Pacific Journal of Mathematics

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Volume 262 No. 1 March 2013

PACIFIC JOURNAL OF MATHEMATICS
Vol. 262, No. 1, 2013
dx.doi.org/10.2140/pjm.2013.262.35

A LOWER BOUND FOR EIGENVALUES OF THE POLY-LAPLACIAN WITH ARBITRARY ORDER

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We study eigenvalues of the poly-Laplacian of arbitrary order on a bounded domain in an n-dimensional Euclidean space. We obtain a lower bound for these eigenvalues, significantly improving on that of Levine and Protter. In particular, the result of Melas (2003) is subsumed.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with piecewise smooth boundary $\partial\Omega$ in an n-dimensional Euclidean space $\mathbb{R}^n$. Let λ_i be the i-th eigenvalue of the Dirichlet eigenvalue problem of the poly-Laplacian with arbitrary order:

$$\begin{cases} (-\Delta)^l u = \lambda u & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = \cdots = \dfrac{\partial^{l-1} u}{\partial \nu^{l-1}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

where Δ is the Laplacian in $\mathbb{R}^n$ and ν denotes the outward unit normal vector field of the boundary $\partial\Omega$. It is well known that the spectrum of this eigenvalue problem is real and discrete:

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty,$$

where each λ_i has finite multiplicity and is repeated according to its multiplicity.

Let $V(\Omega)$ denote the volume of Ω and let B_n denote the volume of the unit ball in $\mathbb{R}^n$. When $l = 1$, the eigenvalue problem (1-1) is called a fixed membrane problem. In this case, one has Weyl's asymptotic formula

$$\lambda_k \sim \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to +\infty. \tag{1-2}$$

Cheng was partially supported by JSPS Grant-in-Aid for Scientific Research (B) no. 24340013. Qi was partially supported by NSFC grant no. 11171091. Wei was partially supported by NSFC grant no. 11001087 and the project of Pear River New Star of Guangzhou (grant no. 2012J2200028). Wei is the corresponding author.
MSC2010: 35P15.
Keywords: eigenvalue problem, lower bound for eigenvalues, poly-Laplacian with arbitrary order.

From the above asymptotic formula, one can obtain

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i \sim \frac{n}{n+2}\frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}}, \quad k \to +\infty. \tag{1-3}$$

Pólya [1961] proved that

$$\lambda_k \geq \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \dots, \tag{1-4}$$

if Ω is a tiling domain in $\mathbb{R}^n$. Moreover, he proposed the following:

Conjecture of Pólya. If Ω is a bounded domain in $\mathbb{R}^n$, then the k-th eigenvalue λ_k of the fixed membrane problem satisfies

$$\lambda_k \geq \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \dots. \tag{1-5}$$

Berezin [1972] and Lieb [1980] gave a partial solution to this conjecture. Li and Yau [1983] proved that

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i \geq \frac{n}{n+2}\frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \dots. \tag{1-6}$$

Formula (1-3) shows that (1-6) is sharp in the sense of averages. From (1-6), one can derive

$$\lambda_k \geq \frac{n}{n+2}\frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \dots, \tag{1-7}$$

which gives a partial solution for the conjecture of Pólya with a factor $\frac{n}{n+2}$. Melas [2003] has improved the estimate (1-6) to

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i \geq \frac{n}{n+2}\frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}} + \frac{1}{24(n+2)}\frac{V(\Omega)}{I(\Omega)}, \quad \text{for } k = 1, 2, \dots, \tag{1-8}$$

where

$$I(\Omega) = \min_{a\in\mathbb{R}^n}\int_{\Omega}|x-a|^2\,dx$$

is called *the moment of inertia* of Ω.

