Estimates for lower bounds of eigenvalues of the poly-Laplacian and quadratic polynomial operator of the Laplacian

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Abstract. In this paper, we investigate the Dirchlet eigenvalue problems of poly-Laplacian with any order and quadratic polynomial operator of the Laplacian. We give some estimates for lower bounds of the sums of their first k eigenvalues which improve the previous results.

1 Introduction

Let Ω be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$, where $n \geq 2$. The Dirichlet eigenvalue problem of the poly-Laplacian is described by

$$\begin{cases} (-\Delta)^l u = \lambda u, & \text{on } \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}|_{\partial\Omega} = \cdots = \frac{\partial^{l-1} u}{\partial \nu^{l-1}}|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where Δ is the Laplacian and ν denotes the outward unit normal vector field of $\partial\Omega$. As we known, this problem has a real and discrete spectrum: $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty$, where each eigenvalue repeats with its multiplicity.

When $l = 1$, problem (1.1) is called the Dirichlet Laplacian problem or the fixed membrane problem. The asymptotic behavior of its k-th eigenvalue λ_k relates to geometric properties of Ω when $k \to \infty$. In fact, the following Weyl's asymptotic formula holds

$$\lambda_k \sim \frac{(2\pi)^2}{(\omega_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{as } k \to \infty, \tag{1.2}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$ and $V(\Omega)$ denotes the volume of Ω. In 1961, Pólya [13] proved that

$$\lambda_k \geq \frac{(2\pi)^2}{(\omega_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}} \tag{1.3}$$

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holds on tiling domains in $\mathbb{R}^2$. His proof also works on tiling domains in $\mathbb{R}^n$. Moreover, he conjectured that (1.3) holds for any bounded domain in $\mathbb{R}^n$. Berezin [3] and Lieb [10] made some contributions to the partial solution of this conjecture. In 1983, Li and Yau [9] proved the following so-called Li-Yau inequality

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j \geq \frac{n}{n+2}\frac{(2\pi)^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}}. \tag{1.4}$$

In 2000, Laptev and Weidl [7] pointed out that (1.4) can be derived by the Legendre transform of a result derived by Berezin [3]. Hence, (1.4) is also called the Berezin-Li-Yau inequality. In 2003, adding an additional positive term to the right-hand side of (1.4), Melas [11] improved (1.4) to

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j \geq \frac{n}{n+2}\frac{(2\pi)^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}} + \frac{1}{24(n+2)}\frac{V(\Omega)}{I(\Omega)}, \tag{1.5}$$

where $I(\Omega) = \min_{a\in\mathbb{R}^n}\int_\Omega |x-a|^2dx$ is the moment of inertia of Ω. Recently, Ilyin [6] obtained the following asymptotic lower bound for eigenvalues of problem (1.1):

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j \geq \frac{n}{n+2}\frac{(2\pi)^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}} + \frac{n}{48}\frac{V(\Omega)}{I(\Omega)}\Big(1-\varepsilon_n(k)\Big), \tag{1.6}$$

where $0 \leq \varepsilon_n(k) = O(k^{-\frac{2}{n}})$ is a infinitesimal of $k^{-\frac{2}{n}}$. Moreover, he derived some explicit inequalities for the particular cases of $n = 2, 3, 4$:

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j \geq \frac{n}{n+2}\frac{(2\pi)^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}} + \frac{n}{48}\beta_n\frac{V(\Omega)}{I(\Omega)}, \tag{1.7}$$

where $\beta_2 = \frac{119}{120}$, $\beta_3 = 0.986$ and $\beta_4 = 0.983$.

When $l = 2$, problem (1.1) is called the clamped plate problem. Agmon [1] and Pleijel [12] obtained

$$\lambda_k \sim \frac{(2\pi)^4}{(\omega_n V(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}}, \quad \text{as } k \to +\infty. \tag{1.8}$$

In 1985, Levine and Protter [8] proved:

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j \geq \frac{n}{n+4}\frac{(2\pi)^4}{(\omega_n V(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}}. \tag{1.9}$$

For the special case of $n = 2$, Ilyin [6] proved

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j \geq \frac{16\pi^2}{3(V(\Omega))^2}k^2 + \frac{12095\pi}{3\cdot 12096 I(\Omega)}k. \tag{1.10}$$

In 2011, Cheng and Wei [5] strengthened (1.9) to

$$\begin{aligned}\frac{1}{k}\sum_{j=1}^{k}\lambda_j \geq & \frac{n}{n+4}\frac{(2\pi)^4}{(\omega_n V(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}} \\ & + \frac{n}{n+2}\left[\frac{n+2}{12n(n+4)} - \frac{1}{1152n^2(n+4)}\right]\frac{(2\pi)^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}\frac{V(\Omega)}{I(\Omega)}k^{\frac{2}{n}} \\ & + \left[\frac{1}{576n(n+4)} - \frac{1}{27648n^2(n+2)(n+4)}\right]\left(\frac{V(\Omega)}{I(\Omega)}\right)^2.\end{aligned} \tag{1.11}$$

When $l \geq 3$, Levine and Protter [8] proved

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j \geq \frac{n}{n+2l}\frac{(2\pi)^{2l}}{(\omega_n V(\Omega))^{\frac{2l}{n}}}k^{\frac{2l}{n}}. \tag{1.12}$$

Recently, adding l terms of lower order of $k^{\frac{2l}{n}}$ to its right-hand side of (1.12), Cheng, Qi and Wei [4] derived

$$\begin{aligned}\frac{1}{k}\sum_{j=1}^{k}\lambda_j \geq & \frac{n}{n+2l}\frac{(2\pi)^{2l}}{(\omega_n V(\Omega))^{\frac{2l}{n}}}k^{\frac{2l}{n}} \\ & + \frac{n}{(n+2l)}\sum_{p=1}^{l}\frac{l+1-p}{(24)^p n\cdots(n+2p-2)}\frac{(2\pi)^{2(l-p)}}{(\omega_n V(\Omega))^{\frac{2(l-p)}{n}}}\left(\frac{V(\Omega)}{I(\Omega)}\right)^p k^{\frac{2(l-p)}{n}}.\end{aligned} \tag{1.13}$$

When $l = 1$, (1.13) becomes (1.5).

