$

where each eigenvalue is repeated according to its multiplicity. We have following estimates for eigenvalues of the Dirichlet eigenvalue problem (5.1).

Theorem 5.1. *Let Ω be a bounded domain with a piecewise smooth boundary $\partial\Omega$ in an n-dimensional complete self-shrinker M^n in $\mathbf{R}^{n+p}$. Then, eigenvalues of the Dirichlet eigenvalue problem* (5.1) *satisfy*

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)\left(\lambda_i + \frac{2n - \inf_\Omega |X|^2}{4}\right).$$

Proof. By making use of the same proof as in the proof of Theorem 1.1, we can prove Theorem 5.1 if one notices to count the number of eigenvalues from 1. □

From the recursion formula of [3], we can give an upper bound for eigenvalue λ_{k+1}.

Theorem 5.2. *Let Ω be a bounded domain with a piecewise smooth boundary $\partial\Omega$ in an n-dimensional complete self-shrinker M^n in $\mathbf{R}^{n+p}$. Then, eigenvalues of the*

Dirichlet eigenvalue problem (5.1) *satisfy, for any* $k \geq 1$,

$$\lambda_{k+1} + \frac{2n - \inf_\Omega |X|^2}{4} \leq \left(1 + \frac{a(\min\{n, k-1\})}{n}\right) \times \left(\lambda_1 + \frac{2n - \inf_\Omega |X|^2}{4}\right) k^{2/n},$$

where the bound of $a(m)$ *can be formulated as*:

$$\begin{cases} a(0) \leq 4, \\ a(1) \leq 2.64, \\ a(m) \leq 2.2 - 4\log\left(1 + \frac{1}{50}(m-3)\right), \quad \text{for } m \geq 2. \end{cases}$$

In particular, for $n \geq 41$ *and* $k \geq 41$, *we have*

$$\lambda_{k+1} + \frac{2n - \inf_\Omega |X|^2}{4} \leq \left(\lambda_1 + \frac{2n - \inf_\Omega |X|^2}{4}\right) k^{2/n}.$$

Remark 5.1. For the Euclidean space $\mathbf{R}^n$, the differential operator $\mathfrak{L}$ is called Ornstein–Uhlenbeck operator in stochastic analysis. Since the Euclidean space $\mathbf{R}^n$ is a complete self-shrinker in $\mathbf{R}^{n+1}$, our theorems also give estimates for eigenvalues of the Dirichlet eigenvalue problem of the Ornstein–Uhlenbeck operator.

For the Dirichlet eigenvalue problem (5.1), components x_A's of the position vector X are not eigenfunctions corresponding to the eigenvalue 1 because they do not satisfy the boundary condition. In order to prove Theorem 5.2, we need to obtain the following estimates for lower-order eigenvalues.

Proposition 5.1. *Let* Ω *be a bounded domain with a piecewise smooth boundary* $\partial\Omega$ *in an* n*-dimensional complete self-shrinker* M^n *in* $\mathbf{R}^{n+p}$. *Then, eigenvalues of the Dirichlet eigenvalue problem* (5.1) *satisfy*

$$\sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \leq \left(2n - \inf_\Omega |X|^2\right) + 4\lambda_1.$$

Proof. Let u_j be an eigenfunction corresponding to the eigenvalue λ_j such that

$$\begin{cases} \mathfrak{L} u_j = -\lambda_j u_j & \text{in } \Omega \\ u_j = 0, & \text{on } \partial\Omega \\ \int_\Omega u_i u_j e^{-\frac{|X|^2}{2}} dv = \delta_{ij} & \text{for any } i, j = 1, 2, \ldots. \end{cases} \tag{5.2}$$

We consider an $(n+p) \times (n+p)$-matrix $B = (b_{AB})$ defined by

$$b_{AB} = \int_\Omega x_A u_1 u_{B+1} e^{-\frac{|X|^2}{2}} dv.$$

From the orthogonalization of Gram and Schmidt, there exist an upper triangle matrix $R = (R_{AB})$ and an orthogonal matrix $Q = (q_{AB})$ such that $R = QB$.

Thus,

$$R_{AB} = \sum_{C=1}^{n+p} q_{AC} b_{CB}$$
$$= \int_\Omega \sum_{C=1}^{n+p} q_{AC} x_C u_1 u_{B+1} = 0 \quad \text{for } 1 \leq B < A \leq n+p. \tag{5.3}$$

Defining $y_A = \sum_{C=1}^{n+p} q_{AC} x_C$, we have

$$\int_\Omega y_A u_1 u_{B+1} = \int_\Omega \sum_{C=1}^{n+p} q_{AC} x_C u_1 u_{B+1} = 0 \quad \text{for } 1 \leq B < A \leq n+p. \tag{5.4}$$

