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Upper and lower bounds for eigenvalues of the clamped plate problem ☆

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ABSTRACT

In this paper, we study estimates for eigenvalues of the clamped plate problem. A sharp upper bound for eigenvalues is given and the lower bound for eigenvalues in Cheng and Wei (2011) [4] is improved.

1. Introduction

A membrane has its transverse vibration governed by equation

$$\Delta u = -\lambda u, \quad \text{in } \Omega$$

with the boundary condition

$$u = 0, \quad \text{on } \partial\Omega,$$

where Δ is the Laplacian in $\mathbf{R}^n$ and Ω is a bounded domain in $\mathbf{R}^n$. It is classical that there is a countable sequence of eigenvalues

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$$0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \to \infty,$$

and a sequence of corresponding eigenfunctions $u_1, u_2, \ldots, u_k, \ldots$ such that

$$\Delta u_k = -\lambda_k u_k, \quad \text{in } \Omega.$$

The eigenfunctions form an orthonormal basis of $L^2(\Omega)$.

On the other hand, the vibration of a stiff plate differs from that of a membrane not only in the equation which governs its motion but also in the way the plate is fastened to its boundary. A plate spanning a domain Ω in $\mathbf{R}^n$ has its transverse vibrations governed by

$$\begin{cases} \Delta^2 u = \Gamma u, & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ν denotes the outward unit normal to the boundary $\partial\Omega$. Namely, not only is the rim of the plate firmly fastened to the boundary, but the plate is clamped so that lateral motion can occur at the edge. One calls it *a clamped plate problem*. It is known that this problem has a real and discrete spectrum

$$0 < \Gamma_1 \leqslant \Gamma_2 \leqslant \cdots \leqslant \Gamma_k \leqslant \cdots \to +\infty,$$

where each Γ_i has finite multiplicity which is repeated according to its multiplicity.

For the eigenvalues of the clamped plate problem (1.1), Agmon [1] and Pleijel [13] gave the following asymptotic formula,

$$\Gamma_k \sim \frac{16\pi^4}{(B_n \operatorname{vol}(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}, \quad k \to \infty.$$

This implies that

$$\frac{1}{k}\sum_{j=1}^{k} \Gamma_j \sim \frac{n}{n+4} \frac{16\pi^4}{(B_n \operatorname{vol}(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}, \quad k \to \infty, \tag{1.2}$$

where B_n denotes the volume of the unit ball in $\mathbf{R}^n$. Furthermore, Levine and Protter [9] proved that the eigenvalues of the clamped plate problem (1.1) satisfy

$$\frac{1}{k}\sum_{j=1}^{k} \Gamma_j \geqslant \frac{n}{n+4} \frac{16\pi^4}{(B_n \operatorname{vol}(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}.$$

The formula (1.2) shows that the coefficient of $k^{\frac{4}{n}}$ is the best possible constant. Thus, it will be interesting and very important to find the second term on k of the asymptotic expansion formula of Γ_k. The authors [4] have made effort for this problem. We have improved the result due to Levine and Protter [9] by adding to its right hand side two terms of lower order in k.

On the other hand, if one can obtain an upper bound with optimal order of k for eigenvalue Γ_k, then one can know the exact second term on k. From our knowledge, there is no any result on upper bounds for eigenvalue Γ_k with optimal order of k. In [6], Cheng and Yang have established a recursion formula in order to obtain upper bounds for eigenvalues of the Dirichlet eigenvalue problem of the Laplacian. Hence, if one can get a sharper universal inequality for eigenvalues of the clamped plate problem, we can also derive an upper bound for eigenvalue Γ_k by making use of the recursion formula due to Cheng and Yang [6]. On the investigation of universal inequalities for eigenvalues of the clamped plate problem, Payne, Pólya and Weinberger [12] proved

$$\Gamma_{k+1} - \Gamma_k \leqslant \frac{8(n+2)}{n^2 k} \sum_{i=1}^{k} \Gamma_i.$$

Chen and Qian [3] and Hook [7], independently, extended the above inequality to

$$\frac{n^2k^2}{8(n+2)} \leqslant \sum_{i=1}^{k} \frac{\Gamma_i^{\frac{1}{2}}}{\Gamma_{k+1} - \Gamma_i} \sum_{i=1}^{k} \Gamma_i^{\frac{1}{2}}.$$

Recently, answering a question of Ashbaugh [2], Cheng and Yang [5] have proved the following remarkable estimate:

$$\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i) \leqslant \left(\frac{8(n+2)}{n^2}\right)^{\frac{1}{2}} \sum_{i=1}^{k} \big(\Gamma_i(\Gamma_{k+1} - \Gamma_i)\big)^{\frac{1}{2}}.$$

Furthermore, Wang and Xia [14] have proved

$$\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 \leqslant \frac{8(n+2)}{n^2} \sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)\Gamma_i.$$

The first author has conjectured the following:

Conjecture. *Eigenvalue Γ_j's of the clamped plate problem* (1.1) *satisfy*

$$\sum_{j=1}^{k} (\Gamma_{k+1} - \Gamma_j)^2 \leqslant \frac{8}{n} \sum_{j=1}^{k} (\Gamma_{k+1} - \Gamma_j)\Gamma_j. \tag{1.3}$$

If one can solve the above conjecture, then from the recursion formula of Cheng and Yang [6], we can derive an upper bound for the eigenvalue Γ_k with the optimal order of k. But it seems to be hard to solve this conjecture.