In this paper, we obtain the following result for problem (1.1).

Theorem 1. *Let Ω be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$. Denote by λ_j the j-th eigenvalue of problem (1.1). Then we have*

$$\begin{aligned}\frac{1}{k}\sum_{j=1}^{k}\lambda_j \geq & \frac{n}{n+2l}\frac{(2\pi)^{2l}}{(\omega_n V(\Omega))^{\frac{2l}{n}}}k^{\frac{2l}{n}} \\ & + \frac{nl}{48}\frac{(2\pi)^{2l-2}}{(\omega_n V(\Omega))^{\frac{2l-2}{n}}}\frac{V(\Omega)}{I(\Omega)}k^{\frac{2l-2}{n}}\Big(1-\varepsilon_n(k)\Big),\end{aligned} \tag{1.14}$$

where $0 \leq \varepsilon_n(k) = O(k^{-\frac{2}{n}})$ is a infinitesimal of $k^{-\frac{2}{n}}$.

Remark 1.1. *Taking $l = 1$ in (1.14), we obtain (1.6). Moreover, the second term on the right-hand side of (1.13) is*

$$\frac{l}{24(n+2l)}\frac{(2\pi)^{2l-2}}{(\omega_n V(\Omega))^{\frac{2l-2}{n}}}\frac{V(\Omega)}{I(\Omega)}k^{\frac{2l-2}{n}}.$$

Hence, the second term on the right-hand side of (1.14) is $\frac{n(n+2l)}{2}$ times larger than that of (1.13). Thus, for large k, (1.14) is sharper than (1.13).

Furthermore, we investigate the following Dirichlet eigenvalue problem of quadratic polynomial operator of the Laplacian:

$$\begin{cases}\Delta^2 u - a\Delta u = \Gamma u, & \text{on } \Omega,\\ u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0,\end{cases} \tag{1.15}$$

where a is a nonnegative constant. Levine and Protter [8] proved that the eigenvalues of this problem satisfy

$$\Gamma_k \geq \frac{n}{n+4}\frac{(2\pi)^4}{(\omega_n V(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}} + \frac{na}{n+2}\frac{(2\pi)^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}}. \tag{1.16}$$

In this paper, we derive the following results for problem (1.15).

Theorem 2. *Let Ω be a bounded domain in $\mathbb{R}^n$. Denote by Γ_j the j-th eigenvalue of problem (1.15). Then we have*

$$\begin{aligned}\frac{1}{k}\sum_{j=1}^{k}\Gamma_j \geq & \frac{n}{n+4}\frac{(2\pi)^4}{(\omega_n V(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}} + \left(\frac{n}{24}\frac{V(\Omega)}{I(\Omega)} + \frac{na}{n+2}\right)\frac{(2\pi)^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}}\\ & + \left[-\frac{n(n^2-4)}{3840}\frac{V(\Omega)}{I(\Omega)} + \frac{na}{48}\right]\frac{V(\Omega)}{I(\Omega)}\Big(1-\varepsilon_n(k)\Big),\end{aligned} \tag{1.17}$$

where $0 \leq \varepsilon_n(k) = O(k^{-\frac{2}{n}})$ is a infinitesimal of $k^{-\frac{2}{n}}$.

For the special cases of $n = 2, 3, 4$, we prove the following sharper result:

Theorem 3. *Denote by Γ_j the j-th eigenvalue of problem (1.15) on a bounded domain Ω in $\mathbb{R}^n$, where $n = 2, 3, 4$. Then we have*

$$\begin{aligned}\frac{1}{k}\sum_{j=1}^{k}\Gamma_j \geq & \frac{n}{n+4}\frac{(2\pi)^4}{(\omega_n V(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}} + \left(\frac{n}{24}\alpha_n\frac{V(\Omega)}{I(\Omega)} + \frac{na}{n+2}\right)\frac{(2\pi)^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}}\\ & + \frac{na}{48}\beta_n\frac{V(\Omega)}{I(\Omega)},\end{aligned} \tag{1.18}$$

where $\alpha_2 = \frac{12095}{12096}$, $\beta_2 = \frac{119}{120}$, $\alpha_3 = 0.991$, $\beta_3 = 0.986$, $\alpha_4 = 0.985$ and $\beta_4 = 0.983$.

Making a modification in the proof of Theorem 3, we can get the following result:

Theorem 4. *Denote by Γ_j the j-th eigenvalue of problem (1.15) on a bounded domain Ω in $\mathbb{R}^n$, where $n = 3, 4$. Then we have*

$$\begin{aligned}\frac{1}{k}\sum_{j=1}^{k}\Gamma_j \geq & \frac{n}{n+4}\frac{(2\pi)^4}{(\omega_n V(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}} + \left(\frac{n}{24}\frac{V(\Omega)}{I(\Omega)} + \frac{na}{n+2}\right)\frac{(2\pi)^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}}\\ & + \left[-\frac{n(n^2-4)}{3840}\frac{V(\Omega)}{I(\Omega)} + \frac{na}{48}\beta_n\right]\frac{V(\Omega)}{I(\Omega)}.\end{aligned} \tag{1.19}$$

Remark 1.2. *Taking $a = 0$ in (1.17), (1.18) and (1.19), we can get some results for the clamped plate problem.*

2 Proofs of the main results

In order to prove Theorem 1, we need the following lemma derived by Ilyin [6].