When $l = 2$, the eigenvalue problem (1-1) is called the clamped plate problem. For the eigenvalues of the clamped plate problem, it follows from [Agmon 1965] and [Pleijel 1950] that

$$\lambda_k \sim \frac{16\pi^4}{(B_n V(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}}, \quad k \to +\infty. \tag{1-9}$$

This implies that

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i \sim \frac{n}{n+4}\frac{16\pi^4}{(B_nV(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}}, \quad k\to+\infty. \tag{1-10}$$

Furthermore, Levine and Protter [1985] proved that the eigenvalues of the clamped plate problem satisfy

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i \ge \frac{n}{n+4}\frac{16\pi^4}{(B_nV(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}}. \tag{1-11}$$

Formula (1-10) shows that the coefficient of $k^{\frac{4}{n}}$ is the best possible constant. Very recently, Cheng and Wei [2011] obtained the following improvement of (1-11):

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i \ge \frac{n}{n+4}\frac{16\pi^4}{(B_nV(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}} + c_n\frac{n}{n+2}\frac{4\pi^2}{(B_nV(\Omega))^{\frac{2}{n}}}\frac{V(\Omega)}{I(\Omega)}k^{\frac{2}{n}} + d_n\left(\frac{V(\Omega)}{I(\Omega)}\right)^2, \tag{1-12}$$

where c_n and d_n are constants depending only on the dimension n.

When $l\ge 3$, Levine and Protter [1985] proved that

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i \ge \frac{n}{n+2l}\frac{(2\pi)^{2l}}{(B_nV(\Omega))^{\frac{2l}{n}}}k^{\frac{2l}{n}}, \quad \text{for } k=1,2,\dots. \tag{1-13}$$

From the above formula, one can obtain

$$\lambda_k \ge \frac{n}{n+2l}\frac{(2\pi)^{2l}}{(B_nV(\Omega))^{\frac{2l}{n}}}k^{\frac{2l}{n}}, \quad \text{for } k=1,2,\dots. \tag{1-14}$$

In this paper we investigate eigenvalues of the Dirichlet eigenvalue problem (1-1) for the Laplacian with any order. We give a significant improvement of (1-13) by adding l lower-order terms than $k^{2l/n}$ to its right-hand side. In fact, we prove:

Theorem. *Let Ω be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$. Let λ_i, $i=1,2,\dots,$ be the i-th eigenvalue of the eigenvalue problem* (1-1). *Then*

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j \ge \frac{n}{n+2l}\frac{(2\pi)^{2l}}{(B_nV(\Omega))^{\frac{2l}{n}}}k^{\frac{2l}{n}} + \frac{n}{n+2l}\sum_{p=1}^{l}\frac{(l+1-p)}{(24)^p n\cdots(n+2p-2)}\frac{(2\pi)^{2(l-p)}}{(B_nV(\Omega))^{\frac{2(l-p)}{n}}}\left(\frac{V(\Omega)}{I(\Omega)}\right)^p k^{\frac{2(l-p)}{n}}.$$

Remark. If we take $l=1$, we obtain the inequality (1-8).

2. Proof of the Theorem

Before giving the proof, we introduce some definitions and basic facts about symmetric decreasing rearrangements.

For a bounded domain $\Omega \subset \mathbb{R}^n$, *the moment of inertia* of Ω is defined by

$$I(\Omega) = \min_{a \in \mathbb{R}^n} \int_\Omega |x-a|^2\, dx.$$

By translating the origin, we may assume that

$$I(\Omega) = \int_\Omega |x|^2\, dx.$$

Let Ω^* be the symmetric rearrangement of Ω, that is, Ω^* is the open ball centered at the origin with the same volume as Ω. Then

$$\Omega^* = \left\{ x \in \mathbb{R}^n;\ |x| < \left(\frac{V(\Omega)}{B_n} \right)^{\frac{1}{n}} \right\}.$$

By using the symmetric rearrangement Ω^* of Ω, we have

$$I(\Omega) = \int_\Omega |x|^2\, dx \geq \int_{\Omega^*} |x|^2\, dx = \frac{n}{n+2} V(\Omega) \left(\frac{V(\Omega)}{B_n} \right)^{\frac{2}{n}}. \tag{2-1}$$

Let f be a nonnegative continuous function on Ω. We consider its *distribution function* $\mu_f(t)$ defined by

$$\mu_f(t) = \mathrm{Vol}(\{x \in \Omega;\ f(x) > t\}).$$

The distribution function can be viewed as a function from $[0, +\infty)$ to $[0, V(\Omega)]$. The *symmetric decreasing rearrangement* f^* *of* f is defined by

$$f^*(x) = \inf\{t \geq 0;\ \mu_f(t) < B_n |x|^n\}, \quad \text{for } x \in \Omega^*.$$