In this paper, we will try to use a fact that eigenfunctions of the clamped plate problem (1.1) form an orthonormal basis of the Sobolev space $W_0^{2,2}(\Omega)$ to get an upper bound for eigenvalues of the clamped plate problem (1.1). A similar fact for the Dirichlet eigenvalue problem of the Laplacian is also used by Li and Yau [10] and Kröger [8]. Furthermore, we will give an improvement of the lower bound for eigenvalues in [4].

Let Ω be a bounded domain with a smooth boundary $\partial\Omega$ in the n-dimensional Euclidean space $\mathbf{R}^n$. Let $d(x) = \mathrm{dist}(x, \partial\Omega)$ denote the distance function from the point x to the boundary $\partial\Omega$ of Ω. We define

$$\Omega_r = \left\{ x \in \Omega \;\middle|\; d(x) < \frac{1}{r} \right\}.$$

Theorem 1.1. *Let Ω be a bounded domain with a smooth boundary $\partial\Omega$ in $\mathbf{R}^n$. Then there exists a constant $r_0 > 0$ such that eigenvalues of the clamped plate problem* (1.1) *satisfy*

$$\frac{1}{k}\sum_{j=1}^{k}\Gamma_j \leqslant \frac{1+\frac{4(n+4)(n^2+2n+6)}{n+2}\frac{\mathrm{vol}(\Omega_{r_0})}{\mathrm{vol}(\Omega)}}{(1-\frac{\mathrm{vol}(\Omega_{r_0})}{\mathrm{vol}(\Omega)})^{\frac{n+4}{n}}}\frac{n}{n+4}\frac{16\pi^4}{(B_n\,\mathrm{vol}(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}}, \tag{1.4}$$

for $k \geqslant \mathrm{vol}(\Omega)r_0^n$.

Remark 1.1. Since $\mathrm{vol}(\Omega_{r_0}) \to 0$ when $r_0 \to \infty$, we know that the upper bound in Theorem 1.1 is sharp in the sense of the asymptotic formula due to Agmon and Pleijel.

Corollary 1.1. *Let* Ω *be a bounded domain with a smooth boundary* $\partial\Omega$ *in* $\mathbf{R}^n$. *If there exists a constant* c_0 *such that*

$$\mathrm{vol}(\Omega_r) \leqslant c_0\,\mathrm{vol}(\Omega)^{\frac{n-1}{n}}\frac{1}{r}$$

for $r > \mathrm{vol}(\Omega)^{\frac{-1}{n}}$, *then there exists a constant* r_0 *such that eigenvalues of the clamped plate problem* (1.1) *satisfy*

$$\frac{1}{k}\sum_{j=1}^{k}\Gamma_j \leqslant \frac{n}{n+4}\frac{16\pi^4}{(B_n\,\mathrm{vol}(\Omega))^{\frac{4}{n}}}\left(k^{\frac{4}{n}}+c_0c(n)k^{\frac{3}{n}}\right), \tag{1.5}$$

for $k = \mathrm{vol}(\Omega)r_0^n > c_0^n$, *where* $c(n)$ *is a constant depending only on* n.

Theorem 1.2. *Let* Ω *be a bounded domain with a piecewise smooth boundary* $\partial\Omega$ *in* $\mathbf{R}^n$. *Eigenvalue* Γ_j*'s of the clamped plate problem* (1.1) *satisfy*

$$\begin{aligned}\frac{1}{k}\sum_{j=1}^{k}\Gamma_j \geqslant{}& \frac{n}{n+4}\frac{16\pi^4}{(B_n\,\mathrm{vol}(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}}\\ &+\frac{n+2}{12n(n+4)}\frac{\mathrm{vol}(\Omega)}{I(\Omega)}\frac{n}{n+2}\frac{4\pi^2}{(B_n\,\mathrm{vol}(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}}\\ &+\frac{(n+2)^2}{1152n(n+4)^2}\left(\frac{\mathrm{vol}(\Omega)}{I(\Omega)}\right)^2,\end{aligned} \tag{1.6}$$

where $I(\Omega)$ *is the moment of inertia of* Ω.

2. Upper bounds for eigenvalues

In this section, we will study the upper bounds for eigenvalues of the clamped plate problem (1.1).