Lemma 1. *Let*

$$\Psi_s(r) = \begin{cases} M, & for \quad 0 \leq r \leq s; \\ M - L(r-s), & for \quad s \leq r \leq s + \dfrac{M}{L}; \\ 0, & for \quad r \geq s + \dfrac{M}{L}. \end{cases}$$

Suppose that $\int_0^{+\infty} r^b \Psi_s(r)dr = m^*$ *and* $d \geq b$. *Then for any decreasing and absolutely continuous function* F *satisfying the conditions*

$$0 \leq F \leq M, \quad \int_0^{+\infty} r^b F(r)dr = m^*, \quad 0 \leq -F' \leq L, \tag{2.1}$$

the following inequality holds:

$$\int_0^{+\infty} r^d F(r)dr \geq \int_0^{+\infty} r^d \Psi_s(r)dr. \tag{2.2}$$

Now we give the proof of Theorem 1.

Proof of Thereom 1 Let u_j be an orthonormal eigenfuction corresponding to the j-th eigenvalue λ_j of problem (1.1). Denote by $\widehat{u}_j(\xi)$ the Fourier transform of $u_j(x)$, which is defined by

$$\widehat{u}_j(\xi) = (2\pi)^{-\frac{n}{2}} \int_\Omega u_j(x) e^{ix\cdot\xi} dx. \tag{2.3}$$

It follows from Plancherel's Theorem that

$$\int_\Omega \widehat{u}_j(\xi)\widehat{u}_q(\xi)d\xi = \delta_{jq}. \tag{2.4}$$

Set $h(\xi) = \sum_{j=1}^k |\widehat{u}_j(\xi)|^2$. From (2.4) and Bessel's inequality, one can get

$$h(\xi) = \sum_{j=1}^k |\widehat{u}_j(\xi)|^2 \leq (2\pi)^{-n} \int_\Omega |e^{ix\cdot\xi}|^2 dx = (2\pi)^{-n} V(\Omega). \tag{2.5}$$

Moreover, Parsevel's identity implies that

$$\int_{\mathbb{R}^n} h(\xi)d\xi = \sum_{j=1}^k \int_\Omega |u_j(x)|^2 dx = k. \tag{2.6}$$

Since

$$\nabla \widehat{u}_j(\xi) = (2\pi)^{-\frac{n}{2}} \int_\Omega ix u_j(x) e^{ix\cdot\xi} dx,$$

we have

$$\sum_{j=1}^{k}|\nabla\widehat{u}_j(\xi)|^2\le(2\pi)^{-n}\int_{\Omega}|ixe^{ix\cdot\xi}|^2dx=(2\pi)^{-n}I(\Omega). \tag{2.7}$$

It follows from (2.5) and (2.7) that

$$|\nabla h(\xi)|\le 2\big(\sum_{j=1}^{k}|\widehat{u}_j(\xi)|^2\big)^{\frac{1}{2}}\big(\sum_{j=1}^{k}|\nabla\widehat{u}_j(\xi)|^2\big)^{\frac{1}{2}}\le 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)}. \tag{2.8}$$

Denote by $h^*(\xi)=\psi(|\xi|)$ the symmetric decreasing rearrangement (see [2,14]) of h. From

$$k=\sum_{j=1}^{k}\int_{\Omega}|u_j(x)|^2dx=\int_{\mathbb{R}^n}h(\xi)d\xi=\int_{\mathbb{R}^n}h^*(\xi)d\xi=n\omega_n\int_0^{+\infty}r^{n-1}\psi(r)dr,$$

we get

$$\int_0^{+\infty}r^{n-1}\psi(r)dr=\frac{k}{n\omega_n}. \tag{2.9}$$

At the same time, using integration by parts and Parseval's identity, we have

$$\begin{aligned}\int_{\mathbb{R}^n}|\xi|^{2l}h(\xi)d\xi=&\sum_{j=1}^{k}\sum_{p_1,\cdots,p_l=1}^{n}\int_{\mathbb{R}^n}\left|(2\pi)^{-\frac{n}{2}}\int_{\Omega}\xi_{p_1}\cdots\xi_{p_l}u_j(x)e^{ix\cdot\xi}dx\right|^2d\xi\\ =&\sum_{j=1}^{k}\sum_{p_1,\cdots,p_l=1}^{n}\int_{\mathbb{R}^n}\left|(2\pi)^{-\frac{n}{2}}\int_{\Omega}\frac{\partial^l u_j(x)}{\partial x_{p_1}\cdots\partial x_{p_l}}e^{ix\cdot\xi}dx\right|^2d\xi\\ =&\sum_{j=1}^{k}\sum_{p_1,\cdots,p_l=1}^{n}\int_{\mathbb{R}^n}\left|\frac{\widehat{\partial^l u_j(\xi)}}{\partial x_{p_1}\cdots\partial x_{p_l}}\right|^2d\xi\\ =&\sum_{j=1}^{k}\sum_{p_1,\cdots,p_l=1}^{n}\int_{\mathbb{R}^n}\left(\frac{\partial^l u_j(x)}{\partial x_{p_1}\cdots\partial x_{p_l}}\right)^2dx\\ =&\sum_{j=1}^{k}\int_{\Omega}u_j(x)(-\Delta)^l u_j(x)dx.\end{aligned} \tag{2.10}$$

Thus, it yields

$$\sum_{j=1}^{k}\lambda_j=\int_{\mathbb{R}^n}|\xi|^{2l}h(\xi)d\xi. \tag{2.11}$$

Making use of (2.11) and the properties of symmetric decreasing rearrangement, we obtain

$$\sum_{j=1}^{k}\lambda_j=\int_{\mathbb{R}^n}|\xi|^{2l}h(\xi)d\xi\ge\int_{\mathbb{R}^n}|\xi|^{2l}h^*(\xi)d\xi=n\omega_n\int_0^{+\infty}r^{n+2l-1}\psi(r)dr. \tag{2.12}$$