By definition, we know that $f^*(x)$ is a radially symmetric function and

$$\mathrm{Vol}\big(\{x \in \Omega;\ f(x) > t\}\big) = \mathrm{Vol}\big(\{x \in \Omega^*;\ f^*(x) > t\}\big) \quad \text{for all } t > 0.$$

Let $f^*(x) = \phi(|x|)$. Then one gets that $\phi : [0, +\infty) \to [0, \sup f]$ is a decreasing function of $|x|$. We may assume that ϕ is absolutely continuous. It is well known that

$$\int_\Omega f(x)\, dx = \int_{\Omega^*} f^*(x)\, dx = n B_n \int_0^{+\infty} s^{n-1} \phi(s)\, ds \tag{2-2}$$

and

$$\int_\Omega |x|^{2l} f(x)\, dx \geq \int_{\Omega^*} |x|^{2l} f^*(x)\, dx = n B_n \int_0^{+\infty} s^{n+2l-1} \phi(s)\, ds. \tag{2-3}$$

Good sources of further information on rearrangements are [Bandle 1980; Pólya and Szegő 1951].

One gets from the coarea formula that

$$\mu_f(t) = \int_t^{\sup f} \int_{\{f=s\}} |\nabla f|^{-1}\, d\sigma_s\, ds.$$

Since f^* is radial, we have

$$\begin{aligned}\mu_f(\phi(s)) &= \mathrm{Vol}\{x \in \Omega;\ f(x) > \phi(s)\} = \mathrm{Vol}\{x \in \Omega^*;\ f^*(x) > \phi(s)\}\\ &= \mathrm{Vol}\{x \in \Omega^*;\ \phi(|x|) > \phi(s)\} = B_n s^n.\end{aligned}$$

It follows that

$$nB_n s^{n-1} = \mu_f'(\phi(s))\phi'(s)$$

for almost every s. Putting $\tau := \sup|\nabla f|$, we obtain from the above equations and the isoperimetric inequality that

$$-\mu_f'(\phi(s)) = \int_{\{f=\phi(s)\}} |\nabla f|^{-1}\, d\sigma_{\phi(s)} \geq \tau^{-1}\, \mathrm{Vol}_{n-1}(\{f=\phi(s)\}) \geq \tau^{-1} nB_n s^{n-1}.$$

Therefore, one obtains, for almost every s,

$$-\tau \leq \phi'(s) \leq 0. \tag{2-4}$$

In order to prove our theorem, we need the following lemma.

Lemma. *Let $b \geq 1$ and η, $A > 0$, and let $\psi : [0, +\infty) \to [0, +\infty)$ be a decreasing, absolutely continuous function such that*

$$-\eta \leq \psi'(s) \leq 0, \quad A = \int_0^{+\infty} s^{b-1}\psi(s)\, ds.$$

For any positive integer l, let

$$A_l := \int_0^{+\infty} s^{b+2l-1}\psi(s)\, ds.$$

Then, we have

$$A_l \geq \frac{1}{b+2l}\left[(bA)^{\frac{b+2l}{b}}\psi(0)^{-\frac{2l}{b}} + \sum_{p=1}^{l} \frac{(l+1-p)(bA)^{\frac{b+2(l-p)}{b}}\psi(0)^{\frac{2pb-2(l-p)}{b}}}{6^p b \cdots (b+2p-2)\eta^{2p}}\right].$$

Proof. The proof is by induction. Firstly, one can get from the lemma of [Melas 2003] that

$$A_1 = \int_0^{+\infty} s^{b+1}\psi(s)\, ds \geq \frac{1}{b+2}\left[(bA)^{\frac{b+2}{b}}\psi(0)^{-\frac{2}{b}} + \frac{A\psi(0)^2}{6\eta^2}\right]. \tag{2-5}$$

To prove the induction step, we assume the statement holds for $l = r$, that is,

$$A_r \geq \frac{1}{b+2r}\left[(bA)^{\frac{b+2r}{b}}\psi(0)^{-\frac{2r}{b}}+\sum_{p=1}^{r}\frac{(r+1-p)(bA)^{\frac{b+2(r-p)}{b}}\psi(0)^{\frac{2pb-2(r-p)}{b}}}{6^p b\cdots(b+2p-2)\eta^{2p}}\right].$$