Proof of Theorem 1.1. Since $d(x)$ is the distance function from the point x to the boundary $\partial\Omega$ of Ω, we define a function f_r for any fixed r by

$$f_r(x) = \begin{cases} 1, & x\in\Omega,\ d(x)\geqslant\frac{1}{r},\\ r^2d^2(x), & x\in\Omega,\ d(x)<\frac{1}{r},\\ 0, & \text{the other}.\end{cases} \tag{2.1}$$

Let u_j be an orthonormal eigenfunction corresponding to the eigenvalue Γ_j, that is, u_j satisfies

$$\begin{cases} \Delta^2 u_j = \Gamma_j u_j, & \text{in } \Omega, \\ u_j = \dfrac{\partial u_j}{\partial \nu} = 0, & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega u_i(x) u_j(x)\, dx = \delta_{ij}, & \text{for any } i, j. \end{cases}$$

Thus, $\{u_j\}$ forms an orthonormal basis of the Sobolev space $W_0^{2,2}(\Omega)$. For an arbitrary fixed point $z \in \mathbf{R}^n$ and $r > 0$, a function

$$g_{r,z}(x) = e^{i\langle z,x\rangle} f_r(x), \tag{2.2}$$

with $i = \sqrt{-1}$, belongs to the Sobolev space $W_0^{2,2}(\Omega)$. Hence, we have

$$g_{r,z}(x) = \sum_{j=1}^{\infty} a_{r,j}(z) u_j(x), \tag{2.3}$$

where

$$a_{r,j}(z) = \int_\Omega g_{r,z}(x) u_j(x)\, dx. \tag{2.4}$$

Defining a function

$$\varphi_k(x) = g_{r,z}(x) - \sum_{j=1}^{k} a_{r,j}(z) u_j(x), \tag{2.5}$$

we have $\varphi_k = \frac{\partial \varphi_k}{\partial \nu} = 0$ on $\partial\Omega$ and

$$\int_\Omega \varphi_k(x) u_j(x)\, dx = 0, \quad \text{for } j = 1, 2, \ldots, k.$$

Therefore, φ_k is a trial function. From the Rayleigh–Ritz formula, we have

$$\Gamma_{k+1} \int_\Omega \left|\varphi_k(x)\right|^2 dx \leqslant \int_\Omega \left|\Delta \varphi_k(x)\right|^2 dx. \tag{2.6}$$

From the definition of φ_k and (2.1), we have

$$\int_\Omega \left|\varphi_k(x)\right|^2 dx = \int_\Omega \left| g_{r,z}(x) - \sum_{j=1}^{k} a_{r,j}(z) u_j(x) \right|^2$$

$$= \int_\Omega \left|f_r(x)\right|^2 dx - \sum_{j=1}^k \left|a_{r,j}(z)\right|^2$$

$$\geqslant \mathrm{vol}(\Omega) - \mathrm{vol}(\Omega_r) - \sum_{j=1}^k \left|a_{r,j}(z)\right|^2. \tag{2.7}$$

From (2.5) and Stokes' formula, we infer

$$\begin{aligned}
\int_\Omega \left|\Delta\varphi_k(x)\right|^2 dx &= \int_\Omega \left|\Delta g_{r,z}(x) - \sum_{j=1}^k a_{r,j}(z)\Delta u_j(x)\right|^2 dx \\
&= \int_\Omega \left(\left|\Delta g_{r,z}(x)\right|^2 + \left|\sum_{j=1}^k a_{r,j}(z)\Delta u_j(x)\right|^2\right) dx \\
&\quad - \int_\Omega \left(\Delta g_{r,z}(x)\sum_{j=1}^k \overline{a_{r,j}(z)}\Delta u_j(x) + \overline{\Delta g_{r,z}(x)}\sum_{j=1}^k a_{r,j}(z)\Delta u_j(x)\right) dx \\
&= \int_\Omega \left|\Delta g_{r,z}(x)\right|^2 dx - \sum_{j=1}^k \Gamma_j \left|a_{r,j}(z)\right|^2 \\
&= \int_\Omega \left|-|z|^2 f_r(x) + 2i\langle z, \nabla f_r(x)\rangle + \Delta f_r(x)\right|^2 dx - \sum_{j=1}^k \Gamma_j \left|a_{r,j}(z)\right|^2 \\
&= \int_\Omega \left\{\left(-|z|^2 f_r(x) + \Delta f_r(x)\right)^2 + 4\langle z, \nabla f_r(x)\rangle^2\right\} dx - \sum_{j=1}^k \Gamma_j \left|a_{r,j}(z)\right|^2
\end{aligned} \tag{2.8}$$

since

$$\Delta g_{r,z}(x) = e^{i\langle z,x\rangle}\left(-|z|^2 f_r(x) + 2i\langle z, \nabla f_r(x)\rangle + \Delta f_r(x)\right).$$

According to the definition of the function f_r, we have

$$\Delta f_r(x) = \begin{cases} 0, & x \in \Omega,\ d(x) \geqslant \frac{1}{r}, \\ r^2 \Delta d^2(x), & x \in \Omega,\ d(x) < \frac{1}{r}, \\ 0, & \text{the other.} \end{cases}$$

Hence, we obtain, from the Schwarz inequality and $|\nabla d(x)|^2 = 1$,

$$\begin{aligned}
&\int_\Omega \left\{\left(-|z|^2 f_r(x) + \Delta f_r(x)\right)^2 + 4\langle z, \nabla f_r(x)\rangle^2\right\} dx \\
&\quad \leqslant |z|^4 \mathrm{vol}(\Omega) + 24r^2|z|^2 \mathrm{vol}(\Omega_r) + \int_{\Omega_r} \left(\Delta f_r(x)\right)^2 dx.
\end{aligned} \tag{2.9}$$