Noticing (2.5), (2.8) and (2.9), we can apply Lemma 1 to ψ with $b = n-1$ and $d = n+2l-1$. Therefore, using (2.12), we have

$$\sum_{j=1}^{k} \lambda_j \geq n\omega_n \int_0^{+\infty} r^{n+2l-1}\psi_(r)dr \geq n\omega_n \int_0^{+\infty} r^{n+2l-1}\Psi_s(r)dr \tag{2.13}$$

with $M = (2\pi)^{-n}V(\Omega)$, $m_* = \frac{k}{n\omega_n}$ and $L = 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)}$. Set $t = \frac{Ls}{M}$. Combining (2.9) and

$$\int_0^{+\infty} r^{n-1}\psi(r)dr = \int_0^{+\infty} r^{n-1}\Psi_s(r)dr = \frac{M^{n+1}}{n(n+1)L^n}\Big[(t+1)^{n+1} - t^{n+1}\Big],$$

it yields

$$(t+1)^{n+1} - t^{n+1} = k_*, \tag{2.14}$$

where

$$k_* = k\frac{(n+1)L^n}{\omega_n M^{n+1}}.$$

Set $\eta = t - \frac{1}{2}$. Then (2.14) becomes

$$(\eta + \frac{1}{2})^{n+1} - (\eta - \frac{1}{2})^{n+1} = k_*. \tag{2.15}$$

The asymptotic expansion for the unique positive root of (2.15) is

$$\eta(k_*) = \zeta - \frac{n-1}{24}\zeta^{-1} + \frac{(n-1)(n-3)(2n+1)}{5760}\zeta^{-3} + \cdots, \tag{2.16}$$

where $\zeta = \left(\frac{k_*}{n+1}\right)^{\frac{1}{n}}$. Then we can deduce

$$\begin{aligned}
&\Big(t(k^*)+1\Big)^{n+2l+1} - t(k^*)^{n+2l+1}\\
=&\binom{n+2l+1}{1}\zeta^{n+2l} + 2\left[\frac{1}{2^3}\binom{n+2l+1}{3} - \frac{n-1}{48}\binom{n+2l+1}{2}\binom{2}{1}\right]\zeta^{n+2l-2}\\
&+ 2\left[\frac{1}{2^5}\binom{n+2l+1}{5} - \frac{1}{2^3}\frac{n-1}{24}\binom{n+2l+1}{4}\binom{4}{1} + \frac{1}{2}\frac{(n-1)^2}{24^2}\binom{n+2l+1}{3}\binom{3}{1}\right.\\
&\left.+ \frac{1}{2}\frac{(n-1)(n-3)(2n+1)}{5760}\binom{n+2l+1}{2}\binom{2}{1}\right]\zeta^{n+2l-4} + \cdots\\
=&(n+2l+1)\left[\zeta^{n+2l} + \frac{l(n+2l)}{12}\zeta^{n+2l-2} + \frac{(n+2l)C(n,l)}{5760}\zeta^{n+2l-4} + \cdots\right],
\end{aligned} \tag{2.17}$$

where $\binom{q}{t} = \frac{q!}{t!(q-t)!}$ and

$$\begin{aligned}
C(n,l) =&(n+2l-1)\Big[(n+2l-2)(6l-7n+1) + 5(n-1)^2\Big]\\
&+ (n-1)(n-3)(2n+1).
\end{aligned}$$

Using (2.17), we get

$$\begin{aligned}
&n\omega_n\int_0^{+\infty} r^{n+2l-1}\Psi_s(r)dr\\
=&\frac{n\omega_n M^{n+2l+1}}{(n+2l)(n+2l+1)L^{n+2l}}\Big[\big(t(k_*)+1\big)^{n+2l+1}-t(k_*)^{n+2l+1}\Big]\\
=&\frac{n\omega_n M^{n+2l+1}}{(n+2l)L^{n+2l}}\Big[(\frac{k_*}{n+1})^{\frac{n+2l}{n}}+\frac{l(n+2l)}{12}(\frac{k_*}{n+1})^{\frac{n+2l-2}{n}}\\
&+\frac{(n+2l)C(n,l)}{5760}(\frac{k_*}{n+1})^{\frac{n+2l-4}{n}}+\cdots\Big].
\end{aligned} \tag{2.18}$$

Substituting $k_*=k\frac{(n+1)L^n}{\omega_n M^{n+1}}$, $M=(2\pi)^{-n}V(\Omega)$ and $L=2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)}$ into (2.18), we have

$$\begin{aligned}
&n\omega_n\int_0^{+\infty} r^{n+2l-1}\Psi_s(r)dr\\
=&\frac{n}{n+2l}\omega_n^{-\frac{2l}{n}}M^{-\frac{2l}{n}}k^{1+\frac{2l}{n}}+\frac{nl}{12}\omega_n^{-\frac{2l-2}{n}}\frac{M^{2-\frac{2l-2}{n}}}{L^2}k^{1+\frac{2l-2}{n}}\\
&+\frac{nC(n,l)}{5760}\omega_n^{-\frac{2l-4}{n}}\frac{M^{4-\frac{2l-4}{n}}}{L^4}k^{1+\frac{2l-4}{n}}+O(k^{1+\frac{2l-6}{n}})\\
=&\frac{n}{n+2l}\frac{(2\pi)^{2l}}{(\omega_n V(\Omega))^{\frac{2l}{n}}}k^{1+\frac{2l}{n}}+\frac{nl}{48}\frac{(2\pi)^{2l-2}}{(\omega_n V(\Omega))^{\frac{2l-2}{n}}}\frac{V(\Omega)}{I(\Omega)}k^{1+\frac{2l-2}{n}}\\
&+\frac{nC(n,l)}{92160}\frac{(2\pi)^{2l-4}}{(\omega_n V(\Omega))^{\frac{2l-4}{n}}}\left(\frac{V(\Omega)}{I(\Omega)}\right)^2k^{1+\frac{2l-4}{n}}+O(k^{1+\frac{2l-6}{n}}).
\end{aligned} \tag{2.19}$$

Inserting (2.19) into (2.13), we know that (1.14) is true. This completes the proof of Theorem 1. □