Since the formula (2-5) holds for any $b \geq 1$, we have

$$\begin{aligned}
&A_{r+1}\\
&=\int_0^{+\infty} s^{b+2r+1}\psi(s)\,ds\\
&\geq \frac{1}{b+2r+2}\left\{[(b+2r)A_r]^{\frac{b+2r+2}{b+2r}}\psi(0)^{-\frac{2}{b+2r}}+\frac{A_r\psi(0)^2}{6\eta^2}\right\}\\
&\geq \frac{\psi(0)^{-\frac{2}{b+2r}}}{b+2r+2}\\
&\qquad\times\left[(bA)^{\frac{b+2r}{b}}\psi(0)^{-\frac{2r}{b}}+\sum_{p=1}^{r}\frac{(r+1-p)(bA)^{\frac{b+2(r-p)}{b}}\psi(0)^{\frac{2pb-2(r-p)}{b}}}{6^p b\cdots(b+2p-2)\eta^{2p}}\right]^{\frac{b+2r+2}{b+2r}}\\
&\quad+\frac{1}{(b+2r)(b+2r+2)}\sum_{p=1}^{r}\frac{(r+1-p)(bA)^{\frac{b+2(r-p)}{b}}\psi(0)^{\frac{2(p+1)b-2(r-p)}{b}}}{6^{p+1}b\cdots(b+2p-2)\eta^{2p+2}}\\
&\quad+\frac{(bA)^{\frac{b+2r}{b}}\psi(0)^{\frac{2b-2r}{b}}}{6(b+2r)(b+2r+2)\eta^2}\\
&=\frac{\psi(0)^{-\frac{2}{b+2r}}}{b+2r+2}\left[(bA)p^{\frac{b+2r}{b}}\psi(0)^{-\frac{2r}{b}}\right]^{\frac{b+2r+2}{b+2r}}\\
&\qquad\times\left[1+\sum_{p=1}^{r}\frac{(r+1-p)(bA)^{\frac{-2p}{b}}\psi(0)^{\frac{2pb+2p}{b}}}{6^p b\cdots(b+2p-2)\eta^{2p}}\right]^{\frac{b+2r+2}{b+2r}}+\frac{(bA)^{\frac{b+2r}{b}}\psi(0)^{\frac{2b-2r}{b}}}{6(b+2r)(b+2r+2)\eta^2}\\
&\quad+\frac{1}{(b+2r)(b+2r+2)}\sum_{p=2}^{r+1}\frac{(r+2-p)(bA)^{\frac{b+2r-2p+2}{b}}\psi(0)^{\frac{2pb-2r+2p-2}{b}}}{6^p b\cdots(b+2p-4)\eta^{2p}}\\
&=\frac{(bA)^{\frac{b+2r+2}{b}}\psi(0)^{-\frac{2r+2}{b}}}{b+2r+2}\left[1+\sum_{p=1}^{r}\frac{(r+1-p)(bA)^{\frac{-2p}{b}}\psi(0)^{\frac{2pb+2p}{b}}}{6^p b\cdots(b+2p-2)\eta^{2p}}\right]^{\frac{b+2r+2}{b+2r}}\\
&\quad+\frac{(bA)^{\frac{b+2r}{b}}\psi(0)^{\frac{2b-2r}{b}}}{6(b+2r)(b+2r+2)\eta^2}+\frac{(bA)\psi(0)^{2(r+1)}}{6^{r+1}b\cdots(b+2r+2)\eta^{2(r+1)}}\\
&\quad+\frac{1}{(b+2r)(b+2r+2)}\sum_{p=2}^{r}\frac{(r+2-p)(bA)^{\frac{b+2(r+1-p)}{b}}\psi(0)^{\frac{2pb-2(r+1-p)}{b}}}{6^p b\cdots(b+2p-4)\eta^{2p}}.
\end{aligned}$$

