

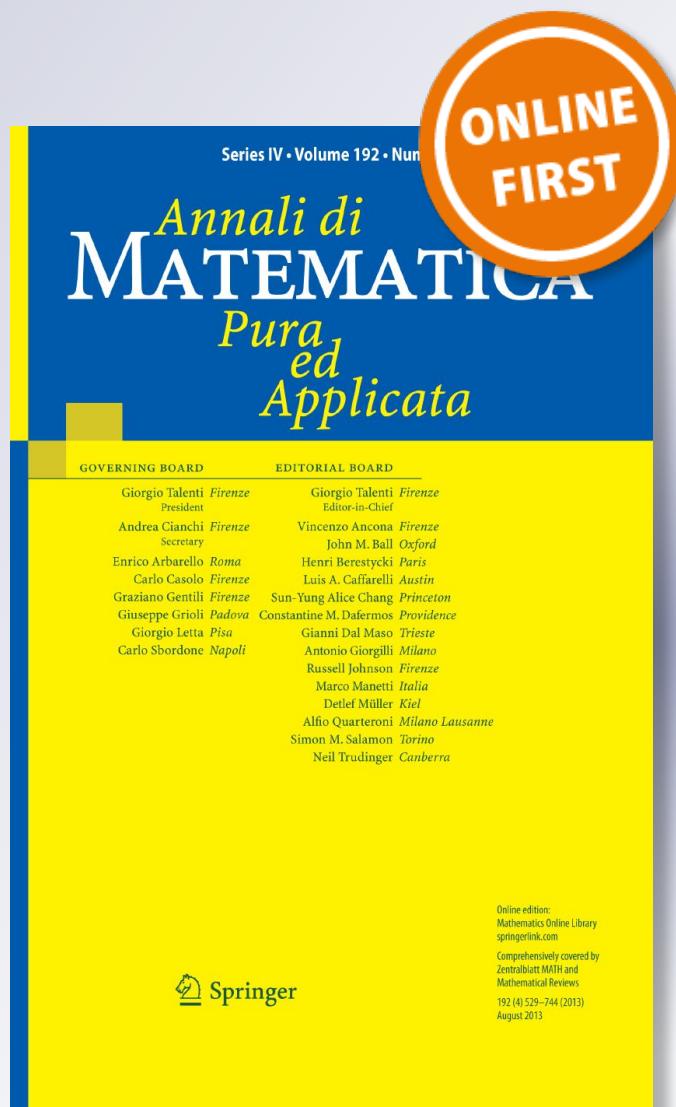
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Inequalities for eigenvalues of the buckling problem of arbitrary order

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Abstract This paper studies eigenvalues of the buckling problem of arbitrary order on bounded domains in Euclidean spaces and spheres. We prove universal bounds for the k th eigenvalue in terms of the lower ones independent of the domains. Our results strengthen the recent work in Jost et al. (*Commun Partial Differ Equ* 35:1563–1589, 2010) and generalize Cheng–Yang’s recent estimates (Cheng and Yang in *Trans Am Math Soc* 364:6139–6158, 2012) on the buckling eigenvalues of order two to arbitrary order.

Keywords Universal inequality for eigenvalues · The buckling problem of arbitrary order · Euclidean space · Sphere

Mathematics Subject Classification 35P15 · 53C20

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1 Introduction

Let Ω be a bounded domain with smooth boundary in an $n(\geq 2)$ -dimensional Riemannian manifold M and denote by Δ the Laplace operator acting on functions on M . Let v be the outward unit normal vector field of $\partial\Omega$, and let us consider the following eigenvalue problems :

$$\Delta u = -\lambda u \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega, \quad (1.1)$$

$$\Delta^2 u = -\Lambda \Delta u \quad \text{in } \Omega, \quad u = \frac{\partial u}{\partial v} = 0, \quad \text{on } \partial\Omega. \quad (1.2)$$

They are called the *fixed membrane problem* and the *buckling problem*, respectively. It should be mentioned that the buckling problem (1.2) has interpretations in physics, that is, it describes the critical buckling load of a clamped plate subjected to a uniform compressive force around its boundary. Let

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots,$$

$$0 < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \dots$$

denote the successive eigenvalues for (1.1) and (1.2), respectively. Here each eigenvalue is repeated according to its multiplicity. An important theme of geometric analysis is to estimate these (and other) eigenvalues. When Ω is a bounded domain in an n -dimensional Euclidean space \mathbf{R}^n , Payne, Pólya and Weinberger (cf. [32,33]) proved the bound

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{kn} \sum_{i=1}^k \lambda_i, \quad k = 1, 2, \dots. \quad (1.3)$$

Inequality of this type is called a universal inequality since it does not depend on Ω .

On the other hand, Payne, Pólya and Weinberger also studied eigenvalues of the buckling problem (1.2) for bounded domains in \mathbf{R}^n and proved (cf. [32,33])

$$\Lambda_2/\Lambda_1 < 3 \quad \text{for } \Omega \subset \mathbf{R}^2.$$

For $\Omega \subset \mathbf{R}^n$, this reads

$$\Lambda_2/\Lambda_1 < 1 + 4/n.$$

Furthermore, Payne, Pólya and Weinberger proposed the following

Problem 1 (cf. [32,33]). *Can one obtain a universal inequality for the eigenvalues of the buckling problem (1.2) on a bounded domain in \mathbf{R}^n which is similar to the universal inequality (1.3) for the eigenvalues of the fixed membrane problem (1.1)?*

With respect to the above problem, Hile and Yeh [27] obtained

$$\frac{\Lambda_2}{\Lambda_1} \leq \frac{n^2 + 8n + 20}{(n + 2)^2} \quad \text{for } \Omega \subset \mathbf{R}^n.$$

Ashbaugh [2] proved :

$$\sum_{i=1}^n \Lambda_{i+1} \leq (n + 4)\Lambda_1. \quad (1.4)$$

This inequality has been improved to the following form in [30]:

$$\sum_{i=1}^n \Lambda_{i+1} + \frac{4(\Lambda_2 - \Lambda_1)}{n+4} \leq (n+4)\Lambda_1. \quad (1.5)$$

By introducing a new method of constructing trial functions, Cheng and Yang [12] have obtained the following universal inequality:

$$\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4(n+2)}{n^2} \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i. \quad (1.6)$$

Thus, the problem proposed by Payne, Pólya and Weinberger has been solved affirmatively. By making use of the asymptotic formula of Weyl for eigenvalues of the Dirichlet eigenvalue problem of the Laplacian and one of Agmon [1] and Pleijel [34] for eigenvalues of the clamped plate problem, we can have the asymptotic formula of eigenvalues for the buckling problem according to the variational characterization for eigenvalues of the buckling problem:

$$\Lambda_k \sim \frac{4\pi^2}{(\omega_n \text{vol}(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \rightarrow \infty,$$

where ω_n denotes the volume of the unit ball in \mathbf{R}^n . By the results of Li and Yau [31] and the variational characterization for eigenvalues, one can obtain a lower bound for eigenvalues of the buckling problem:

$$\frac{1}{k} \sum_{j=1}^k \Lambda_j \geq \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \text{vol}(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}.$$

On the other hand, by making use of the recursion formula of Cheng and Yang [14], one can obtain an upper bound for eigenvalues of the buckling problem, which is sharp in the sense of the order of k , if one can get a sharp universal inequality for eigenvalues of the buckling problem as the following:

Conjecture. Eigenvalues of the buckling problem on a bounded domain in a Euclidean space \mathbf{R}^n satisfy the following universal inequality:

$$\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i, \quad (1.7)$$

which is proposed by Cheng and Yang [14]. Therefore, the next landmark goal for the study on eigenvalues of the buckling problem will be to prove the above sharp universal inequality. Recently, Cheng and Yang [17] have made an important breakthrough for it. They have obtained

$$\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4(n + \frac{4}{3})}{n^2} \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i. \quad (1.8)$$

In this paper, we will investigate eigenvalues of the buckling problem of arbitrary order:

$$\begin{cases} (-\Delta)^l u = -\Lambda \Delta u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial v} = \cdots = \frac{\partial^{l-1} u}{\partial v^{l-1}} = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.9)$$

where Ω is a bounded domain in a Euclidean space and l is any integer no less than 2. Yang type inequalities for eigenvalues of the problem (1.9) have been obtained recently in [29]. We conjecture that the following sharp universal inequality:

$$\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{2l}{n} \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i \quad (1.10)$$

holds for eigenvalues of the problem (1.9). The main purpose of this paper is to attack the above problem. We prove the following:

Theorem 1.1 *Let Λ_i be the i th eigenvalue of the buckling problem (1.9), where Ω is a bounded domain with smooth boundary in \mathbf{R}^n . Then for any positive non-increasing monotone sequence $\{\delta_i\}_{i=1}^k$, we have*

$$\begin{aligned} & n \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \\ & \leq \sum_{i=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \left(2l^2 + \left(n - \frac{14}{3} \right) l + \frac{8}{3} - n \right) \Lambda_i^{(l-2)/(l-1)} \\ & \quad + \sum_{i=1}^k \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \Lambda_i^{1/(l-1)}. \end{aligned} \quad (1.11)$$

Remark 1.1 Taking

$$\delta_1 = \delta_2 = \dots = \delta_k = \left\{ \frac{\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i^{1/(l-1)}}{(2l^2 + (n - \frac{14}{3}) l + \frac{8}{3} - n) \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \Lambda_i^{(l-2)/(l-1)}} \right\}^{1/2}$$

in (1.11), we have

$$\begin{aligned} & \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{2(2l^2 + (n - \frac{14}{3}) l + \frac{8}{3} - n)^{1/2}}{n} \\ & \quad \times \left\{ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \Lambda_i^{(l-2)/(l-1)} \right\}^{1/2} \left\{ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i^{1/(l-1)} \right\}^{1/2}, \end{aligned} \quad (1.12)$$

which improves the inequality (1.13) in [29]. From (1.12), we can obtain a quadratic inequality about $\Lambda_1, \dots, \Lambda_{k+1}$.

Corollary 1.1 *For any $k \geq 1$, the first $k+1$ eigenvalues of the buckling problem (1.9) with $\Omega \subset \mathbf{R}^n$ satisfy the following inequality*

$$\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4(2l^2 + (n - \frac{14}{3}) l + \frac{8}{3} - n)}{n^2} \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i. \quad (1.13)$$

Remark 1.2 When $l = 2$, (1.13) becomes Cheng–Yang’s inequality (1.8).

Furthermore, we prove the following universal inequality for eigenvalues of the buckling problem of arbitrary order on spherical domains.

Theorem 1.2 Let $l \geq 2$ and let Λ_i be the i th eigenvalue of the buckling problem:

$$\begin{cases} (-\Delta)^l u = -\Lambda \Delta u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial v} = \cdots = \frac{\partial^{l-1} u}{\partial v^{l-1}} = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.14)$$

where Ω is a domain with smooth boundary in S^n . For each $q = 1, \dots$, define the polynomials Φ_q inductively by

$$\Phi_1(t) = t - 1, \quad \Phi_2(t) = t^2 - (n + 5)t - (n - 2), \quad (1.15)$$

$$\Phi_q(t) = (2t - 2)\Phi_{q-1}(t) - (t^2 + 2t - n(n - 2))\Phi_{q-2}(t), \quad q = 3, \dots \quad (1.16)$$

Set

$$\Phi_{l-1}(t) = t^{l-1} - a_{l-2}t^{l-2} + \cdots + (-1)^{l-2}a_1t - (n - 2)^{l-2}. \quad (1.17)$$

Then for any positive integer k and any positive non-increasing monotone sequence $\{\delta_i\}_{i=1}^k$, we have

$$\begin{aligned} & \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \left(2 + \frac{n-2}{\Lambda_i^{1/(l-1)} - (n-2)} \right) \\ & \leq \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \delta_i S_i + \sum_{i=1}^k \frac{(\Lambda_{k+1} - \Lambda_i)}{\delta_i} \left(\Lambda_i^{1/(l-1)} + \frac{(n-2)^2}{4} \right), \end{aligned} \quad (1.18)$$

where

$$S_i = \Lambda_i \left(1 - \frac{1}{\Lambda_i^{1/(l-1)} - (n-2)} \right) + (-1)^l (n-2)^{l-2} + \sum_{j=1}^{l-2} a_j^+ \Lambda_i^{j/(l-1)}, \quad (1.19)$$

with $a_j^+ = \max\{a_j, 0\}$ and when $l = 2$ we use the convention that $\sum_{j=1}^{l-2} a_j^+ \Lambda_i^{j/(l-1)} = 0$.

Remark 1.3 When $l = 2$, the inequality (1.18) is stronger than one of Cheng and Yang in [17].

Remark 1.4 Universal inequalities for eigenvalues of various elliptic operators have been studied extensively in recent years. For the developments in this direction, we refer to [1–30, 33–42] and the references therein.

2 Proofs of the results

First we recall a method of constructing trial functions developed by Cheng–Yang (cf. [12, 29, 37]). Let M be an n -dimensional complete submanifold in \mathbf{R}^m . Denote by $\langle \cdot, \cdot \rangle$ the canonical metric on \mathbf{R}^m as well as that induced on M . Denote by Δ and ∇ the Laplacian and the gradient operator of M , respectively. Let Ω be a bounded domain with smooth boundary in M and let v be the outward unit normal vector field of $\partial\Omega$. For functions $f, g \in W_0^{1,2}(\Omega)$ the *Dirichlet inner product* $(f, g)_D$ of f and g is given by

$$(f, g)_D = \int_{\Omega} \langle \nabla f, \nabla g \rangle.$$

The Dirichlet norm of a function f is defined by

$$\|f\|_D = \{(f, f)_D\}^{1/2} = \left(\int_{\Omega} |\nabla f|^2 \right)^{1/2}.$$

Consider the eigenvalue problem

$$\begin{cases} (-\Delta)^l u = -\Lambda \Delta u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial v} = \dots = \frac{\partial^{l-1} u}{\partial v^{l-1}} = 0, & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

Let

$$0 < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \dots$$

denote the successive eigenvalues, where each eigenvalue is repeated according to its multiplicity.

Let u_i be the i th orthonormal eigenfunction of the problem (2.1) corresponding to the eigenvalue Λ_i , $i = 1, 2, \dots$, that is,

$$\begin{cases} (-\Delta)^l u_i = -\Lambda_i \Delta u_i, & \text{in } \Omega, \\ u_i = \frac{\partial u_i}{\partial v} = \dots = \frac{\partial^{l-1} u_i}{\partial v^{l-1}} = 0, & \text{on } \partial\Omega, \\ (u_i, u_j)_D = \int_{\Omega} \langle \nabla u_i, \nabla u_j \rangle = \delta_{ij}, & \forall i, j. \end{cases} \quad (2.2)$$

Consider the subspace $W_{0,l}^{1,2}(\Omega)$ of $W_0^{1,2}(\Omega)$ defined by

$$W_{0,l}^{1,2}(\Omega) = \left\{ f \in W_0^{1,2}(\Omega) : f|_{\partial\Omega} = \frac{\partial f}{\partial v} \Big|_{\partial\Omega} = \dots = \frac{\partial^{l-1} f}{\partial v^{l-1}} \Big|_{\partial\Omega} = 0 \right\}. \quad (2.3)$$

The eigenfunctions $\{u_i\}_{i=1}^{\infty}$ defined in (2.2) form a complete orthonormal basis for the Hilbert space $W_{0,l}^{1,2}(\Omega)$. If $\phi \in W_{0,l}^2(\Omega)$ satisfies $(\phi, u_j)_D = 0$, $\forall j = 1, 2, \dots, k$, then the Rayleigh–Ritz inequality tells us that

$$\Lambda_{k+1} \|\phi\|_D^2 \leq \int_{\Omega} \phi (-\Delta)^l \phi. \quad (2.4)$$

For vector-valued functions $F = (f_1, f_2, \dots, f_m)$, $G = (g_1, g_2, \dots, g_m) : \Omega \rightarrow \mathbf{R}^m$, we define an inner product (F, G) by

$$(F, G) \equiv \int_{\Omega} \langle F, G \rangle = \int_{\Omega} \sum_{\alpha=1}^m f_{\alpha} g_{\alpha}. \quad (2.5)$$

The norm of F is given by

$$\|F\| = (F, F)^{1/2} = \left\{ \int_{\Omega} \sum_{\alpha=1}^m f_{\alpha}^2 \right\}^{1/2}.$$

Let

$$\mathbf{L}^2(\Omega) = \{F : \Omega \rightarrow \mathbf{R}^m, \|F\| < +\infty\}.$$

Observe that a vector field on Ω can be regarded as a vector-valued function from Ω to \mathbf{R}^m . Let $\mathbf{L}_{0,1}^2(\Omega) \subset \mathbf{L}^2(\Omega)$ be the subspace of $\mathbf{L}^2(\Omega)$ spanned by the vector-valued functions $\{\nabla u_i\}_{i=1}^{\infty}$, which form a complete orthonormal basis for the Hilbert space $\mathbf{L}_{0,1}^2(\Omega)$. For any $f \in W_{0,l}^2(\Omega)$, we have $\nabla f \in \mathbf{L}_{0,1}^2(\Omega)$, and for any $X \in \mathbf{L}_{0,1}^2(\Omega)$, there exists a function $f \in W_{0,l}^2(\Omega)$ such that $X = \nabla f$.