Proof of Thereom 2 It follows from (2.10) that

$$\begin{aligned}
\sum_{j=1}^k\Gamma_j=&\sum_{j=1}^k\int_\Omega u_j(x)\Big(\Delta^2u_j(x)-a\Delta u_j(x)\Big)dx\\
=&\int_{\mathbb{R}^n}|\xi|^4h(\xi)d\xi+a\int_{\mathbb{R}^n}|\xi|^2h(\xi)d\xi\\
\geq&\int_{\mathbb{R}^n}|\xi|^4h^*(\xi)d\xi+a\int_{\mathbb{R}^n}|\xi|^2h^*(\xi)d\xi\\
=&n\omega_n\Big(\int_0^{+\infty}r^{n+3}\psi(r)dr+a\int_0^{+\infty}r^{n+1}\psi(r)dr\Big).
\end{aligned} \tag{2.20}$$

Then, applying Lemma 1 to ψ and using (2.20), we obtain

$$\sum_{j=1}^k\Gamma_j\geq n\omega_n\Big(\int_0^{+\infty}r^{n+3}\Psi_s(r)dr+a\int_0^{+\infty}r^{n+1}\Psi_s(r)dr\Big). \tag{2.21}$$

Observe that $C(n,l) = -24n^2 + 96$ when $l = 2$ and $C(n,l) = -4(3n+2)(n-1)$ when $l = 1$. Therefore, from (2.19), we have

$$\begin{aligned} & n\omega_n\Big(\int_0^{+\infty} r^{n+3}\Psi_s(r)dr + a\int_0^{+\infty} r^{n+1}\Psi_s(r)dr\Big) \\ =& \frac{n}{n+4}\frac{(2\pi)^4}{(\omega_n V(\Omega))^{\frac{4}{n}}}k^{1+\frac{4}{n}} + \Big(\frac{n}{24}\frac{V(\Omega)}{I(\Omega)} + \frac{na}{n+2}\Big)\frac{(2\pi)^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}k^{1+\frac{2}{n}} \\ &+ \Big[-\frac{n(n^2-4)}{3840}\frac{V(\Omega)}{I(\Omega)} + a\frac{n}{48}\Big]\frac{V(\Omega)}{I(\Omega)}k + O(k^{1-\frac{2}{n}}). \end{aligned} \tag{2.22}$$

Then it is easy to find that (1.17) holds. This completes the proof of Theorem 2. □

Proof of Thereom 3 When $n = 2$, making use of (1.7) and (1.10), we have

$$\begin{aligned} \sum_{j=1}^k \Gamma_j =& \int_{\mathbb{R}^n} |\xi|^4 h(\xi)d\xi + a\int_{\mathbb{R}^n} |\xi|^2 h(\xi)d\xi \\ \geq& 2\omega_2\int_0^{+\infty} r^5\Psi_s(r)dr + 2a\omega_2\int_0^{+\infty} r^3\Psi_s(r)dr \\ \geq& \frac{1}{3}\frac{(2\pi)^4}{(\omega_2 V(\Omega))^2}k^3 + \Big(\frac{\alpha_2}{12I(\Omega)} + \frac{a}{2V(\Omega)}\Big)\frac{(2\pi)^2}{\omega_2}k^2 + \frac{a}{24}\beta_2\frac{V(\Omega)}{I(\Omega)}k, \end{aligned} \tag{2.23}$$

where $\alpha_2 = \frac{12095}{12096}$ and $\beta_2 = \frac{119}{120}$.

When $n = 3$, it follows from (2.21) that

$$\sum_{j=1}^k \Gamma_j \geq 3\omega_3\int_0^{+\infty} r^6\Psi_s(r)dr + 3a\omega_3\int_0^{+\infty} r^4\Psi_s(r)dr. \tag{2.24}$$

Now we make an estimate for the lower bound of $\int_0^{+\infty} r^6\Psi_s(r)dr$. Since

$$\int_0^{+\infty} r^6\Psi_s(r)dr = \frac{M^8}{56L^7}\Big[\big(t(k_*)+1\big)^8 - t(k_*)^8\Big],$$

we need to estimate $\big(t(k_*)+1\big)^8 - t(k_*)^8$. The equation (2.14) becomes $(t+1)^4 - t^4 = k_*$ when $n = 3$. Its positive root $t(k_*)$ is

$$t(k_*) = \frac{1}{2}\big(\rho(k_*) - \varrho(k_*)\big) - \frac{1}{2},$$

where $\rho(k_*) = \big(k_* + \sqrt{k_*^2 + \frac{1}{27}}\big)^{\frac{1}{3}}$ and $\varrho(k_*) = \big(-k_* + \sqrt{k_*^2 + \frac{1}{27}}\big)^{\frac{1}{3}}$. Set $\vartheta(k_*) = \frac{1}{2}\big(\rho(k_*) - \varrho(k_*)\big)$. Then we have

$$\begin{aligned} \big(t(k_*)+1\big)^8 - t(k_*)^8 =& 8\vartheta(k_*)^7 + 14\vartheta(k_*)^5 + \frac{7}{2}\vartheta(k_*)^3 + \frac{1}{8}\vartheta(k_*) \\ =& \frac{1}{16}\Big[\big(\rho(k_*) - \varrho(k_*)\big)^7 + 7\big(\rho(k_*) - \varrho(k_*)\big)^5 \\ &+ 7\big(\rho(k_*) - \varrho(k_*)\big)^3 + \big(\rho(k_*) - \varrho(k_*)\big)\Big]. \end{aligned} \tag{2.25}$$