It follows from the Taylor formula that

$$
\begin{aligned}
&A_{r+1}\\
&\geq \frac{1}{b+2r+2}(bA)^{\frac{b+2r+2}{b}}\psi(0)^{-\frac{2r+2}{b}}\\
&\quad\times\left[1+\frac{b+2r+2}{b+2r}\sum_{p=1}^{r}\frac{(r+1-p)(bA)^{\frac{-2p}{b}}\psi(0)^{\frac{2pb+2p}{b}}}{6^p b\cdots(b+2p-2)\eta^{2p}}\right]\\
&\quad+\frac{(bA)^{\frac{b+2r}{b}}\psi(0)^{\frac{2b-2r}{b}}}{6(b+2r)(b+2r+2)\eta^2}+\frac{(bA)\psi(0)^{2(r+1)}}{6^{r+1}b\cdots(b+2r+2)\eta^{2(r+1)}}\\
&\quad+\frac{1}{(b+2r)(b+2r+2)}\sum_{p=2}^{r}\frac{(r+2-p)(bA)^{\frac{b+2(r+1-p)}{b}}\psi(0)^{\frac{2pb-2(r+1-p)}{b}}}{6^p b\cdots(b+2p-4)\eta^{2p}}\\
&=\frac{1}{b+2r+2}(bA)^{\frac{b+2r+2}{b}}\psi(0)^{-\frac{2r+2}{b}}\\
&\quad+\frac{1}{b+2r}\sum_{p=1}^{r}\frac{(r+1-p)(bA)^{\frac{b+2(r+1-p)}{b}}\psi(0)^{\frac{2pb-2(r+1-p)}{b}}}{6^p b\cdots(b+2p-2)\eta^{2p}}\\
&\quad+\frac{(bA)^{\frac{b+2r}{b}}\psi(0)^{\frac{2b-2r}{b}}}{6(b+2r)(b+2r+2)\eta^2}+\frac{(bA)\psi(0)^{2(r+1)}}{6^{r+1}b\cdots(b+2r+2)\eta^{2(r+1)}}\\
&\quad+\frac{1}{(b+2r)(b+2r+2)}\sum_{p=2}^{r}\frac{(r+2-p)(bA)^{\frac{b+2(r+1-p)}{b}}\psi(0)^{\frac{2pb-2(r+1-p)}{b}}}{6^p b\cdots(b+2p-4)\eta^{2p}}\\
&=\frac{1}{b+2r+2}(bA)^{\frac{b+2r+2}{b}}\psi(0)^{-\frac{2r+2}{b}}\\
&\quad+\left[\frac{r}{b(b+2r)}+\frac{1}{(b+2r)(b+2r+2)}\right]\frac{1}{6\eta^2}(bA)^{\frac{b+2r}{b}}\psi(0)^{\frac{2b-2r}{b}}\\
&\quad+\sum_{p=2}^{r}\left[\frac{r+1-p}{b+2r}+\frac{(r+2-p)(b+2p-2)}{(b+2r)(b+2r+2)}\right]\frac{(bA)^{\frac{b+2(r+1-p)}{b}}\psi(0)^{\frac{2pb-2(r+1-p)}{b}}}{6^p b\cdots(b+2p-2)\eta^{2p}}\\
&\quad+\frac{(bA)\psi(0)^{2(r+1)}}{6^{r+1}b\cdots(b+2r+2)\eta^{2(r+1)}}\\
&\geq \frac{1}{b+2(r+1)}(bA)^{\frac{b+2(r+1)}{b}}\psi(0)^{-\frac{2(r+1)}{b}}\\
&\quad+\frac{1}{b+2(r+1)}\sum_{p=1}^{r+1}\frac{(r+2-p)(bA)^{\frac{b+2(r+1-p)}{b}}\psi(0)^{\frac{2pb-2(r+1-p)}{b}}}{6^p b\cdots(b+2p-2)\eta^{2p}}.
\end{aligned}
$$

This completes the proof of the lemma. □

Proof of the Theorem. Let u_j be an orthonormal eigenfunction corresponding to the eigenvalue λ_j, that is, u_j satisfies

$$\begin{cases} (-\Delta)^l u_j = \lambda_j u_j, & \text{in } \Omega, \\ u_j = \dfrac{\partial u_j}{\partial \nu} = \cdots = \dfrac{\partial^{l-1} u_j}{\partial \nu^{l-1}} = 0, & \text{on } \partial\Omega, \\ \int_\Omega u_i u_j = \delta_{ij}, & \text{for any } i,\ j. \end{cases} \tag{2-6}$$