Lemma 2.1 (cf. [29,30]) Let u_i and Λ_i , $i = 1, 2, \dots$, be as in (2.2), then

$$0 \leq \int_{\Omega} u_i (-\Delta)^k u_i \leq \Lambda_i^{(k-1)/(l-1)}, \quad k = 1, \dots, l-1. \quad (2.6)$$

We are now ready to prove the main results in this paper.

Proof of Theorem 1.1. With the notations as above, we consider now the special case that Ω is a bounded domain in \mathbf{R}^n . Denote by x_1, \dots, x_n the coordinate functions of \mathbf{R}^n and let us decompose the vector-valued functions $x_\alpha \nabla u_i$ as

$$x_\alpha \nabla u_i = \nabla h_{\alpha i} + W_{\alpha i}, \quad (2.7)$$

where $h_{\alpha i} \in W_{0,l}^{1,2}(\Omega)$, $\nabla h_{\alpha i}$ is the projection of $x_\alpha \nabla u_i$ in $\mathbf{L}_{0,1}^2(\Omega)$ and $W_{\alpha i} \perp \mathbf{L}_{0,1}^2(\Omega)$. Thus we have

$$W_{\alpha i}|_{\partial\Omega} = 0, \quad \text{and} \quad (W_{\alpha i}, \nabla u) = \int_{\Omega} \langle W_{\alpha i}, \nabla u \rangle = 0, \quad \text{for any } u \in W_{0,l}^2(\Omega) \quad (2.8)$$

and from the discussions in [12] and [37] we know that

$$\operatorname{div} W_{\alpha i} = 0, \quad (2.9)$$

where for a vector field Z on Ω , $\operatorname{div} Z$ denotes the divergence of Z . \square

For each $\alpha = 1, \dots, n$, $i = 1, \dots, k$, consider the functions $\phi_{\alpha i} : \Omega \rightarrow \mathbf{R}$, given by

$$\phi_{\alpha i} = h_{\alpha i} - \sum_{j=1}^k a_{\alpha ij} u_j, \quad (2.10)$$

where

$$a_{\alpha ij} = \int_{\Omega} x_\alpha \langle \nabla u_i, \nabla u_j \rangle = a_{\alpha ji}. \quad (2.11)$$

We have

$$\phi_{\alpha i}|_{\partial\Omega} = \frac{\partial \phi_{\alpha i}}{\partial \nu} \Big|_{\partial\Omega} = \dots = \frac{\partial^{l-1} \phi_{\alpha i}}{\partial \nu^{l-1}} \Big|_{\partial\Omega} = 0, \quad (2.12)$$

$$(\phi_{\alpha i}, u_j)_D = \int_{\Omega} \langle \nabla \phi_{\alpha i}, \nabla u_j \rangle = 0, \quad \forall j = 1, \dots, k. \quad (2.13)$$

It then follows from the Rayleigh–Ritz inequality for Λ_{k+1} that

$$\Lambda_{k+1} \int_{\Omega} |\nabla \phi_{\alpha i}|^2 \leq \int_{\Omega} \phi_{\alpha i} (-\Delta)^l \phi_{\alpha i}, \quad \forall \alpha = 1, \dots, n, \quad i = 1, \dots, k. \quad (2.14)$$

After some calculations, we have (cf. (2.36) in [29])

$$\begin{aligned} \int_{\Omega} \phi_{\alpha i} (-\Delta)^l \phi_{\alpha i} &= \int_{\Omega} (-1)^l \left\{ (-l+1) u_i \Delta^{l-1} u_i + (2l^2 - 4l + 3) (\Delta^{l-2} u_i)_{,\alpha} u_{i,\alpha} \right\} \\ &\quad + \Lambda_i \left\{ \int_{\Omega} x_\alpha^2 |\nabla u_i|^2 - \int_{\Omega} u_i^2 \right\} - \sum_{j=1}^k \Lambda_j a_{\alpha ij}^2. \end{aligned} \quad (2.15)$$

It is easy to see that

$$||x_\alpha \nabla u_i||^2 = ||\nabla h_{\alpha i}||^2 + ||W_{\alpha i}||^2, \quad ||\nabla h_{\alpha i}||^2 = ||\nabla \phi_{\alpha i}||^2 + \sum_{j=1}^k a_{\alpha ij}^2, \quad (2.16)$$

where for a vector field Z on Ω , $\|Z\|^2 = \int_{\Omega} |Z|^2$. Combining (2.14)–(2.16), we infer

$$\begin{aligned} & (\Lambda_{k+1} - \Lambda_i) \|\nabla \phi_{\alpha i}\|^2 \\ & \leq \int_{\Omega} (-1)^l \left\{ (-l+1)u_i \Delta^{l-1} u_i + (2l^2 - 4l + 3)(\Delta^{l-2} u_i)_{,\alpha} u_{i,\alpha} \right\} \\ & \quad - \Lambda_i (\|u_i\|^2 - \|W_{\alpha i}\|^2) + \sum_{j=1}^k (\Lambda_i - \Lambda_j) a_{\alpha i j}^2. \end{aligned} \quad (2.17)$$

Observe that $\nabla(x_{\alpha} u_i) = u_i \nabla x_{\alpha} + x_{\alpha} \nabla u_i \in \mathbf{L}_{0,1}^2(\Omega)$. Set $y_{\alpha i} = x_{\alpha} u_i - h_{\alpha i}$; then

$$u_i \nabla x_{\alpha} = \nabla y_{\alpha i} - W_{\alpha i}.$$

and so

$$\|u_i\|^2 = \|u_i \nabla x_{\alpha}\|^2 = \|W_{\alpha i}\|^2 + \|\nabla y_{\alpha i}\|^2. \quad (2.18)$$

Substituting (2.18) into (2.17), we get

$$\begin{aligned} & (\Lambda_{k+1} - \Lambda_i) \|\nabla \phi_{\alpha i}\|^2 \\ & \leq \int_{\Omega} (-1)^l \left\{ (-l+1)u_i \Delta^{l-1} u_i + (2l^2 - 4l + 3)(\Delta^{l-2} u_i)_{,\alpha} u_{i,\alpha} \right\} \\ & \quad - \Lambda_i \|\nabla y_{\alpha i}\|^2 + \sum_{j=1}^k (\Lambda_i - \Lambda_j) a_{\alpha i j}^2. \end{aligned}$$

Summing on α from 1 to n , we have

$$\begin{aligned} & (\Lambda_{k+1} - \Lambda_i) \sum_{\alpha=1}^n \|\nabla \phi_{\alpha i}\|^2 \\ & \leq \int_{\Omega} (-1)^l \left\{ n(-l+1)u_i \Delta^{l-1} u_i + (2l^2 - 4l + 3)\langle \nabla(\Delta^{l-2} u_i), \nabla u_i \rangle \right\} \\ & \quad - \Lambda_i \sum_{\alpha=1}^n \|\nabla y_{\alpha i}\|^2 + \sum_{\alpha=1}^n \sum_{j=1}^k (\Lambda_i - \Lambda_j) a_{\alpha i j}^2 \\ & = (2l^2 + (n-4)l + 3 - n) \int_{\Omega} u_i (-\Delta)^{l-1} u_i - \Lambda_i \sum_{\alpha=1}^n \|\nabla y_{\alpha i}\|^2 + \sum_{\alpha=1}^n \sum_{j=1}^k (\Lambda_i - \Lambda_j) a_{\alpha i j}^2. \end{aligned} \quad (2.19)$$

Using the divergence theorem, one can show that (cf. [12,29])

$$-2 \int_{\Omega} x_{\alpha} \langle \nabla u_i, \nabla \langle \nabla u_i, \nabla x_{\alpha} \rangle \rangle = 1. \quad (2.20)$$