Observe that

$$\rho(k_*)\cdot\varrho(k_*)=\frac{1}{3}. \tag{2.26}$$

Then, using (2.26), we have

$$\begin{aligned}&\big(\rho(k_*)-\varrho(k_*)\big)^7\\ =&\rho(k_*)\big(\rho(k_*)^6+7\varrho(k_*)^6\big)+21\rho(k_*)^2\varrho(k_*)^2\big(\rho(k_*)^3-\varrho(k_*)^3\big)\\ &-35\rho(k_*)^3\varrho(k_*)^3\big(\rho(k_*)-\varrho(k_*)\big)-\varrho(k_*)\big(7\rho(k_*)^6+\varrho(k_*)^6\big)\\ =&\big(k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{1}{3}}\big(16k_*^2-12k_*\sqrt{k_*^2+\frac{1}{27}}-1\big)+\frac{14}{3}k_*\\ &-\big(-k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{1}{3}}\big(16k_*^2+12k_*\sqrt{k_*^2+\frac{1}{27}}-1\big),\end{aligned} \tag{2.27}$$

$$\begin{aligned}&\big(\rho(k_*)-\varrho(k_*)\big)^5\\ =&\rho(k_*)^5-5\rho(k_*)\varrho(k_*)\big(\rho(k_*)^3-\varrho(k_*)^3\big)+10\rho(k_*)^2\varrho(k_*)^2\big(\rho(k_*)-\varrho(k_*)\big)-\varrho(k_*)^5\\ =&\big(k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{5}{3}}-\big(-k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{5}{3}}-\frac{10}{3}k_*+\frac{10}{9}\big(k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{1}{3}}\\ &-\frac{10}{9}\big(-k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{1}{3}}\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}&7\big(\rho(k_*)-\varrho(k_*)\big)^3+\big(\rho(k_*)-\varrho(k_*)\big)\\ =&7\big[\rho(k_*)^3-\varrho(k_*)^3-3\rho(k_*)\varrho(k_*)\big(\rho(k_*)-\varrho(k_*)\big)\big]+\big(\rho(k_*)-\varrho(k_*)\big)\\ =&14k_*-6\big(k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{1}{3}}+6\big(-k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{1}{3}}.\end{aligned} \tag{2.29}$$

Substituting (2.27-2.29) into (2.25), we obtain

$$\begin{aligned}&\big(t(k_*)+1\big)^8-t(k_*)^8\\ =&\big(k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{1}{3}}\big(k_*^2-\frac{3}{4}k_*\sqrt{k_*^2+\frac{1}{27}}\big)-\frac{7}{16}\big(-k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{5}{3}}\\ &+\Big[\frac{7}{16}\big(k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{5}{3}}-\big(-k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{1}{3}}\big(k_*^2+\frac{3}{4}k_*\sqrt{k_*^2+\frac{1}{27}}\big)\Big]\\ &+\frac{7}{144}\Big[\big(k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{1}{3}}-\big(-k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{1}{3}}\Big]-\frac{7}{24}k_*.\end{aligned} \tag{2.30}$$

Now we make some estimates for some terms in the right hand side of (2.30). The first term can be estimated as follows:

$$\big(k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{1}{3}}\big(k_*^2-\frac{3}{4}k_*\sqrt{k_*^2+\frac{1}{27}}\big)\geq\frac{2^{\frac{1}{3}}}{4}k_*^{\frac{7}{3}}-\frac{2^{\frac{1}{3}}}{72}k_*^{\frac{1}{3}}. \tag{2.31}$$

Here we use the inequality $\sqrt{k_*^2+\frac{1}{27}} \le k_* + \frac{1}{54k_*}$ since k_* is large. The second term is

$$-\frac{7}{16}\big(-k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{5}{3}} \ge -\frac{7\cdot 2^{\frac{1}{3}}}{16\cdot 54\cdot 18}k_*^{-\frac{5}{3}}. \tag{2.32}$$

The third term is

$$\begin{aligned}&\frac{7}{16}\big(k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{5}{3}} - \big(-k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{1}{3}}\big(k_*^2+\frac{3}{4}k_*\sqrt{k_*^2+\frac{1}{27}}\big)\\ \ge&\frac{7\cdot 2^{\frac{2}{3}}}{12}k_*^{\frac{5}{3}} - \frac{2^{\frac{2}{3}}}{432}k_*^{-\frac{1}{3}}.\end{aligned} \tag{2.33}$$

The fourth term is

$$\frac{7}{144}\Big[\big(k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{1}{3}} - \big(-k_*+\sqrt{k_*^2+\frac{1}{27}}\big)^{\frac{1}{3}}\Big] \ge \frac{7\cdot 2^{\frac{1}{3}}}{144}k_*^{\frac{1}{3}} - \frac{7\cdot 2^{\frac{2}{3}}}{16\cdot 54}k_*^{-\frac{1}{3}}. \tag{2.34}$$

Therefore, using (2.31-2.34) in (2.30), we have

$$\begin{aligned}\big(t(k_*)+1\big)^8 - t(k_*)^8 \ge&\frac{2^{\frac{1}{3}}}{4}k_*^{\frac{7}{3}} + \frac{7\cdot 2^{\frac{2}{3}}}{12}k_*^{\frac{5}{3}} - \frac{7}{24}k_* + \frac{5\cdot 2^{\frac{1}{3}}}{144}k_*^{\frac{1}{3}} - \frac{2^{\frac{2}{3}}}{96}k_*^{-\frac{1}{3}}\\ &- \frac{7\cdot 2^{\frac{1}{3}}}{16\cdot 54\cdot 18}k_*^{-\frac{5}{3}}\\ \ge&\frac{2^{\frac{1}{3}}}{4}k_*^{\frac{7}{3}} + \frac{7\cdot 2^{\frac{2}{3}}}{12}k_*^{\frac{5}{3}} - \frac{7}{24}k_*.\end{aligned} \tag{2.35}$$