For a point $x \in \Omega$, there is a point $y = y(x) \in \partial\Omega$ such that $d(x) = \mathrm{dist}(x, y)$, then we know that

$$\Delta d^2(x) = 2n - \sum_{j=1}^{n-1} \frac{2}{1 - \kappa_j d(x)}, \tag{2.10}$$

where $\kappa_1, \kappa_2, \dots, \kappa_{n-1}$ are the principal curvatures of $\partial\Omega$ at the point y. Since the boundary $\partial\Omega$ of the domain Ω is smooth and a compact hypersurface, one has that all of κ_j are bounded. Without loss of generality, we can assume that $|\kappa_j(y)| \leqslant \kappa$ for any $y \in \partial\Omega$, $1 \leqslant j \leqslant n-1$, then it follows that if $r \geqslant r_0 > n\kappa$, then we see from (2.10)

$$0 < \Delta d^2(x) < 2n, \quad x \in \Omega_r$$

and

$$\int_{\Omega_r} \big(\Delta f_r(x)\big)^2 \, dx \leqslant 4n^2 r^4 \operatorname{vol}(\Omega_r).$$

Hence, if $r > r_0$, then we obtain

$$\int_{\Omega} \big|\Delta\varphi_k(x)\big|^2 \, dx \leqslant |z|^4 \operatorname{vol}(\Omega) + 24r^2|z|^2 \operatorname{vol}(\Omega_r) + 4n^2 r^4 \operatorname{vol}(\Omega_r) - \sum_{j=1}^{k} \Gamma_j \big|a_{r,j}(z)\big|^2. \tag{2.11}$$

From (2.6), (2.7) and (2.11), we have

$$\begin{aligned} &\Gamma_{k+1}\big(\operatorname{vol}(\Omega) - \operatorname{vol}(\Omega_r)\big) \\ &\quad \leqslant |z|^4 \operatorname{vol}(\Omega) + 24r^2|z|^2 \operatorname{vol}(\Omega_r) + 4n^2 r^4 \operatorname{vol}(\Omega_r) + \sum_{j=1}^{k} (\Gamma_{k+1} - \Gamma_j)\big|a_{r,j}(z)\big|^2, \end{aligned} \tag{2.12}$$

here $r > r_0$.

Let $B_n(r)$ denote the ball with a radius r and the origin 0 in $\mathbf{R}^n$. By integrating the above inequality on the variable z on the ball $B_n(r)$, we derive

$$\begin{aligned} &r^n B_n\big(\operatorname{vol}(\Omega) - \operatorname{vol}(\Omega_r)\big)\Gamma_{k+1} \\ &\quad \leqslant r^{n+4} B_n\left(\frac{n}{n+4}\operatorname{vol}(\Omega) + 24\frac{n}{n+2}\operatorname{vol}(\Omega_r) + 4n^2 \operatorname{vol}(\Omega_r)\right) \\ &\qquad + \sum_{j=1}^{k} (\Gamma_{k+1} - \Gamma_j) \int_{B_n(r)} \big|a_{r,j}(z)\big|^2 \, dz, \quad r > r_0. \end{aligned} \tag{2.13}$$

From Parseval's identity for Fourier transform, we have

$$\begin{aligned} \int_{B_n(r)} \big|a_{r,j}(z)\big|^2 \, dz &\leqslant \int_{\mathbf{R}^n} \big|a_{r,j}(z)\big|^2 \, dz \\ &= \int_{\mathbf{R}^n} \left| \int_{\mathbf{R}^n} e^{i\langle z,x\rangle} f_r(x) u_j(x) \, dx \right|^2 dz \end{aligned}$$

$$= (2\pi)^n \int_{\mathbf{R}^n} \left| \widehat{f_r u_j}(z) \right|^2 dz = (2\pi)^n \int_{\mathbf{R}^n} \left| f_r(x) u_j(x) \right|^2 dx$$
$$\leqslant (2\pi)^n. \tag{2.14}$$

We obtain

$$\begin{aligned} &r^n B_n \big(\mathrm{vol}(\Omega) - \mathrm{vol}(\Omega_r) \big) \Gamma_{k+1} \\ &\quad \leqslant r^{n+4} B_n \left(\frac{n}{n+4} \mathrm{vol}(\Omega) + 24 \frac{n}{n+2} \mathrm{vol}(\Omega_r) + 4n^2 \mathrm{vol}(\Omega_r) \right) \\ &\qquad + (2\pi)^n \sum_{j=1}^{k} (\Gamma_{k+1} - \Gamma_j), \quad r > r_0. \end{aligned} \tag{2.15}$$