Set

$$d_{\alpha i j} = \int_{\Omega} \langle \nabla \langle \nabla u_i, \nabla x_{\alpha} \rangle, \nabla u_j \rangle,$$

then $d_{\alpha i j} = -d_{\alpha j i}$ and we have from (2.7), (2.8), (2.10) and (2.20) that

$$\begin{aligned} 1 &= -2 \int_{\Omega} \langle \nabla h_{\alpha i}, \nabla \langle \nabla u_i, \nabla x_{\alpha} \rangle \rangle \\ &= -2 \int_{\Omega} \langle \nabla \phi_{\alpha i}, \nabla \langle \nabla u_i, \nabla x_{\alpha} \rangle \rangle - 2 \sum_{j=1}^k a_{\alpha i j} d_{\alpha i j}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &(\Lambda_{k+1} - \Lambda_i)^2 \left(1 + 2 \sum_{j=1}^k a_{\alpha i j} d_{\alpha i j} \right) \\ &= (\Lambda_{k+1} - \Lambda_i)^2 \int_{\Omega} (-2) \left\langle \nabla \phi_{\alpha i}, \nabla u_{i,\alpha} - \sum_{j=1}^k d_{\alpha i j} \nabla u_j \right\rangle \\ &\leq \delta_i (\Lambda_{k+1} - \Lambda_i)^3 \|\nabla \phi_{\alpha i}\|^2 + \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \left(\|\nabla u_{i,\alpha}\|^2 - \sum_{j=1}^k d_{\alpha i j}^2 \right), \quad (2.21) \end{aligned}$$

where $u_{i,\alpha} = \langle \nabla u_i, \nabla x_{\alpha} \rangle$. Summing on α from 1 to n , we have by using (2.19) that

$$\begin{aligned} &(\Lambda_{k+1} - \Lambda_i)^2 \left(n + 2 \sum_{\alpha=1}^n \sum_{j=1}^k a_{\alpha i j} d_{\alpha i j} \right) \\ &\leq \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \left((2l^2 + (n-4)l + 3 - n) \int_{\Omega} u_i (-\Delta)^{l-1} u_i - \Lambda_i \sum_{\alpha=1}^n \|\nabla y_{\alpha i}\|^2 \right. \\ &\quad \left. + \sum_{\alpha=1}^n \sum_{j=1}^k (\Lambda_i - \Lambda_j) a_{\alpha i j}^2 \right) + \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \\ &\quad \times \left(\sum_{\alpha=1}^n \|\nabla u_{i,\alpha}\|^2 - \sum_{\alpha=1}^n \sum_{j=1}^k d_{\alpha i j}^2 \right). \end{aligned}$$

Summing on i from 1 to k and noticing the fact that $a_{\alpha i j} = a_{\alpha j i}$, $d_{\alpha i j} = -d_{\alpha j i}$, one gets

$$\begin{aligned} &n \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 - 2 \sum_{\alpha=1}^n \sum_{i,j=1}^k (\Lambda_{k+1} - \Lambda_i)(\Lambda_i - \Lambda_j) a_{\alpha i j} d_{\alpha i j} \\ &\leq \sum_{i=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \left((2l^2 + (n-4)l + 3 - n) \int_{\Omega} u_i (-\Delta)^{l-1} u_i - \Lambda_i \sum_{\alpha=1}^n \|\nabla y_{\alpha i}\|^2 \right) \\ &\quad - \sum_{\alpha=1}^n \sum_{i,j=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)(\Lambda_i - \Lambda_j)^2 a_{\alpha i j}^2 - \sum_{\alpha=1}^n \sum_{i,j=1}^k \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) d_{\alpha i j}^2 \\ &\quad + \sum_{i=1}^n \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \sum_{\alpha=1}^n \|\nabla u_{i,\alpha}\|^2 \\ &\quad + \sum_{\alpha=1}^n \sum_{i,j=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)(\Lambda_i - \Lambda_j)^2 a_{\alpha i j}^2 + \sum_{\alpha=1}^n \sum_{i,j=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)^2 (\Lambda_i - \Lambda_j) a_{\alpha i j}^2. \quad (2.22) \end{aligned}$$

Since $\{\delta_i\}_{i=1}^k$ is a non-increasing monotone sequence, we have

$$\begin{aligned} & \sum_{\alpha=1}^n \sum_{i,j=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i) (\Lambda_i - \Lambda_j)^2 a_{\alpha ij}^2 + \sum_{\alpha=1}^n \sum_{i,j=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)^2 (\Lambda_i - \Lambda_j) a_{\alpha ij}^2 \\ &= \frac{1}{2} \sum_{\alpha=1}^n \sum_{i,j=1}^k (\Lambda_{k+1} - \Lambda_i) (\Lambda_{k+1} - \Lambda_j) (\Lambda_i - \Lambda_j) (\delta_i - \delta_j) a_{\alpha ij}^2 \leq 0. \end{aligned}$$

We conclude from (2.22) that

$$\begin{aligned} & n \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \\ & \leq \sum_{i=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \left((2l^2 + (n-4)l + 3 - n) \int_{\Omega} u_i (-\Delta)^{l-1} u_i - \Lambda_i \sum_{\alpha=1}^n \|\nabla y_{\alpha i}\|^2 \right) \\ & \quad + \sum_{i=1}^n \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \sum_{\alpha=1}^n \|\nabla u_{i,\alpha}\|^2. \end{aligned} \tag{2.23}$$

It follows from the divergence theorem and Lemma 2.1 that

$$\begin{aligned} \sum_{\alpha=1}^k \|\nabla u_{i,\alpha}\|^2 &= - \int_{\Omega} \sum_{\alpha=1}^k u_{i,\alpha} \Delta u_{i,\alpha} \\ &= - \int_{\Omega} \sum_{\alpha=1}^k u_{i,\alpha} (\Delta u_i)_{,\alpha} \\ &= \int_{\Omega} \sum_{\alpha=1}^k u_{i,\alpha\alpha} \Delta u_i \\ &= \int_{\Omega} (\Delta u_i)^2 \\ &= \int_{\Omega} u_i \Delta^2 u_i \\ &\leq \Lambda_i^{1/(l-1)}, \end{aligned}$$

where $u_{i,\alpha\alpha} = \frac{\partial^2 u_i}{\partial x_{\alpha}^2}$. Thus, we have

$$\begin{aligned} & n \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \\ & \leq \sum_{i=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \left((2l^2 + (n-4)l + 3 - n) \int_{\Omega} u_i (-\Delta)^{l-1} u_i - \sum_{\alpha=1}^n \Lambda_i \|\nabla y_{\alpha i}\|^2 \right) \\ & \quad + \sum_{i=1}^n \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \Lambda_i^{1/(l-1)}. \end{aligned} \tag{2.24}$$

Before we can finish the proof of Theorem 1.2, we shall need two lemmas.

Lemma 2.2 For any i , we have

$$(n - 2l - 2) \int_{\Omega} u_i (-\Delta)^{l-1} u_i = n \Lambda_i \|u_i\|^2 - 4 \Lambda_i \sum_{\alpha=1}^n \|\nabla y_{\alpha i}\|^2. \quad (2.25)$$

Proof When $l = 2$, the above formula has been proved by Cheng and Yang [17]. We only consider the case that $l > 2$. In this case, we conclude from the boundary condition on u_i that $y_{\alpha i}|_{\partial\Omega} = \nabla y_{\alpha i}|_{\partial\Omega} = \Delta y_{\alpha i}|_{\partial\Omega} = 0$. Using the divergence theorem, we have

$$\begin{aligned} & \int_{\Omega} x_{\alpha} u_i \left\langle \nabla x_{\alpha}, \nabla (\Delta^{l-1} u_i) \right\rangle \\ &= \int_{\Omega} x_{\alpha} u_i \Delta^{l-1} \langle \nabla x_{\alpha}, \nabla u_i \rangle \\ &= \int_{\Omega} \Delta^{l-1} (x_{\alpha} u_i) \langle \nabla x_{\alpha}, \nabla u_i \rangle \\ &= - \int_{\Omega} \left\langle u_i \nabla x_{\alpha}, \nabla (\Delta^{l-1} (x_{\alpha} u_i)) \right\rangle \\ &= - \int_{\Omega} \left\langle \nabla y_{\alpha i}, \nabla (\Delta^{l-1} (x_{\alpha} u_i)) \right\rangle \\ &= \int_{\Omega} y_{\alpha i} \Delta^l (x_{\alpha} u_i) \\ &= \int_{\Omega} y_{\alpha i} \left(2l \left\langle \nabla (\Delta^{l-1} u_i), \nabla x_{\alpha} \right\rangle + x_{\alpha} \Delta^l u_i \right) \\ &= -2l \int_{\Omega} \Delta^{l-1} u_i \langle \nabla y_{\alpha i}, \nabla x_{\alpha} \rangle + \Lambda_i (-1)^{l-1} \int_{\Omega} y_{\alpha i} x_{\alpha} \Delta u_i, \end{aligned} \quad (2.26)$$