Thus, $\{u_j\}_{j=1}^{\infty}$ forms an orthonormal basis of $L^2(\Omega)$. We define a function φ_j by

$$\varphi_j(x) = \begin{cases} u_j(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2-7}$$

The Fourier transform $\widehat{\varphi}_j(z)$ of $\varphi_j(x)$ is then given by

$$\widehat{\varphi}_j(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi_j(x) e^{i\langle x,z\rangle}\, dx = (2\pi)^{-n/2} \int_\Omega u_j(x) e^{i\langle x,z\rangle}\, dx. \tag{2-8}$$

We fix $k \geq 1$ and set

$$f(z) = \sum_{j=1}^{k} \left|\widehat{\varphi}_j(z)\right|^2, \quad \text{for } z \in \mathbb{R}^n.$$

From Bessel's inequality, it follows that

$$\begin{aligned} 0 \leq f(z) = \sum_{j=1}^{k} \left|\widehat{\varphi}_j(z)\right|^2 &= (2\pi)^{-n} \sum_{j=1}^{k} \left| \int_\Omega u_j(x) e^{i\langle x,z\rangle}\, dx \right|^2 \\ &\leq (2\pi)^{-n} \int_\Omega \left|e^{i\langle x,z\rangle}\right|^2 dx = (2\pi)^{-n} V(\Omega). \end{aligned} \tag{2-9}$$

By Parseval's identity, we have

$$\begin{aligned} \int_{\mathbb{R}^n} f(z)\, dz = \sum_{j=1}^{k} \int_{\mathbb{R}^n} \left|\widehat{\varphi}_j(z)\right|^2 dz &= \sum_{j=1}^{k} \int_{\mathbb{R}^n} \varphi_j^2(x)\, dx \\ &= \sum_{j=1}^{k} \int_\Omega u_j^2(x)\, dx = k. \end{aligned} \tag{2-10}$$

Furthermore, we deduce from integration by parts and Parseval's identity that

$$\begin{aligned}
\text{(2-11)}\quad & \int_{\mathbb{R}^n} |z|^{2l} f(z)\,dz \\
&= \sum_{j=1}^{k} \int_{\mathbb{R}^n} |z|^{2l} |\widehat{\varphi}_j(z)|^2\,dz \\
&= \sum_{j=1}^{k} \int_{\mathbb{R}^n} |z|^{2l} \left| (2\pi)^{-n/2} \int_\Omega u_j(x) e^{i\langle x,z\rangle}\,dx \right|^2 dz \\
&= \sum_{j=1}^{k} \sum_{r_1,\dots,r_l=1}^{n} \int_{\mathbb{R}^n} \left| (2\pi)^{-n/2} \int_\Omega z_{r_1}\cdots z_{r_l} u_j(x) e^{i\langle x,z\rangle}\,dx \right|^2 dz \\
&= \sum_{j=1}^{k} \sum_{r_1,\dots,r_l=1}^{n} \int_{\mathbb{R}^n} \left| (2\pi)^{-n/2} \int_\Omega u_j(x) \frac{\partial^l e^{i\langle x,z\rangle}}{\partial x_{r_1}\cdots\partial x_{r_l}}\,dx \right|^2 dz \\
&= \sum_{j=1}^{k} \sum_{r_1,\dots,r_l=1}^{n} \int_{\mathbb{R}^n} \left| (2\pi)^{-n/2} \int_\Omega \frac{\partial^l u_j(x)}{\partial x_{r_1}\cdots\partial x_{r_l}} e^{i\langle x,z\rangle}\,dx \right|^2 dz \\
&= \sum_{j=1}^{k} \sum_{r_1,\dots,r_l=1}^{n} \int_{\mathbb{R}^n} \left| \widehat{\frac{\partial^l u_j}{\partial x_{r_1}\cdots\partial x_{r_l}}} \right|^2 dz \\
&= \sum_{j=1}^{k} \sum_{r_1,\dots,r_l=1}^{n} \int_{\mathbb{R}^n} \left(\frac{\partial^l u_j}{\partial x_{r_1}\cdots\partial x_{r_l}} \right)^2 dx \\
&= \sum_{j=1}^{k} \int_\Omega u_j (-\Delta)^l u_j\,dx = \sum_{j=1}^{k} \lambda_j.
\end{aligned}$$