Here we used the fact that $k_* \ge 1$. In fact, noticing $k_* \ge \frac{(n+1)(4\pi)^n}{\omega_n^2}(\frac{n}{n+2})^{\frac{n}{2}}$, it is not difficult to observe that $k_* = 4kL^3(\omega_3)^{-1}M^{-4} \ge \tau := \frac{432\sqrt{15}\pi}{25} \approx 210.25$ when $n=3$. Hence, when $\alpha \ge \frac{7}{24}\tau^{-\frac{2}{3}}$, the following inequality

$$\alpha k_*^{\frac{5}{3}} \ge \frac{7}{24}k_*,$$

holds for $k_* \in [\tau, +\infty)$. Since $1-\frac{6}{7}\cdot 2^{\frac{1}{3}}\alpha \le 1-\frac{1}{4}\cdot 2^{\frac{1}{3}}\tau^{-\frac{2}{3}} \approx 0.9911$, we can conclude that

$$\big(t(k_*)+1\big)^8 - t(k_*)^8 \ge \frac{2^{\frac{1}{3}}}{4}k_*^{\frac{7}{3}} + \frac{7\cdot 2^{\frac{2}{3}}}{12}\alpha_3 k_*^{\frac{5}{3}}, \tag{2.36}$$

where $\alpha_3 = 0.991$. Therefore, using (2.36), we derive

$$\begin{aligned}3\omega_3\int_0^{+\infty} r^6\Psi_s(r)dr =&\frac{3\omega_3 M^8}{56L^7}\Big[\big(t(k_*)+1\big)^8 - t(k_*)^8\Big]\\ \ge&\frac{3\cdot 2^{\frac{1}{3}}\omega_3 M^8}{224L^7}k_*^{\frac{7}{3}} + \frac{2^{\frac{2}{3}}\omega_3 M^8}{32L^7}\alpha_3 k_*^{\frac{5}{3}}\\ =&\frac{3}{7}\frac{(2\pi)^4}{(\omega_3 V(\Omega))^{\frac{4}{3}}}k^{\frac{7}{3}} + \frac{1}{8}\alpha_3\frac{(2\pi)^2}{(\omega_3 V(\Omega))^{\frac{2}{3}}}\frac{V(\Omega)}{I(\Omega)}k^{\frac{5}{3}}.\end{aligned} \tag{2.37}$$

At the same time, it follows from (1.7) that

$$3\omega_3 \int_0^{+\infty} r^4 \Psi_s(r)dr \geq \frac{3}{5}\frac{(2\pi)^2}{(\omega_3 V(\Omega))^{\frac{2}{3}}} k^{\frac{5}{3}} + \frac{1}{16}\beta_3 \frac{V(\Omega)}{I(\Omega)} k, \tag{2.38}$$

where $\beta_3 = 0.986$. Substituting (2.37) and (2.38) into (2.24), we obtain

$$\begin{aligned}\sum_{j=1}^{k} \Gamma_j \geq & \frac{3}{7}\frac{(2\pi)^4}{(\omega_3 V(\Omega))^{\frac{4}{3}}} k^{\frac{7}{3}} + \left(\frac{1}{8}\alpha_3 \frac{V(\Omega)}{I(\Omega)} + \frac{3a}{5}\right)\frac{(2\pi)^2}{(\omega_3 V(\Omega))^{\frac{2}{3}}} k^{\frac{5}{3}} \\ & + \frac{a}{16}\beta_3 \frac{V(\Omega)}{I(\Omega)} k.\end{aligned} \tag{2.39}$$

When $n = 4$, it follows from (2.21) that

$$\sum_{j=1}^{k} \Gamma_j \geq 4\omega_4 \int_0^{+\infty} r^7 \Psi_s(r)dr + 4a\omega_4 \int_0^{+\infty} r^5 \Psi_s(r)dr. \tag{2.40}$$

Now we make an estimate for the lower bound of $\int_0^{+\infty} r^7 \Psi_s(r)dr$. Since

$$\int_0^{+\infty} r^7 \Psi_s(r)dr = \frac{M^9}{72L^8}\left[\big(t(k_*) + 1\big)^9 - t(k_*)^9\right],$$

we need to estimate $\big(t(k_*)+1\big)^9 - t(k_*)^9$. The equation (2.14) becomes $(t+1)^5 - t^5 = k_*$ when $n = 4$. Its positive root $t(k_*)$ is

$$t(k_*) = \theta(k_*) - \frac{1}{2},$$

where $\theta(k_*) = \sqrt{\frac{\sqrt{20k_*+5}}{10} - \frac{1}{4}}$. Then we have

$$\begin{aligned}&\big(t(k_*) + 1\big)^9 - t(k_*)^9 \\ =& 9\theta(k_*)^8 + 21\theta(k_*)^6 + \frac{63}{8}\theta(k_*)^4 + \frac{9}{16}\theta(k_*)^2 + \frac{1}{2^8} \\ =& \frac{9}{25}k_*^2 + \frac{6}{25}k_*\sqrt{20k_* + 5} - \frac{18}{25}k_* + \frac{3}{50}\sqrt{20k_* + 5} - \frac{7}{50} \\ \geq& \frac{9}{25}k_*^2 + \frac{12\sqrt{5}}{25}k_*^{\frac{3}{2}} - \frac{18}{25}k_*.\end{aligned} \tag{2.41}$$

Here we used the fact that $k_* \geq 1$. In fact, noticing $k_* \geq \frac{(n+1)(4\pi)^n}{\omega_n^2}(\frac{n}{n+2})^{\frac{n}{2}}$, it is not difficult to observe that $k_* = 5kL^4(\omega_4)^{-1}M^{-5} \geq \sigma := \frac{5\cdot 2^{12}}{9} \approx 2275.56$ when $n = 4$. Hence, when $\alpha \geq \frac{18}{25}\sigma^{-\frac{1}{2}}$, the following inequality

$$\alpha k_*^{\frac{3}{2}} \geq \frac{18}{25}k_*,$$

holds for $k_* \in [\sigma, +\infty)$. Since $1 - \frac{5\sqrt{5}}{12}\alpha \le 1 - \frac{3\sqrt{5}}{10}\sigma^{-\frac{1}{2}} \approx 0.9859$, we can conclude that

$$\big(t(k_*)+1\big)^9 - t(k_*)^9 \ge \frac{9}{25}k_*^2 + \frac{12\sqrt{5}}{25}\alpha_4 k_*^{\frac{3}{2}}, \tag{2.42}$$