$$\begin{aligned} & \int_{\Omega} y_{\alpha i} x_{\alpha} \Delta u_i \\ &= - \int_{\Omega} \langle \nabla y_{\alpha i}, x_{\alpha} \nabla u_i \rangle - \int_{\Omega} y_{\alpha i} \langle \nabla x_{\alpha}, \nabla u_i \rangle \\ &= - \int_{\Omega} \langle \nabla y_{\alpha i}, x_{\alpha} \nabla u_i \rangle + \int_{\Omega} \langle \nabla y_{\alpha i}, u_i \nabla x_{\alpha} \rangle \\ &= - \int_{\Omega} \langle \nabla y_{\alpha i}, x_{\alpha} \nabla u_i \rangle + \|\nabla y_{\alpha i}\|^2, \end{aligned} \quad (2.27)$$

$$\begin{aligned} & \int_{\Omega} \langle \nabla y_{\alpha i}, x_{\alpha} \nabla u_i \rangle \\ &= \int_{\Omega} \langle \nabla y_{\alpha i}, \nabla h_{\alpha i} \rangle \\ &= \int_{\Omega} \langle \nabla y_{\alpha i}, \nabla (x_{\alpha} u_i) - \nabla y_{\alpha i} \rangle \\ &= \int_{\Omega} \langle \nabla y_{\alpha i}, \nabla (x_{\alpha} u_i) \rangle - \|\nabla y_{\alpha i}\|^2 \\ &= \int_{\Omega} \langle u_i \nabla x_{\alpha}, \nabla (x_{\alpha} u_i) \rangle - \|\nabla y_{\alpha i}\|^2 \end{aligned}$$

$$\begin{aligned}
&= ||u_i||^2 + \int_{\Omega} \langle u_i \nabla x_{\alpha}, x_{\alpha} \nabla u_i \rangle - ||\nabla y_{\alpha i}||^2 \\
&= ||u_i||^2 - \frac{1}{4} \int_{\Omega} u_i^2 \Delta x_{\alpha}^2 - ||\nabla y_{\alpha i}||^2 \\
&= \frac{1}{2} ||u_i||^2 - ||\nabla y_{\alpha i}||^2
\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}
&\int_{\Omega} \Delta^{l-1} u_i \langle \nabla y_{\alpha i}, \nabla x_{\alpha} \rangle \\
&= - \int_{\Omega} y_{\alpha i} \left\langle \nabla \left(\Delta^{l-1} u_i \right), \nabla x_{\alpha} \right\rangle \\
&= - \int_{\Omega} y_{\alpha i} \Delta \left\langle \nabla \left(\Delta^{l-2} u_i \right), \nabla x_{\alpha} \right\rangle \\
&= \int_{\Omega} \left\langle \nabla y_{\alpha i}, \nabla \left\langle \nabla \left(\Delta^{l-2} u_i \right), \nabla x_{\alpha} \right\rangle \right\rangle \\
&= \int_{\Omega} \left\langle u_i \nabla x_{\alpha}, \nabla \left\langle \nabla \left(\Delta^{l-2} u_i \right), \nabla x_{\alpha} \right\rangle \right\rangle \\
&= - \int_{\Omega} \left\langle \nabla x_{\alpha}, \nabla \left(\Delta^{l-2} u_i \right) \right\rangle \langle \nabla u_i, \nabla x_{\alpha} \rangle.
\end{aligned} \tag{2.29}$$

It follows from (2.26)–(2.29) that

$$\begin{aligned}
&\int_{\Omega} x_{\alpha} u_i \left\langle \nabla x_{\alpha}, \nabla \left(\Delta^{l-1} u_i \right) \right\rangle \\
&= 2l \int_{\Omega} \left\langle \nabla x_{\alpha}, \nabla \left(\Delta^{l-2} u_i \right) \right\rangle \langle \nabla u_i, \nabla x_{\alpha} \rangle + (-1)^{l-1} \Lambda_i \left(-\frac{1}{2} ||u_i||^2 + 2 ||\nabla y_{\alpha i}||^2 \right).
\end{aligned} \tag{2.30}$$

Since

$$\Delta^{l-1}(x_{\alpha} u_i) = 2(l-1) \left\langle \nabla \left(\Delta^{l-2} u_i \right), \nabla x_{\alpha} \right\rangle + x_{\alpha} \Delta^{l-1} u_i,$$

we get

$$\begin{aligned}
&\int_{\Omega} x_{\alpha} u_i \left\langle \nabla x_{\alpha}, \nabla \left(\Delta^{l-1} u_i \right) \right\rangle \\
&= \int_{\Omega} x_{\alpha} u_i \Delta^{l-1} \langle \nabla u_i, \nabla x_{\alpha} \rangle \\
&= \int_{\Omega} \Delta^{l-1}(x_{\alpha} u_i) \langle \nabla u_i, \nabla x_{\alpha} \rangle \\
&= \int_{\Omega} \left(2(l-1) \left\langle \nabla \left(\Delta^{l-2} u_i \right), \nabla x_{\alpha} \right\rangle + x_{\alpha} \Delta^{l-1} u_i \right) \langle \nabla u_i, \nabla x_{\alpha} \rangle.
\end{aligned} \tag{2.31}$$

On the other hand, we have

$$\int_{\Omega} x_{\alpha} u_i \left\langle \nabla x_{\alpha}, \nabla \left(\Delta^{l-1} u_i \right) \right\rangle = - \int_{\Omega} \Delta^{l-1} u_i (u_i + x_{\alpha} \langle \nabla u_i, \nabla x_{\alpha} \rangle). \tag{2.32}$$

We obtain from (2.31) and (2.32) that

$$\begin{aligned} & \int_{\Omega} x_{\alpha} u_i \left\langle \nabla x_{\alpha}, \nabla (\Delta^{l-1} u_i) \right\rangle \\ &= \int_M \left\{ (l-1) \left\langle \nabla (\Delta^{l-2} u_i), \nabla x_{\alpha} \right\rangle \langle \nabla u_i, \nabla x_{\alpha} \rangle - \frac{1}{2} u_i \Delta^{l-1} u_i \right\}. \end{aligned} \quad (2.33)$$

Combining (2.30) and (2.33), we infer

$$\begin{aligned} & \int_M \left\{ (l-1) \left\langle \nabla x_{\alpha}, \nabla (\Delta^{l-2} u_i) \right\rangle \langle \nabla x_{\alpha}, \nabla u_i \rangle - \frac{1}{2} u_i \Delta^{l-1} u_i \right\} \\ &= 2l \int_{\Omega} \left\langle \nabla x_{\alpha}, \nabla (\Delta^{l-2} u_i) \right\rangle \langle \nabla u_i, \nabla x_{\alpha} \rangle + (-1)^{l-1} \Lambda_i \left(-\frac{1}{2} \|u_i\|^2 + 2 \|\nabla y_{\alpha i}\|^2 \right). \end{aligned} \quad (2.34)$$

Summing on α , we get (2.25). \square

Lemma 2.3 *For any i , we have*

$$\sum_{\alpha=1}^n \|W_{\alpha i}\|^2 \geq \frac{n-1}{\Lambda_i^{1/(l-1)}}. \quad (2.35)$$

Proof Using the definition of $W_{\alpha i}$ and the divergence theorem and noticing (2.20), we have