Therefore, the functions φ_A defined by

$$\varphi_A = (y_A - a_A) u_1, \quad a_A = \int_\Omega y_A u_1^2 e^{-\frac{|X|^2}{2}} dv \quad \text{for } 1 \leq A \leq n+p$$

satisfy

$$\int_\Omega \varphi_A u_B = 0 \quad \text{for } 1 \leq B \leq A \leq n+p.$$

Therefore, φ_A is a trial function. From the Rayleigh–Ritz inequality, we have, for $1 \leq A \leq n+p$,

$$\lambda_{A+1} \leq \frac{-\int_\Omega \varphi_A \mathfrak{L} \varphi_A e^{-\frac{|X|^2}{2}} dv}{\int_\Omega (\varphi_A)^2 e^{-\frac{|X|^2}{2}} dv}. \tag{5.5}$$

From the definition of φ_A, we derive

$$\begin{aligned}
\mathfrak{L}\varphi_A &= \Delta \varphi_A - \langle X, \nabla \varphi_A \rangle \\
&= \Delta\{(y_A - a_A)u_1\} - \langle X, \nabla\{(y_A - a_A)u_1\}\rangle \\
&= y_A \mathfrak{L} u_1 + u_1 \mathfrak{L} y_A + 2\nabla y_A \cdot \nabla u_1 - a_A \mathfrak{L} u_1 \\
&= -\lambda_1 y_A u_1 - u_1 y_A + 2\nabla y_A \cdot \nabla u_1 + a_A \lambda_1 u_1.
\end{aligned}$$

Thus, (5.5) can be written as

$$(\lambda_{A+1} - \lambda_1)\|\varphi_A\|^2 \leq \int_\Omega (y_A u_1 - 2\nabla y_A \cdot \nabla u_1)\varphi_A e^{-\frac{|X|^2}{2}} dv. \tag{5.6}$$

From the Cauchy–Schwarz inequality, we obtain

$$\left(\int_\Omega (y_A u_1 - 2\nabla y_A \cdot \nabla u_1)\varphi_A e^{-\frac{|X|^2}{2}} dv\right)^2 \leq \|\varphi_A\|^2 \|y_A u_1 - 2\nabla y_A \cdot \nabla u_1\|^2.$$

Multiplying the above inequality by $(\lambda_{A+1} - \lambda_1)$, we infer, from (5.6),

$$(\lambda_{A+1} - \lambda_1)\left(\int_\Omega (y_A u_1 - 2\nabla y_A \cdot \nabla u_1)\varphi_A e^{-\frac{|X|^2}{2}} dv\right)^2$$

$$\leq (\lambda_{A+1} - \lambda_1)\|\varphi_A\|^2 \|y_A u_1 - 2\nabla y_A \cdot \nabla u_1\|^2$$
$$\leq \left(\int_\Omega (y_A u_1 - 2\nabla y_A \cdot \nabla u_1)\varphi_A e^{-\frac{|X|^2}{2}} dv \right) \|y_A u_1 - 2\nabla y_A \cdot \nabla u_1\|^2. \quad (5.7)$$

Hence, we derive

$$(\lambda_{A+1} - \lambda_1) \int_\Omega (y_A u_1 - 2\nabla y_A \cdot \nabla u_1)\varphi_A e^{-\frac{|X|^2}{2}} dv \leq \|y_A u_1 - 2\nabla y_A \cdot \nabla u_1\|^2. \quad (5.8)$$

Since

$$\sum_{A=1}^{n+p} y_A^2 = \sum_{A=1}^{n+p} x_A^2 = |X|^2,$$

we infer

$$\sum_{A=1}^{n+p} \|y_A u_1 - 2\nabla y_A \cdot \nabla u_1\|^2$$
$$= \sum_{A=1}^{n+p} \int_\Omega (y_A^2 u_1^2 - 4y_A u_1 \nabla y_A \cdot \nabla u_1 + 4(\nabla y_A \cdot \nabla u_1)^2) e^{-\frac{|X|^2}{2}} dv$$
$$= \int_\Omega (|X|^2 u_1^2 - \nabla |X|^2 \cdot \nabla u_1^2 + 4\nabla u_1 \cdot \nabla u_1) e^{-\frac{|X|^2}{2}} dv$$
$$= \int_\Omega (|X|^2 u_1^2 + \mathfrak{L}|X|^2 u_1^2 + 4\nabla u_1 \cdot \nabla u_1) e^{-\frac{|X|^2}{2}} dv$$
$$= \int_\Omega (2n - |X|^2) u_1^2 e^{-\frac{|X|^2}{2}} dv + 4\lambda_1 \leq \left(2n - \inf_\Omega |X|^2\right) + 4\lambda_1. \quad (5.9)$$