Since

$$\nabla\widehat{\varphi}_j(z) = (2\pi)^{-n/2} \int_\Omega i x u_j(x) e^{i\langle x,z\rangle}\,dx, \tag{2-12}$$

we obtain from Bessel's inequality that

$$\sum_{j=1}^{k} \left|\nabla\widehat{\varphi}_j(z)\right|^2 \le (2\pi)^{-n} \int_\Omega \left| i x e^{i\langle x,z\rangle} \right|^2 dx = (2\pi)^{-n} I(\Omega). \tag{2-13}$$

It follows from (2-9), (2-13) and the Cauchy–Schwarz inequality that, for every $z \in \mathbb{R}^n$,

$$\begin{aligned}
\text{(2-14)}\quad \left|\nabla f(z)\right| &\le 2 \left(\sum_{j=1}^{k} \left|\widehat{\varphi}_j(z)\right|^2 \right)^{1/2} \left(\sum_{j=1}^{k} \left|\nabla\widehat{\varphi}_j(z)\right|^2 \right)^{1/2} \\
&\le 2(2\pi)^{-n} \sqrt{V(\Omega) I(\Omega)}.
\end{aligned}$$

Using the symmetric decreasing rearrangement f^* of f and noting that

$$f^*(x) = \phi(|x|), \quad \tau = \sup|\nabla f| \le 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)} := \eta,$$

we obtain, from (2-4),

$$-\eta \le -\tau \le \phi'(s) \le 0 \tag{2-15}$$

for almost every s. According to (2-2) and (2-10), we infer

$$k = \int_{\mathbb{R}^n} f(z)\,dz = \int_{\mathbb{R}^n} f^*(z)\,dz = nB_n \int_0^{+\infty} s^{n-1}\phi(s)\,ds. \tag{2-16}$$

From (2-3) and (2-11), we obtain

$$\begin{aligned}\sum_{j=1}^{k} \lambda_j &= \int_{\mathbb{R}^n} |z|^{2l} f(z)\,dz \ge \int_{\mathbb{R}^n} |z|^{2l} f^*(z)\,dz \\ &= nB_n \int_0^{+\infty} s^{n+2l-1}\phi(s)\,ds.\end{aligned} \tag{2-17}$$

Now, we can apply the Lemma to the function ϕ with

$$b = n, \quad A = \frac{k}{nB_n}, \quad \eta = 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)}. \tag{2-18}$$

We conclude that

$$\begin{aligned}\sum_{j=1}^{k} \lambda_j \ge{}& \frac{nB_n}{n+2l}\Big(\frac{k}{B_n}\Big)^{\frac{n+2l}{n}} \phi(0)^{-\frac{2l}{n}} \\ &+ \frac{nB_n}{n+2l}\sum_{p=1}^{l} \frac{(l+1-p)}{6^p n \cdots (n+2p-2)\eta^{2p}} \Big(\frac{k}{B_n}\Big)^{\frac{n+2l-2p}{n}} \phi(0)^{\frac{2pn+2p-2l}{n}}.\end{aligned} \tag{2-19}$$

Note that $0 < \phi(0) \le \sup f \le (2\pi)^{-n} V(\Omega)$. Hence we consider the function F defined by

$$\begin{aligned}F(t) ={}& \frac{nB_n}{n+2l}\Big(\frac{k}{B_n}\Big)^{\frac{n+2l}{n}} t^{-\frac{2l}{n}} \\ &+ \frac{nB_n}{n+2l}\sum_{p=1}^{l} \frac{(l+1-p)}{6^p n \cdots (n+2p-2)\eta^{2p}} \Big(\frac{k}{B_n}\Big)^{\frac{n+2l-2p}{n}} t^{\frac{2pn+2p-2l}{n}},\end{aligned} \tag{2-20}$$

for $t \in (0, (2\pi)^{-n} V(\Omega)]$. From (2-1), we have

$$\eta \ge (2\pi)^{-n} B_n^{-\frac{1}{n}} V(\Omega)^{\frac{n+1}{n}}. \tag{2-21}$$