$$\begin{aligned} & \int_{\Omega} \langle \nabla x_{\alpha}, W_{\alpha i} \rangle \Delta u_i \\ &= - \int_{\Omega} \langle \nabla u_i, \nabla \langle \nabla x_{\alpha}, W_{\alpha i} \rangle \rangle \\ &= - \int_{\Omega} \langle \nabla u_i, \nabla (\langle x_{\alpha} \nabla u_i - \nabla h_{\alpha i}, \nabla x_{\alpha} \rangle) \rangle \\ &= - \|\langle \nabla u_i, \nabla x_{\alpha} \rangle\|^2 - \int_{\Omega} x_{\alpha} \langle \nabla u_i, \nabla \langle \nabla u_i, \nabla x_{\alpha} \rangle \rangle + \int_{\Omega} \langle \nabla u_i, \nabla \langle \nabla h_{\alpha i}, \nabla x_{\alpha} \rangle \rangle \\ &= - \|\langle \nabla u_i, \nabla x_{\alpha} \rangle\|^2 - \int_{\Omega} x_{\alpha} \langle \nabla u_i, \nabla \langle \nabla u_i, \nabla x_{\alpha} \rangle \rangle - \int_{\Omega} u_i \Delta \langle \nabla h_{\alpha i}, \nabla x_{\alpha} \rangle \\ &= - \|\langle \nabla u_i, \nabla x_{\alpha} \rangle\|^2 - \int_{\Omega} x_{\alpha} \langle \nabla u_i, \nabla \langle \nabla u_i, \nabla x_{\alpha} \rangle \rangle - \int_{\Omega} u_i \langle \nabla (\Delta h_{\alpha i}), \nabla x_{\alpha} \rangle \\ &= - \|\langle \nabla u_i, \nabla x_{\alpha} \rangle\|^2 - \int_{\Omega} \langle x_{\alpha} \nabla u_i, \nabla \langle \nabla u_i, \nabla x_{\alpha} \rangle \rangle - \int_{\Omega} u_i \langle \nabla (\langle \nabla x_{\alpha}, \nabla u_i \rangle + x_{\alpha} \Delta u_i), \nabla x_{\alpha} \rangle \\ &= - \|\langle \nabla u_i, \nabla x_{\alpha} \rangle\|^2 - \int_{\Omega} \langle \nabla (x_{\alpha} u_i), \nabla \langle \nabla u_i, \nabla x_{\alpha} \rangle \rangle - \int_{\Omega} u_i \langle \nabla (x_{\alpha} \Delta u_i), \nabla x_{\alpha} \rangle \\ &= - \|\langle \nabla u_i, \nabla x_{\alpha} \rangle\|^2 + \int_{\Omega} \langle \nabla u_i, \nabla x_{\alpha} \rangle \Delta (x_{\alpha} u_i) + \int_{\Omega} x_{\alpha} \Delta u_i \langle \nabla u_i, \nabla x_{\alpha} \rangle \\ &= \|\langle \nabla u_i, \nabla x_{\alpha} \rangle\|^2 + 2 \int_{\Omega} x_{\alpha} \Delta u_i \langle \nabla u_i, \nabla x_{\alpha} \rangle \\ &= \|\langle \nabla u_i, \nabla x_{\alpha} \rangle\|^2 - 2 \int_{\Omega} \langle \nabla u_i, \nabla (x_{\alpha} \langle \nabla u_i, \nabla x_{\alpha} \rangle) \rangle \\ &= - \|\langle \nabla u_i, \nabla x_{\alpha} \rangle\|^2 - 2 \int_{\Omega} x_{\alpha} \langle \nabla u_i, \nabla \langle \nabla u_i, \nabla x_{\alpha} \rangle \rangle \\ &= - \|\langle \nabla u_i, \nabla x_{\alpha} \rangle\|^2 + 1. \end{aligned} \quad (2.36)$$

On the other hand, for $\epsilon > 0$, we have

$$\begin{aligned} \int_{\Omega} \langle \nabla x_{\alpha}, W_{\alpha i} \rangle \Delta u_i &= \int_{\Omega} \langle \Delta u_i \nabla x_{\alpha} - \nabla \langle \nabla u_i, \nabla x_{\alpha} \rangle, W_{\alpha i} \rangle \\ &\leq \frac{\epsilon}{2} \|W_{\alpha i}\|^2 + \frac{1}{2\epsilon} \|\Delta u_i \nabla x_{\alpha} - \nabla \langle \nabla u_i, \nabla x_{\alpha} \rangle\|^2. \end{aligned} \quad (2.37)$$

From (2.36), we have

$$\sum_{\alpha=1}^n \int_{\Omega} \langle \nabla x_{\alpha}, W_{\alpha i} \rangle \Delta u_i = n - 1. \quad (2.38)$$

Also, one can check that

$$\sum_{\alpha=1}^n \|\Delta u_i \nabla x_{\alpha} - \nabla \langle \nabla u_i, \nabla x_{\alpha} \rangle\|^2 = (n-1) \int_{\Omega} u_i \Delta^2 u_i \leq (n-1) \Lambda_i^{1/(l-1)}. \quad (2.39)$$

Thus we have from (2.37)–(2.39) that

$$n - 1 \leq \frac{\epsilon}{2} \sum_{\alpha=1}^n \|W_{\alpha i}\|^2 + \frac{n-1}{2\epsilon} \Lambda_i^{1/(l-1)}. \quad (2.40)$$

Taking

$$\epsilon = \sqrt{\frac{(n-1) \Lambda_i^{1/(l-1)}}{\sum_{\alpha=1}^n \|W_{\alpha i}\|^2}},$$

we get (2.35). This completes the proof of Lemma 2.3. \square

Let us continue the proof of Theorem 1.1. Since $\|u_i\|^2 = \|W_{\alpha i}\|^2 + \|\nabla y_{\alpha i}\|^2$, we have from (2.35) that

$$\begin{aligned} n \Lambda_i \|u_i\|^2 &= \Lambda_i \sum_{\alpha=1}^n \|W_{\alpha i}\|^2 + \Lambda_i \sum_{\alpha=1}^n \|\nabla y_{\alpha i}\|^2 \\ &\geq (n-1) \Lambda_i^{(l-2)/(l-1)} + \Lambda_i \sum_{\alpha=1}^n \|\nabla y_{\alpha i}\|^2, \end{aligned} \quad (2.41)$$

which, combining with (2.25), implies that

$$-\Lambda_i \sum_{\alpha=1}^n \|\nabla y_{\alpha i}\|^2 \leq \frac{(n-2l-2)}{3} \int_{\Omega} u_i (-\Delta)^{l-1} u_i - \frac{(n-1)}{3} \Lambda_i^{(l-2)/(l-1)}. \quad (2.42)$$

Substituting (2.42) into (2.24) and using Lemma 2.1, we get

$$\begin{aligned} n \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 &\leq \sum_{i=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \left(\left(2l^2 + (n-4)l + 3 - n + \frac{n-2l-2}{3} \right) \int_{\Omega} u_i (-\Delta)^{l-1} u_i \right. \\ &\quad \left. - \frac{(n-1)}{3} \Lambda_i^{(l-2)/(l-1)} \right) \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^k \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \sum_{\alpha=1}^n \Lambda_i^{1/(l-1)} \\
& \leq \sum_{i=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \left(\left(2l^2 + (n-4)l + 3 - n + \frac{n-2l-2}{3} - \frac{(n-1)}{3} \right) \Lambda_i^{(l-2)/(l-1)} \right) \\
& \quad + \sum_{i=1}^k \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \Lambda_i^{1/(l-1)} \\
& = \sum_{i=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \left(2l^2 + \left(n - \frac{14}{3} \right) l + \frac{8}{3} - n \right) \Lambda_i^{(l-2)/(l-1)} \\
& \quad + \sum_{i=1}^k \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \Lambda_i^{1/(l-1)}.
\end{aligned}$$

This completes the proof of Theorem 1.1.

Proof of Corollary 1.2 By induction, one can show that

$$\begin{aligned}
& \left\{ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \Lambda_i^{(l-2)/(l-1)} \right\} \left\{ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i^{1/(l-1)} \right\} \\
& \leq \left\{ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \right\} \left\{ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i \right\},
\end{aligned}$$

which, combining with (1.12), gives (1.13). \square

Proof of Theorem 1.2 We use the same notations as in the beginning of this section and take M to be the unit n -sphere S^n . Let x_1, x_2, \dots, x_{n+1} be the standard coordinate functions of the Euclidean space \mathbf{R}^{n+1} , then

$$S = \left\{ (x_1, \dots, x_{n+1}) \in \mathbf{R}^{n+1}; \sum_{\alpha=1}^{n+1} x_{\alpha}^2 = 1 \right\}.$$

It is well known that

$$\Delta x_{\alpha} = -nx_{\alpha}, \quad \alpha = 1, \dots, n+1. \quad (2.43)$$

As in the proof of Theorem 1.1, we decompose the vector-valued functions $x_{\alpha} \nabla u_i$ as

$$x_{\alpha} \nabla u_i = \nabla h_{\alpha i} + W_{\alpha i}, \quad (2.44)$$

where $h_{\alpha i} \in W_{0,l}^{1,2}(\Omega)$, $\nabla h_{\alpha i}$ is the projection of $x_{\alpha} \nabla u_i$ in $\mathbf{L}_{0,1}^2(\Omega)$, $W_{\alpha i} \perp \mathbf{L}_{0,1}^2(\Omega)$ and

$$W_{\alpha i}|_{\partial\Omega} = 0, \quad \operatorname{div} W_{\alpha i} = 0. \quad (2.45)$$