On the other hand, from the definition of φ_A, we have

$$\int_\Omega (y_A u_1 - 2\nabla y_A \cdot \nabla u_1)\varphi_A e^{-\frac{|X|^2}{2}} dv$$
$$= \int_\Omega (y_A^2 u_1^2 - a_A y_A u_1^2 + 2a_A u_1 \nabla y_A \cdot \nabla u_1 - 2y_A u_1 \nabla y_A \cdot \nabla u_1) e^{-\frac{|X|^2}{2}} dv$$
$$= \int_\Omega \left(y_A^2 u_1^2 - a_A y_A u_1^2 - a_A \mathfrak{L} y_A u_1^2 + \frac{1}{2}\mathfrak{L} y_A^2 u_1^2 \right) e^{-\frac{|X|^2}{2}} dv$$
$$= \int_\Omega \left(y_A^2 u_1^2 + \frac{1}{2}\mathfrak{L} y_A^2 u_1^2 \right) e^{-\frac{|X|^2}{2}} dv$$
$$= \int_\Omega \nabla y_A \cdot \nabla y_A u_1^2 e^{-\frac{|X|^2}{2}} dv. \quad (5.10)$$

For any point p, we choose a new coordinate system $\bar{X} = (\bar{x}_1, \ldots, \bar{x}_{n+p})$ of $\mathbf{R}^{n+p}$ given by $X - X(p) = \bar{X}O$ such that $(\frac{\partial}{\partial \bar{x}_1})_p, \ldots, (\frac{\partial}{\partial \bar{x}_n})_p$ span T_pM^n and at p,

$g(\frac{\partial}{\partial \bar{x}_i}, \frac{\partial}{\partial \bar{x}_j}) = \delta_{ij}$, where $O = (o_{AB}) \in O(n+p)$ is an $(n+p) \times (n+p)$ orthogonal matrix:

$$\begin{aligned}
\nabla y_A \cdot \nabla y_A = g(\nabla y_A, \nabla y_A) &= \sum_{B,C=1}^{n+p} q_{AB} q_{AC} g(\nabla x_B, \nabla x_C) \\
&= \sum_{B,C=1}^{n+p} q_{AB} q_{AC} g\left(\sum_{D=1}^{n+p} o_{DB} \nabla \bar{x}_D, \sum_{E=1}^{n+p} o_{EC} \nabla \bar{x}_E\right) \\
&= \sum_{B,C,D,E=1}^{n+p} q_{AB} o_{DB} q_{AC} o_{EC} g(\nabla \bar{x}_D, \nabla \bar{x}_E) \\
&= \sum_{j=1}^{n} \left(\sum_{B=1}^{n+p} q_{AB} o_{jB}\right)^2 \leq 1.
\end{aligned}$$

Since OQ is an orthogonal matrix if Q and O are orthogonal matrices, that is, we have

$$\nabla y_A \cdot \nabla y_A \leq 1. \tag{5.11}$$

Thus, we obtain, from (5.10) and (5.11),

$$\begin{aligned}
&\sum_{A=1}^{n+p} (\lambda_{A+1} - \lambda_1) \int_\Omega (y_A u_1 - 2\nabla y_A \cdot \nabla u_1) \varphi_A e^{-\frac{|X|^2}{2}} dv \\
&\quad = \sum_{A=1}^{n+p} (\lambda_{A+1} - \lambda_1) \int_\Omega \nabla y_A \cdot \nabla y_A u_1^2 e^{-\frac{|X|^2}{2}} dv \\
&\quad = \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \int_\Omega \nabla y_j \cdot \nabla y_j u_1^2 e^{-\frac{|X|^2}{2}} dv \\
&\qquad + \sum_{A=n+1}^{n+p} (\lambda_{A+1} - \lambda_1) \int_\Omega \nabla y_A \cdot \nabla y_A u_1^2 e^{-\frac{|X|^2}{2}} dv \\
&\quad \geq \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \int_\Omega \nabla y_j \cdot \nabla y_j u_1^2 e^{-\frac{|X|^2}{2}} dv \\
&\qquad + \sum_{A=n+1}^{n+p} (\lambda_{n+1} - \lambda_1) \int_\Omega \nabla y_A \cdot \nabla y_A u_1^2 e^{-\frac{|X|^2}{2}} dv \\
&\quad = \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \int_\Omega \nabla y_j \cdot \nabla y_j u_1^2 e^{-\frac{|X|^2}{2}} dv \\
&\qquad + (\lambda_{n+1} - \lambda_1) \int_\Omega \left(n - \sum_{j=1}^{n} \nabla y_j \cdot \nabla y_j\right) u_1^2 e^{-\frac{|X|^2}{2}} dv
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \int_\Omega \nabla y_j \cdot \nabla y_j u_1^2 e^{-\frac{|X|^2}{2}} dv \\
&\quad + (\lambda_{n+1} - \lambda_1) \int_\Omega \sum_{j=1}^{n} (1 - \nabla y_j \cdot \nabla y_j) u_1^2 e^{-\frac{|X|^2}{2}} dv \\
&\geq \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1). \qquad (5.12)
\end{aligned}
$$

According to (5.8), (5.9) and (5.12), we obtain

$$
\sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \leq \left(2n - \inf_\Omega |X|^2\right) + 4\lambda_1.
$$

This completes the proof of Proposition 5.1. □

Proof of Theorem 5.2. By making use of Proposition 5.1 and the same proof as in the proof of Theorem 1.2, we can prove Theorem 5.2 if one notices to count the number of eigenvalues from 1. □

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AQ: Please update Refs. (1, 15).

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