Taking $r = 2\pi (\frac{1+k}{B_n(\mathrm{vol}(\Omega)-\mathrm{vol}(\Omega_{r_0}))})^{\frac{1}{n}}$, noting $k \geqslant \mathrm{vol}(\Omega) r_0^n$ and $\frac{2\pi}{(B_n)^{\frac{1}{n}}} > 1$, then we can obtain $r > r_0$ and

$$\begin{aligned} \frac{1}{1+k} \sum_{j=1}^{k+1} \Gamma_j &\leqslant 16\pi^4 \frac{\frac{n}{n+4} \mathrm{vol}(\Omega) + \frac{24n}{n+2} \mathrm{vol}(\Omega_r) + 4n^2 \mathrm{vol}(\Omega_r)}{(\mathrm{vol}(\Omega) - \mathrm{vol}(\Omega_{r_0}))^{\frac{n+4}{n}}} \frac{1}{B_n^{\frac{4}{n}}} (1+k)^{\frac{4}{n}} \\ &\leqslant 16\pi^4 \frac{\frac{n}{n+4} + (24\frac{n}{n+2} + 4n^2) \frac{\mathrm{vol}(\Omega_{r_0})}{\mathrm{vol}(\Omega)}}{(1 - \frac{\mathrm{vol}(\Omega_{r_0})}{\mathrm{vol}(\Omega)})^{\frac{n+4}{n}}} \frac{1}{(B_n \mathrm{vol}(\Omega))^{\frac{4}{n}}} (1+k)^{\frac{4}{n}}. \end{aligned} \tag{2.16}$$

This completes the proof of Theorem 1.1. □

Proof of Corollary 1.1. From (2.16) we have

$$\frac{1}{1+k} \sum_{j=1}^{k+1} \Gamma_j \leqslant \frac{1 + 4(\frac{6}{n+2} + n)(n+4) \frac{\mathrm{vol}(\Omega_r)}{\mathrm{vol}(\Omega)}}{(1 - \frac{\mathrm{vol}(\Omega_{r_0})}{\mathrm{vol}(\Omega)})^{\frac{n+4}{n}}} \frac{n}{n+4} \frac{16\pi^4}{(B_n \mathrm{vol}(\Omega))^{\frac{4}{n}}} (1+k)^{\frac{4}{n}}. \tag{2.17}$$

Since $r = 2\pi (\frac{1+k}{B_n(\mathrm{vol}(\Omega)-\mathrm{vol}(\Omega_{r_0}))})^{\frac{1}{n}}$, we have

$$\frac{\mathrm{vol}(\Omega_r)}{\mathrm{vol}(\Omega)} \leqslant c_0 \frac{B_n^{\frac{1}{n}}}{2\pi} \left(1 - \frac{\mathrm{vol}(\Omega_{r_0})}{\mathrm{vol}(\Omega)} \right)^{\frac{1}{n}} (1+k)^{-\frac{1}{n}}.$$

Taking $c_1 = 4(\frac{6}{n+2} + n)(n+4) \frac{B_n^{\frac{1}{n}}}{2\pi} c_0$, we have

$$\frac{1}{1+k} \sum_{j=1}^{k+1} \Gamma_j \leqslant \frac{1 + c_1 (1 - \frac{\mathrm{vol}(\Omega_{r_0})}{\mathrm{vol}(\Omega)})^{\frac{1}{n}} (1+k)^{-\frac{1}{n}}}{(1 - \frac{\mathrm{vol}(\Omega_{r_0})}{\mathrm{vol}(\Omega)})^{\frac{n+4}{n}}} \frac{n}{n+4} \frac{16\pi^4}{(B_n \mathrm{vol}(\Omega))^{\frac{4}{n}}} (1+k)^{\frac{4}{n}}. \tag{2.18}$$

Since there exists a constant α such that

$$0 < v = \frac{\mathrm{vol}(\Omega_{r_0})}{\mathrm{vol}(\Omega)} \leqslant \frac{c_0}{(1+k)^{\frac{1}{n}}} \leqslant \alpha < 1$$

with $r_0 = (\frac{1+k}{\mathrm{vol}(\Omega)})^{\frac{1}{n}}$, we define a function

$$G(v) = \frac{1 + c_1(1-v)^{\frac{1}{n}}(1+k)^{-\frac{1}{n}}}{(1-v)^{\frac{n+4}{n}}}$$

with $G(0) = 1 + c_1(1+k)^{-\frac{1}{n}}$. Since

$$G'(v) = \frac{1 + \frac{4}{n} + c_1(1+\frac{3}{n})(1-v)^{\frac{1}{n}}(1+k)^{-\frac{1}{n}}}{(1-v)^{\frac{2n+4}{n}}},$$

by Lagrange mean value theorem, there exists $0 < \theta < 1$ such that

$$G(v) = G(0) + G'(\theta v)v.$$

Hence, there exists a constant $c(n)$ only depending on n such that

$$\begin{aligned} G(v) &= G(0) + G'(\theta v)v \\ &= 1 + c_1(1+k)^{-\frac{1}{n}} + \frac{1 + \frac{4}{n} + c_1(1+\frac{3}{n})(1-\theta v)^{\frac{1}{n}}(1+k)^{-\frac{1}{n}}}{(1-\theta v)^{\frac{2n+4}{n}}} v \\ &\leqslant 1 + c_1(1+k)^{-\frac{1}{n}} + \frac{1 + \frac{4}{n} + c_1(1+\frac{3}{n})}{(1-\theta\alpha)^{\frac{2n+4}{n}}} c_0(1+k)^{-\frac{1}{n}} \\ &\leqslant 1 + c_0 c(n)(1+k)^{-\frac{1}{n}}, \end{aligned}$$