We also consider the functions $\phi_{\alpha i} : \Omega \rightarrow \mathbf{R}$, given by

$$\phi_{\alpha i} = h_{\alpha i} - \sum_{j=1}^k b_{\alpha ij} u_j, \quad b_{\alpha ij} = \int_{\Omega} x_{\alpha} \langle \nabla u_i, \nabla u_j \rangle = b_{\alpha ji}. \quad (2.46)$$

Then

$$\phi_{\alpha i}|_{\partial\Omega} = \frac{\partial\phi_{\alpha i}}{\partial\nu}\Big|_{\partial\Omega} = \dots = \frac{\partial^{l-1}\phi_{\alpha i}}{\partial\nu^{l-1}}\Big|_{\partial\Omega} = 0,$$

$$(\phi_{\alpha i}, u_j)_D = \int_{\Omega} \langle \nabla\phi_{\alpha i}, \nabla u_j \rangle = 0, \quad \forall j = 1, \dots, k$$

and we have the basic Rayleigh–Ritz inequality for Λ_{k+1} :

$$\Lambda_{k+1} \int_{\Omega} |\nabla\phi_{\alpha i}|^2 \leq \int_D \phi_{\alpha i} (-\Delta)^l \phi_{\alpha i}, \quad \forall \alpha = 1, \dots, n, \quad i = 1, \dots, k. \quad (2.47)$$

We have

$$\Delta\phi_{\alpha i} = \langle \nabla x_{\alpha}, \nabla u_i \rangle + x_{\alpha} \Delta u_i - \sum_{j=1}^k b_{\alpha ij} \Delta u_j \quad (2.48)$$

and from (2.56) in [29],

$$\begin{aligned} & \int_{\Omega} \phi_{\alpha i} (-\Delta)^l \phi_{\alpha i} \\ &= \int_{\Omega} (-1)^l (\langle \nabla x_{\alpha}, \nabla u_i \rangle + x_{\alpha} \Delta u_i) \Delta^{l-2} (\langle \nabla x_{\alpha}, \nabla u_i \rangle + x_{\alpha} \Delta u_i) - \sum_{j=1}^k \Lambda_j b_{\alpha ij}^2. \end{aligned} \quad (2.49)$$

For a function g on Ω , we have (cf. (2.31) in [37])

$$\Delta \langle \nabla x_{\alpha}, \nabla g \rangle = -2x_{\alpha} \Delta g + \langle \nabla x_{\alpha}, \nabla ((\Delta + n - 2)g) \rangle. \quad (2.50)$$

For each $q = 0, 1, \dots$, thanks to (2.43) and (2.50), there are polynomials F_q and G_q of degree q such that

$$\Delta^q (\langle \nabla x_{\alpha}, \nabla u_i \rangle + x_{\alpha} \Delta u_i) = x_{\alpha} F_q(\Delta) \Delta u_i + \langle \nabla x_{\alpha}, \nabla (G_q(\Delta) u_i) \rangle. \quad (2.51)$$

It is obvious that

$$F_0 = 1, \quad G_0 = 1. \quad (2.52)$$

It follows from (2.43) and (2.50) that

$$\Delta(x_{\alpha} \Delta u_i + \langle \nabla x_{\alpha}, \nabla u_i \rangle) = x_{\alpha} (\Delta - (n + 2)) \Delta u_i + \langle \nabla x_{\alpha}, \nabla ((3\Delta + n - 2) u_i) \rangle, \quad (2.53)$$

which gives

$$F_1(t) = t - (n + 2), \quad G_1(t) = 3t + n - 2. \quad (2.54)$$

Also, when $q \geq 2$, we have (cf. (2.65) and (2.66) in [29])

$$F_q(t) = (2t - 2)F_{q-1}(t) - (t^2 + 2t - n(n - 2))F_{q-2}(t), \quad q = 2, \dots, \quad (2.55)$$

$$G_q(t) = (2t - 2)G_{q-1}(t) - (t^2 + 2t - n(n - 2))G_{q-2}(t), \quad q = 2, \dots. \quad (2.56)$$

□

For each $q = 1, 2, \dots$, let us set

$$\Phi_q(t) = tF_{q-1}(t) - G_{q-1}(t).$$

We conclude from (2.52), (2.54)–(2.56) that the polynomials Φ_q , $q = 1, 2, \dots$, are defined inductively by (1.15) and (1.16). Substituting

$$\Delta^{l-2}(\langle \nabla x_\alpha, \nabla u_i \rangle + x_\alpha \Delta u_i) = x_\alpha F_{l-2}(\Delta) \Delta u_i + \langle \nabla x_\alpha, \nabla(G_{l-2}(\Delta) u_i) \rangle \quad (2.57)$$

into (2.49), we get

$$\begin{aligned} & \int_{\Omega} \phi_{\alpha i} (-\Delta)^l \phi_{\alpha i} \\ &= \int_{\Omega} (-1)^l (\langle \nabla x_\alpha, \nabla u_i \rangle \langle \nabla x_\alpha, \nabla(G_{l-2}(\Delta) u_i) \rangle \\ &\quad + \langle x_\alpha \nabla x_\alpha, \Delta u_i \nabla(G_{l-2}(\Delta) u_i) + (F_{l-2}(\Delta) \Delta u_i) \nabla u_i \rangle) \\ &\quad + \int_{\Omega} (-1)^l x_\alpha^2 \Delta u_i F_{l-2}(\Delta)(\Delta u_i) - \sum_{j=1}^k \Lambda_j b_{\alpha i j}^2. \end{aligned} \quad (2.58)$$

Summing over α and noticing

$$\sum_{\alpha=1}^{n+1} x_\alpha^2 = 1, \quad \sum_{\alpha=1}^{n+1} \langle \nabla x_\alpha, \nabla u_i \rangle \langle \nabla x_\alpha, \nabla(G_{l-2}(\Delta) u_i) \rangle = \langle \nabla u_i, \nabla(G_{l-2}(\Delta) u_i) \rangle, \quad (2.59)$$

we infer

$$\begin{aligned} & \sum_{\alpha=1}^{n+1} \int_{\Omega} \phi_{\alpha i} (-\Delta)^l \phi_{\alpha i} \\ &= \int_{\Omega} (-1)^l \langle \nabla u_i, \nabla(G_{l-2}(\Delta) u_i) \rangle + \int_{\Omega} (-1)^l \Delta u_i F_{l-2}(\Delta)(\Delta u_i) - \sum_{\alpha=1}^{n+1} \sum_{j=1}^k \Lambda_j b_{\alpha i j}^2 \\ &= \int_{\Omega} (-1)^{l-1} u_i \Delta(G_{l-2}(\Delta) u_i) + \int_{\Omega} (-1)^l u_i \Delta(F_{l-2}(\Delta)(\Delta u_i)) - \sum_{\alpha=1}^{n+1} \sum_{j=1}^k \Lambda_j b_{\alpha i j}^2 \\ &= \int_{\Omega} (-1)^l u_i (\Delta F_{l-2}(\Delta) - G_{l-2}(\Delta))(\Delta u_i) - \sum_{\alpha=1}^{n+1} \sum_{j=1}^k \Lambda_j b_{\alpha i j}^2 \\ &= \int_{\Omega} (-1)^l u_i \Phi_{l-1}(\Delta)(\Delta u_i) - \sum_{\alpha=1}^{n+1} \sum_{j=1}^k \Lambda_j b_{\alpha i j}^2 \\ &= \int_{\Omega} (-1)^l u_i \left(\Delta^{l-1} - a_{l-2} \Delta^{l-2} + \dots \right. \\ &\quad \left. + (-1)^{l-2} a_1 \Delta - (n-2)^{l-2} \right) (\Delta u_i) - \sum_{\alpha=1}^{n+1} \sum_{j=1}^k \Lambda_j b_{\alpha i j}^2 \\ &= \Lambda_i + (-1)^l (n-2)^{l-2} + \sum_{j=1}^{l-2} a_j \int_{\Omega} u_i (-\Delta)^{j+1} u_i - \sum_{\alpha=1}^{n+1} \sum_{j=1}^k \Lambda_j b_{\alpha i j}^2. \end{aligned} \quad (2.60)$$

Set

$$H_i = (-1)^l (n-2)^{l-2} + \sum_{j=1}^{l-2} a_j^+ \Lambda_i^{j/(l-1)}, \quad (2.61)$$

then it is easy to check from Lemma 2.1 that

$$(-1)^l(n-2)^{l-2} + \sum_{j=0}^{l-2} a_j \int_{\Omega} u_i (-\Delta)^{j+1} u_i \leq H_i. \quad (2.62)$$