By a direct calculation, one gets from $B_n = \dfrac{2\pi^{n/2}}{n\Gamma(n/2)}$ that

$$\frac{B_n^{4/n}}{(2\pi)^2} < \frac{1}{2}, \tag{2-22}$$

where $\Gamma(\frac{n}{2})$ is the gamma function. Thus, it follows from (2-21) and (2-22) that

$$\begin{aligned}
F'(t) &= \frac{2B_n t^{-\frac{n+2l}{n}}}{n+2l}\Big(\frac{k}{B_n}\Big)^{\frac{n+2l}{n}}\Big[-l+\sum_{p=1}^{l}\frac{(l+1-p)(pn+p-l)t^{\frac{2p(n+1)}{n}}}{6^p n\cdots(n+2p-2)\eta^{2p}}\Big(\frac{k}{B_n}\Big)^{-\frac{2p}{n}}\Big]\\
&\le \frac{2B_n}{n+2l}\Big(\frac{k}{B_n}\Big)^{\frac{n+2l}{n}} t^{-\frac{n+2l}{n}}\Big[-l+\sum_{p>\frac{l}{n+1}}^{l}\frac{(l+1-p)(pn+p-l)}{6^p n\cdots(n+2p-2)}\Big(\frac{B_n^{\frac{4}{n}}}{(2\pi)^2}\Big)^p\Big]\\
&< \frac{2B_n}{n+2l}\Big(\frac{k}{B_n}\Big)^{\frac{n+2l}{n}} t^{-\frac{n+2l}{n}}\Big[-l+\sum_{p>\frac{l}{n+1}}^{l}\frac{(l+1-p)(pn+p-l)}{(12)^p n\cdots(n+2p-2)}\Big]\\
&< \frac{2B_n}{n+2l}\Big(\frac{k}{B_n}\Big)^{\frac{n+2l}{n}} t^{-\frac{n+2l}{n}}\Big[-l+\frac{l(n+1-l)}{12n}+\sum_{\substack{p>\frac{l}{n+1}\\ p\neq 1}}^{l}\frac{p^2 n(n+1)}{(12)^p n\cdots(n+2p-2)}\Big]\\
&< \frac{2B_n}{n+2l}\Big(\frac{k}{B_n}\Big)^{\frac{n+2l}{n}} t^{-\frac{n+2l}{n}}\Big[-l+\frac{l}{12}+\sum_{\substack{p>\frac{l}{n+1}\\ p\neq 1}}^{l}\frac{p^2}{(12)^p}\Big]\\
&< \frac{2B_n}{n+2l}\Big(\frac{k}{B_n}\Big)^{\frac{n+2l}{n}} t^{-\frac{n+2l}{n}}\Big[-l+\frac{l}{12}+\frac{1}{12}\Big] < 0.
\end{aligned}$$

We obtain that $F(t)$ is a decreasing function on $(0, (2\pi)^{-n}V(\Omega)]$. Then we can replace $\phi(0)$ by $(2\pi)^{-n}V(\Omega)$ in (2-19), namely,

$$\begin{aligned}
\sum_{j=1}^{k}\lambda_j &\ge \frac{n}{n+2l}\frac{(2\pi)^{2l}}{(B_nV(\Omega))^{\frac{2l}{n}}}k^{\frac{n+2l}{n}}\\
&\quad+\frac{n}{n+2l}\sum_{p=1}^{l}\frac{(l+1-p)}{6^p n\cdots(n+2p-2)\eta^{2p}}\frac{(V(\Omega))^{\frac{2pn+2p-2l}{n}}}{(2\pi)^{2pn+2p-2n}B_n^{\frac{2l-2p}{n}}}k^{\frac{n+2l-2p}{n}}\\
&= \frac{n}{n+2l}\frac{(2\pi)^{2l}}{(B_nV(\Omega))^{\frac{2l}{n}}}k^{\frac{n+2l}{n}}\\
&\quad+\frac{n}{n+2l}\sum_{p=1}^{l}\frac{(l+1-p)}{24^p n\cdots(n+2p-2)}\frac{(2\pi)^{2(l-p)}}{(B_nV(\Omega))^{\frac{2(l-p)}{n}}}\Big(\frac{V(\Omega)}{I(\Omega)}\Big)^p k^{\frac{n+2(l-p)}{n}}.
\end{aligned}$$

This completes the proof of the Theorem. □

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Received December 14, 2010.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 262 No. 1 March 2013

0030-8730(201303)262:1;1-A