that is,

$$\frac{1 + c_1(1 - \frac{\mathrm{vol}(\Omega_{r_0})}{\mathrm{vol}(\Omega)})^{\frac{1}{n}}(1+k)^{-\frac{1}{n}}}{(1 - \frac{\mathrm{vol}(\Omega_{r_0})}{\mathrm{vol}(\Omega)})^{\frac{n+4}{n}}} \leqslant 1 + c_0 c(n)(1+k)^{-\frac{1}{n}}.$$

Therefore, we obtain

$$\frac{1}{1+k}\sum_{j=1}^{k+1} \Gamma_j \leqslant \frac{n}{n+4} \frac{16\pi^4}{(B_n \,\mathrm{vol}(\Omega))^{\frac{4}{n}}} \left((1+k)^{\frac{4}{n}} + c_0 c(n)(1+k)^{\frac{3}{n}}\right).$$

This finishes the proof of Corollary 1.1. □

3. Lower bounds for eigenvalues

In this section, we will give a proof of Theorem 1.2. The following Lemma 3.1 will play an important role in the proof of Theorem 1.2.

Lemma 3.1. *For constants $b \geqslant 2$, $\eta > 0$, if $\psi : [0, +\infty) \to [0, +\infty)$ is a decreasing function such that*

$$-\eta \leqslant \psi'(s) \leqslant 0$$

and

$$A := \int_0^\infty s^{b-1}\psi(s)\,ds > 0,$$

then, we have

$$\begin{aligned}\int_0^\infty s^{b+3}\psi(s)\,ds &\geqslant \frac{1}{b+4}(bA)^{\frac{b+4}{b}}\psi(0)^{-\frac{4}{b}} + \frac{1}{3b(b+4)\eta^2}(bA)^{\frac{b+2}{b}}\psi(0)^{\frac{2b-2}{b}} \\ &\quad + \frac{(b+2)^2}{72b(b+4)^2\eta^4}A\psi(0)^4 + \frac{q(b)}{\eta^6}(bA)^{\frac{b-2}{b}}\psi(0)^{\frac{6b+2}{b}} \\ &\geqslant \frac{1}{b+4}(bA)^{\frac{b+4}{b}}\psi(0)^{-\frac{4}{b}} + \frac{1}{3b(b+4)\eta^2}(bA)^{\frac{b+2}{b}}\psi(0)^{\frac{2b-2}{b}} \\ &\quad + \frac{(b+2)^2}{72b(b+4)^2\eta^4}A\psi(0)^4, \end{aligned} \tag{3.1}$$

where

$$q(b) = \begin{cases} \frac{(13b^3+56b^2-52b-32)(b+2)^3}{(12)^4b^4(b+4)^4}, & \textit{for } b \geqslant 4 \textit{ or } b = 2, \\ \frac{(4b^3+11b^2-16b+4)(b+2)^3}{3\times(12)^3b^3(b+4)^3\eta^6}, & \textit{for } 2 < b < 4. \end{cases}$$

Proof. By defining

$$\varphi(t) = \frac{\psi(\frac{\psi(0)}{\eta}t)}{\psi(0)},$$

we have $\varphi(0) = 1$ and $-1 \leqslant \varphi'(t) \leqslant 0$. Hence, without loss of generality, we can assume

$$\psi(0) = 1 \quad \text{and} \quad \eta = 1.$$

Define

$$D := \int_0^\infty s^{b+3}\psi(s)\,ds. \tag{3.2}$$

If $D = \infty$, the conclusion is correct. Hence, one can assume that

$$D = \int_0^\infty s^{b+3}\psi(s)\,ds < \infty.$$

Thus, $\lim_{s\to\infty} s^{b+3}\psi(s) = 0$ holds. Putting $h(s) = -\psi'(s)$ for $s \geqslant 0$, we have

$$0 \leqslant h(s) \leqslant 1, \quad \int_0^\infty h(s)\,ds = \psi(0) = 1.$$

By making use of integration by parts, one has

$$\int_0^\infty s^b h(s)\,ds = b\int_0^\infty s^{b-1}\psi(s)\,ds = bA, \tag{3.3}$$

$$\int_0^\infty s^{b+4} h(s)\,ds \leqslant (b+4)D \tag{3.4}$$

since $\psi(s) \geqslant 0$. By the same assertion as in [11], one can infer that there exists an $\epsilon \geqslant 0$ such that

$$\int_\epsilon^{\epsilon+1} s^b\,ds = \int_0^\infty s^b h(s)\,ds = bA, \tag{3.5}$$

$$\int_\epsilon^{\epsilon+1} s^{b+4}\,ds \leqslant \int_0^\infty s^{b+4} h(s)\,ds \leqslant (b+4)D. \tag{3.6}$$