Substituting (2.62) into (2.60), we have

$$\sum_{\alpha=1}^{n+1} \int_{\Omega} \phi_{\alpha i} (-\Delta)^l \phi_{\alpha i} \leq \Lambda_i + H_i - \sum_{\alpha=1}^{n+1} \sum_{j=1}^k \Lambda_j b_{\alpha ij}^2. \quad (2.63)$$

Observe from (2.44) and (2.46) that

$$||x_{\alpha} \nabla u_i||^2 = ||\nabla h_{\alpha i}||^2 + ||W_{\alpha i}||^2 = ||\nabla \phi_{\alpha i}||^2 + ||W_{\alpha i}||^2 + \sum_{j=1}^k b_{\alpha ij}^2. \quad (2.64)$$

Summing over α , one gets

$$1 = \sum_{\alpha=1}^{n+1} \left(||\nabla \phi_{\alpha i}||^2 + ||W_{\alpha i}||^2 + \sum_{j=1}^k b_{\alpha ij}^2 \right). \quad (2.65)$$

Combining (2.47), (2.63) and (2.65), we get

$$\sum_{\alpha=1}^{n+1} (\Lambda_{k+1} - \Lambda_i) ||\nabla \phi_{\alpha i}||^2 \leq H_i + \sum_{\alpha=1}^{n+1} \Lambda_i ||W_{\alpha i}||^2 + \sum_{\alpha=1}^{n+1} \sum_{j=1}^k (\Lambda_i - \Lambda_j) b_{\alpha ij}^2. \quad (2.66)$$

Set

$$Z_{\alpha i} = \nabla \langle \nabla x_{\alpha}, \nabla u_i \rangle - \frac{n-2}{2} x_{\alpha} \nabla u_i, \quad c_{\alpha ij} = \int_{\Omega} \langle \nabla u_j, Z_{\alpha i} \rangle; \quad (2.67)$$

then $c_{\alpha ij} = -c_{\alpha ji}$ (cf. Lemma in [37]). By using the same arguments as in the proof of (2.37) in [37], we have

$$\begin{aligned} & (\Lambda_{k+1} - \Lambda_i)^2 \left(2 ||\langle \nabla x_{\alpha}, \nabla u_i \rangle||^2 + \int_{\Omega} \langle \nabla x_{\alpha}^2, \Delta u_i \nabla u_i \rangle + (n-2) ||x_{\alpha} \nabla u_i||^2 + 2 \sum_{j=1}^k b_{\alpha ij} c_{\alpha ij} \right) \\ & \leq \delta_i (\Lambda_{k+1} - \Lambda_i)^3 ||\nabla \phi_{\alpha i}||^2 + \frac{\Lambda_{k+1} - \Lambda_i}{\delta_i} \left(||Z_{\alpha i}||^2 - \sum_{j=1}^k c_{\alpha ij}^2 \right) + (n-2)(\Lambda_{k+1} - \Lambda_i)^2 ||W_{\alpha i}||^2. \end{aligned} \quad (2.68)$$

Since

$$\sum_{\alpha=1}^{n+1} ||\langle \nabla x_{\alpha}, \nabla u_i \rangle||^2 = \int_{\Omega} |\nabla u_i|^2 = 1, \quad (2.69)$$

we have by summing over α in (2.68) from 1 to $n+1$ that

$$\begin{aligned}
& (\Lambda_{k+1} - \Lambda_i)^2 \left(n + 2 \sum_{\alpha=1}^{n+1} \sum_{j=1}^k b_{\alpha ij} c_{\alpha ij} \right) \\
& \leq \delta_i \sum_{\alpha=1}^{n+1} (\Lambda_{k+1} - \Lambda_i)^3 \|\nabla \phi_{\alpha i}\|^2 + \frac{\Lambda_{k+1} - \Lambda_i}{\delta_i} \sum_{\alpha=1}^{n+1} \left(\|Z_{\alpha i}\|^2 - \sum_{j=1}^k c_{\alpha ij}^2 \right) \\
& \quad + (n-2) \sum_{\alpha=1}^{n+1} (\Lambda_{k+1} - \Lambda_i)^2 \|W_{\alpha i}\|^2. \tag{2.70}
\end{aligned}$$

The following inequalities have been proved in [29]:

$$\Lambda_i^{1/(l-1)} - (n-2) > 0, \tag{2.71}$$

$$\sum_{\alpha=1}^{n+1} \|Z_{\alpha i}\|^2 \leq \Lambda_i^{1/(l-1)} + \frac{(n-2)^2}{4} \tag{2.72}$$

and

$$\sum_{\alpha=1}^{n+1} \|W_{\alpha i}\|^2 \leq 1 - \frac{1}{\Lambda_i^{1/(l-1)} - (n-2)}. \tag{2.73}$$

Thus, we have by combining (2.66), (2.70), (2.72) and (2.73) that

$$\begin{aligned}
& (\Lambda_{k+1} - \Lambda_i)^2 \left(n + 2 \sum_{\alpha=1}^{n+1} \sum_{j=1}^k b_{\alpha ij} c_{\alpha ij} \right) \\
& \leq \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \left(H_i + \sum_{\alpha=1}^{n+1} \sum_{j=1}^k (\Lambda_i - \Lambda_j) b_{\alpha ij}^2 \right) \\
& \quad + \frac{\Lambda_{k+1} - \Lambda_i}{\delta_i} \left(\|Z_{\alpha i}\|^2 - \sum_{\alpha=1}^{n+1} \sum_{j=1}^k c_{\alpha ij}^2 \right) + \sum_{\alpha=1}^{n+1} (\Lambda_{k+1} - \Lambda_i)^2 (\delta_i \Lambda_i + n-2) \|W_{\alpha i}\|^2 \\
& \leq \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \left(H_i + \sum_{\alpha=1}^{n+1} \sum_{j=1}^k (\Lambda_i - \Lambda_j) b_{\alpha ij}^2 \right) \\
& \quad + \frac{\Lambda_{k+1} - \Lambda_i}{\delta_i} \left(\left(\Lambda_i^{1/(l-1)} + \frac{(n-2)^2}{4} \right) - \sum_{\alpha=1}^{n+1} \sum_{j=1}^k c_{\alpha ij}^2 \right) \\
& \quad + (\Lambda_{k+1} - \Lambda_i)^2 (\delta_i \Lambda_i + n-2) \left(1 - \frac{1}{\Lambda_i^{1/(l-1)} - (n-2)} \right). \tag{2.74}
\end{aligned}$$

Since $\{\delta_i\}_{i=1}^k$ is a positive non-increasing monotone sequence, we have

$$\begin{aligned}
& 2 \sum_{\alpha=1}^{n+1} \sum_{i,j=1}^k (\Lambda_{k+1} - \Lambda_i)^2 b_{\alpha ij} c_{\alpha ij} \\
& \geq \sum_{\alpha=1}^{n+1} \sum_{i,j=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)^2 (\Lambda_i - \Lambda_j) b_{\alpha ij}^2 - \sum_{\alpha=1}^{n+1} \sum_{i,j=1}^k \frac{\Lambda_{k+1} - \Lambda_i}{\delta_i} c_{\alpha ij}^2. \tag{2.75}
\end{aligned}$$

Hence, by summing over i from 1 to k in (2.74), we infer

$$\begin{aligned} & n \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \\ & \leq \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \left(\delta_i H_i + (\delta_i \Lambda_i + n - 2) \left(1 - \frac{1}{\Lambda_i^{1/(l-1)} - (n-2)} \right) \right) \\ & \quad + \sum_{i=1}^k \frac{\Lambda_{k+1} - \Lambda_i}{\delta_i} \left(\Lambda_i^{1/(l-1)} + \frac{(n-2)^2}{4} \right). \end{aligned}$$

That is

$$\begin{aligned} & \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \left(2 + \frac{n-2}{\Lambda_i^{1/(l-1)} - (n-2)} \right) \\ & \leq \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \delta_i \left(H_i + \Lambda_i \left(1 - \frac{1}{\Lambda_i^{1/(l-1)} - (n-2)} \right) \right) \\ & \quad + \sum_{i=1}^k \frac{(\Lambda_{k+1} - \Lambda_i)}{\delta_i} \left(\Lambda_i^{1/(l-1)} + \frac{(n-2)^2}{4} \right) \\ & = \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \delta_i S_i + \sum_{i=1}^k \frac{(\Lambda_{k+1} - \Lambda_i)}{\delta_i} \left(\Lambda_i^{1/(l-1)} + \frac{(n-2)^2}{4} \right), \end{aligned}$$

where S_i is given by (1.19). This completes the proof of Theorem 1.2.

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