where $\alpha_4 = 0.985$. Therefore, using (2.42), we deduce

$$\begin{aligned} 4\omega_4 \int_0^{+\infty} r^7 \Psi_s(r)dr &= \frac{\omega_4 M^9}{18L^8}\left[\big(t(k_*)+1\big)^9 - t(k_*)^9\right] \\ &\ge \frac{\omega_4 M^9}{50L^8}k_*^2 + \frac{2\sqrt{5}\omega_4 M^9}{75L^8}\alpha_4 k_*^{\frac{3}{2}} \\ &= \frac{1}{2}\frac{(2\pi)^4}{\omega_4 V(\Omega)}k^2 + \frac{1}{6}\alpha_4 \frac{(2\pi)^2}{(\omega_4 V(\Omega))^{\frac{1}{2}}}\frac{V(\Omega)}{I(\Omega)}k^{\frac{3}{2}}. \end{aligned} \tag{2.43}$$

Meanwhile, from (1.7), we have

$$4\omega_4 \int_0^{+\infty} r^5 \Psi_s(r)dr \ge \frac{2}{3}\frac{(2\pi)^2}{(\omega_4 V(\Omega))^{\frac{1}{2}}}k^{\frac{3}{2}} + \frac{1}{12}\beta_4 \frac{V(\Omega)}{I(\Omega)}k, \tag{2.44}$$

where $\beta_4 = 0.983$. Substituting (2.43) and (2.44) into (2.40), we obtain

$$\begin{aligned} \sum_{j=1}^{k}\Gamma_j \ge& \frac{1}{2}\frac{(2\pi)^4}{\omega_4 V(\Omega)}k^2 + \left(\frac{1}{6}\alpha_4 \frac{V(\Omega)}{I(\Omega)} + \frac{2a}{3}\right)\frac{(2\pi)^2}{(\omega_4 V(\Omega))^{\frac{1}{2}}}k^{\frac{3}{2}} \\ &+ \frac{a}{12}\beta_4 \frac{V(\Omega)}{I(\Omega)}k. \end{aligned} \tag{2.45}$$

Therefore, synthesizing (2.23), (2.39) and (2.45), we conclude that (1.18) is true. This concludes the proof of Theorem 3. □

Proof of Thereom 4 When $n = 3$, using (2.35), we derive

$$\begin{aligned} 3\omega_3 \int_0^{+\infty} r^6 \Psi_s(r)dr \ge& \frac{3\cdot 2^{\frac{1}{3}}\omega_3 M^8}{224L^7}k_*^{\frac{7}{3}} + \frac{2^{\frac{2}{3}}\omega_3 M^8}{32L^7}k_*^{\frac{5}{3}} - \frac{\omega_3 M^8}{64L^7}k_* \\ =& \frac{3}{7}\frac{(2\pi)^4}{(\omega_3 V(\Omega))^{\frac{4}{3}}}k^{\frac{7}{3}} + \frac{1}{8}\frac{(2\pi)^2}{(\omega_3 V(\Omega))^{\frac{2}{3}}}\frac{V(\Omega)}{I(\Omega)}k^{\frac{5}{3}} - \frac{1}{256}\left(\frac{V(\Omega)}{I(\Omega)}\right)^2 k. \end{aligned} \tag{2.46}$$

Substituting (2.38) and (2.46) into (2.24), we have

$$\begin{aligned} \sum_{j=1}^{k}\Gamma_j \ge& \frac{3}{7}\frac{(2\pi)^4}{(\omega_3 V(\Omega))^{\frac{4}{3}}}k^{\frac{7}{3}} + \left(\frac{1}{8}\frac{V(\Omega)}{I(\Omega)} + \frac{3a}{5}\right)\frac{(2\pi)^2}{(\omega_3 V(\Omega))^{\frac{2}{3}}}k^{\frac{5}{3}} \\ &+ \left(-\frac{1}{256}\frac{V(\Omega)}{I(\Omega)} + \frac{a}{16}\beta_3\right)\frac{V(\Omega)}{I(\Omega)}k. \end{aligned} \tag{2.47}$$

When $n = 4$, it follows from (2.41) that

$$\begin{aligned} 4\omega_4 \int_0^{+\infty} r^7 \Psi_s(r) dr \geq & \frac{\omega_4 M^9}{50 L^8} k_*^2 + \frac{2\sqrt{5}\omega_4 M^9}{75 L^8} k_*^{\frac{3}{2}} - \frac{\omega_4 M^9}{25 L^8} k_* \\ = & \frac{1}{2} \frac{(2\pi)^4}{\omega_4 V(\Omega)} k^2 + \frac{1}{6} \frac{(2\pi)^2}{(\omega_4 V(\Omega))^{\frac{1}{2}}} \frac{V(\Omega)}{I(\Omega)} k^{\frac{3}{2}} - \frac{1}{80} \left(\frac{V(\Omega)}{I(\Omega)} \right)^2 k. \end{aligned} \tag{2.48}$$

Substituting (2.44) and (2.48) into (2.40), we obtain

$$\begin{aligned} \sum_{j=1}^{k} \Gamma_j \geq & \frac{1}{2} \frac{(2\pi)^4}{\omega_4 V(\Omega)} k^2 + \left(\frac{1}{6} \frac{V(\Omega)}{I(\Omega)} + \frac{2a}{3} \right) \frac{(2\pi)^2}{(\omega_4 V(\Omega))^{\frac{1}{2}}} k^{\frac{3}{2}} \\ & + \left(-\frac{1}{80} \frac{V(\Omega)}{I(\Omega)} + \frac{a}{12} \beta_4 \right) \frac{V(\Omega)}{I(\Omega)} k. \end{aligned} \tag{2.49}$$

Therefore, combining (2.47) and (2.49), we conclude that (1.19) is true. This completes the proof of Theorem 4. □

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