Since function $f(s)$ defined by

$$f(s) = bs^{b+4} - (b+4)\tau^4 s^b + 4\tau^{b+4} - 4\tau^{b+2}(s-\tau)^2, \quad \text{for any } \tau > 0, \tag{3.7}$$

only has two critical points, one is $s = \tau$, the other one is in the interval $(0, \tau)$, we have $f(s) \geqslant 0$. By integrating the function $f(s)$ from ϵ to $\epsilon + 1$, we deduce, from (3.5) and (3.6),

$$b(b+4)D - (b+4)\tau^4 bA + 4\tau^{b+4} \geqslant \frac{1}{3}\tau^{b+2}, \quad \text{for any } \tau > 0. \tag{3.8}$$

Hence, we have, for any $\tau > 0$,

$$\int_0^\infty s^{b+3}\psi(s)\,ds = D \geqslant \frac{1}{b(b+4)}\left\{(b+4)\tau^4 bA - 4\tau^{b+4} + \frac{1}{3}\tau^{b+2}\right\}. \tag{3.9}$$

For $b \geqslant 4$ or $b = 2$, we have, from Taylor expansion formula,

$$(1+t)^{\frac{4}{b}} \geqslant 1 + \frac{4}{b}t + \frac{2(4-b)}{b^2}t^2 + \frac{2(4-b)(4-2b)}{3b^3}t^3 + \frac{(4-b)(2-b)(4-3b)}{3b^4}t^4,$$

$$(1+t)^{\frac{b+2}{b}} \geqslant 1 + \frac{(b+2)}{b}t + \frac{(b+2)}{b^2}t^2 + \frac{(b+2)(2-b)}{3b^3}t^3.$$

Since it is not hard to prove

$$\frac{1}{b+1} = \int_0^1 s^b\,ds \leqslant \int_0^\infty s^b h(s)\,ds = bA, \tag{3.10}$$

by making use of the inequality $(s^b - 1)(h(s) - \chi(s)) \geqslant 0$ for $s \in [0, \infty)$, where χ is the characteristic function of the interval $[0, 1]$, we have

$$(b+1)bA \geqslant 1.$$

Taking

$$\tau = (bA)^{\frac{1}{b}}\left(1 + \frac{b+2}{12(b+4)}(bA)^{\frac{-2}{b}}\right)^{\frac{1}{b}},$$

we have

$$\begin{aligned}
&(b+4)\tau^4 bA - 4\tau^{b+4} + \frac{1}{3}\tau^{b+2} \\
&\quad = (bA)^{1+\frac{4}{b}}\left(b - \frac{b+2}{3(b+4)}(bA)^{\frac{-2}{b}}\right)\left(1 + \frac{b+2}{12(b+4)}(bA)^{\frac{-2}{b}}\right)^{\frac{4}{b}} \\
&\qquad + \frac{1}{3}(bA)^{1+\frac{2}{b}}\left(1 + \frac{b+2}{12(b+4)}(bA)^{\frac{-2}{b}}\right)^{\frac{b+2}{b}}.
\end{aligned} \tag{3.11}$$

Putting

$$t = \frac{b+2}{12(b+4)}(bA)^{\frac{-2}{b}},$$

we derive, for $b \geqslant 4$ or $b = 2$,

$$\begin{aligned}
&\left(b - \frac{b+2}{3(b+4)}(bA)^{\frac{-2}{b}}\right)\left(1 + \frac{b+2}{12(b+4)}(bA)^{\frac{-2}{b}}\right)^{\frac{4}{b}} \\
&\quad = (b-4t)(1+t)^{\frac{4}{b}} \\
&\quad \geqslant (b-4t)\left(1 + \frac{4}{b}t + \frac{2(4-b)}{b^2}t^2 + \frac{2(4-b)(4-2b)}{3b^3}t^3 \right. \\
&\qquad \left. + \frac{(4-b)(2-b)(4-3b)}{3b^4}t^4\right) \\
&\quad = b - \frac{2(4+b)}{b}t^2 - \frac{4(4-b)(4+b)}{3b^2}t^3 - \frac{(4-b)(2-b)(4+b)}{b^3}t^4 \\
&\qquad - \frac{4(4-b)(2-b)(4-3b)}{3b^4}t^5 \\
&\quad \geqslant b - \frac{2(4+b)}{b}t^2 - \frac{4(4-b)(4+b)}{3b^2}t^3 - \frac{(4-b)(2-b)(4+b)}{b^3}t^4 \\
&\quad = b - \frac{2(4+b)}{b}\left(\frac{b+2}{12(b+4)}(bA)^{\frac{-2}{b}}\right)^2 - \frac{4(4-b)(4+b)}{3b^2}\left(\frac{b+2}{12(b+4)}(bA)^{\frac{-2}{b}}\right)^3 \\
&\qquad - \frac{(4-b)(2-b)(4+b)}{b^3}\left(\frac{b+2}{12(b+4)}(bA)^{\frac{-2}{b}}\right)^4,
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&\left(1+\frac{b+2}{12(b+4)}(bA)^{\frac{-2}{b}}\right)^{\frac{b+2}{b}}\\
&\quad=(1+t)^{\frac{b+2}{b}}\\
&\quad\geqslant 1+\frac{(b+2)}{b}t+\frac{(b+2)}{b^2}t^2+\frac{(b+2)(2-b)}{3b^3}t^3\\
&\quad=1+\frac{2+b}{b}\left(\frac{b+2}{12(b+4)}(bA)^{\frac{-2}{b}}\right)+\frac{2+b}{b^2}\left(\frac{b+2}{12(b+4)}(bA)^{\frac{-2}{b}}\right)^2\\
&\qquad+\frac{(b+2)(2-b)}{3b^3}\left(\frac{b+2}{12(b+4)}(bA)^{\frac{-2}{b}}\right)^3. \qquad (3.13)
\end{aligned}$$

From (3.11), (3.12) and (3.13), we obtain

$$\begin{aligned}
&(b+4)\tau^4 bA-4\tau^{b+4}+\frac{1}{3}\tau^{b+2}\\
&\quad\geqslant b(bA)^{1+\frac{4}{b}}-\frac{2(4+b)}{b}\left(\frac{b+2}{12(b+4)}\right)^2(bA)\\
&\qquad-\frac{4(4-b)(4+b)}{3b^2}\left(\frac{b+2}{12(b+4)}\right)^3(bA)^{1-\frac{2}{b}}\\
&\qquad-\frac{(4-b)(2-b)(4+b)}{b^3}\left(\frac{b+2}{12(b+4)}\right)^4(bA)^{1-\frac{4}{b}}\\
&\qquad+\frac{1}{3}(bA)^{1+\frac{2}{b}}+\frac{2+b}{3b}\left(\frac{b+2}{12(b+4)}\right)(bA)+\frac{2+b}{3b^2}\left(\frac{b+2}{12(b+4)}\right)^2(bA)^{1-\frac{2}{b}}\\
&\qquad+\frac{(b+2)(2-b)}{9b^3}\left(\frac{b+2}{12(b+4)}\right)^3(bA)^{1-\frac{4}{b}}\\
&\quad=b(bA)^{1+\frac{4}{b}}+\frac{1}{3}(bA)^{1+\frac{2}{b}}+\frac{1}{72}\frac{(b+2)^2}{b(b+4)}(bA)\\
&\qquad+\frac{4(b-1)(b+4)}{3b^2}\left(\frac{b+2}{12(b+4)}\right)^3(bA)^{1-\frac{2}{b}}\\
&\qquad+\frac{(b^2-4)(8-3b)}{36b^3}\left(\frac{b+2}{12(b+4)}\right)^3(bA)^{1-\frac{4}{b}}. \qquad (3.14)
\end{aligned}$$

From $b\geqslant 4$ or $b=2$ and (3.10), we have

$$(bA)^{\frac{2}{b}}\geqslant\frac{1}{(b+1)^{\frac{2}{b}}}\geqslant\frac{1}{3}$$

since $(b+1)^{\frac{2}{b}}\leqslant 3$, and

$$\frac{4(b-1)(b+4)}{3b^2}+\frac{(b^2-4)(8-3b)}{36b^3}(bA)^{-\frac{2}{b}}\geqslant\frac{13b^3+56b^2-52b-32}{12b^3}. \qquad (3.15)$$

According to (3.9), (3.14) and (3.15), we obtain

$$\int_0^\infty s^{b+3}\psi(s)\,ds = D$$
$$\geqslant \frac{1}{b(b+4)}\left\{(b+4)\tau^4 bA - 4\tau^{b+4} + \frac{1}{3}\tau^{b+2}\right\}$$
$$\geqslant \frac{1}{b(b+4)}\left\{b(bA)^{1+\frac{4}{b}} + \frac{1}{3}(bA)^{1+\frac{2}{b}} + \frac{1}{72}\frac{(b+2)^2}{b(b+4)}(bA) + \frac{13b^3+56b^2-52b-32}{12b^3}\left(\frac{b+2}{12(b+4)}\right)^3 (bA)^{1-\frac{2}{b}}\right\}.$$

For $2 < b < 4$, we can obtain the following inequality using the same method as the case of $b \geqslant 4$,

$$\int_0^\infty s^{b+3}\psi(s)\,ds = D$$
$$\geqslant \frac{1}{b(b+4)}\left\{b(bA)^{1+\frac{4}{b}} + \frac{1}{3}(bA)^{1+\frac{2}{b}} + \frac{1}{72}\frac{(b+2)^2}{b(b+4)}(bA) + \frac{12b^3+33b^2-48b+12}{9b^3}\left(\frac{b+2}{12(b+4)}\right)^3 (bA)^{1-\frac{2}{b}}\right\}.$$

This finishes the proof of Lemma 3.1. □

Proof of Theorem 1.2. Using the same method as that of [4] and Lemma 3.1, we can prove Theorem 1.2. □

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