SCIENCE CHINA
Mathematics

• ARTICLES •

https://doi.org/10.1007/s11425-018-9464-7

Examples of compact λ-hypersurfaces in Euclidean spaces

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Received October 11, 2018; accepted January 9, 2019

Abstract In this paper, we first construct compact embedded λ-hypersurfaces with the topology of torus which are called λ-torus in Euclidean spaces $\mathbb{R}^{n+1}$. Then, we give many compact immersed λ-hypersurfaces in Euclidean spaces $\mathbb{R}^{n+1}$.

Keywords the weighted area functional, embedded λ-hypersurfaces, λ-torus

MSC(2010) 53C44, 53C42

Citation: Cheng Q-M, Wei G. Examples of compact λ-hypersurfaces in Euclidean spaces . Sci China Math, 2020, 63, https://doi.org/10.1007/s11425-018-9464-7

1 Introduction

We introduce λ-hypersurfaces from two views.

View one. Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional smooth immersed hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. In [5], Cheng and Wei have introduced the weighted volume-preserving mean curvature flow, which is defined as the following: a family $X(t)$ of smooth immersions:

$$X(t) = X(\cdot, t) : M \to \mathbb{R}^{n+1}$$

with $X(0) = X(\cdot, 0) = X(\cdot)$ is called *a weighted volume-preserving mean curvature flow* if they satisfy

$$\frac{\partial X(t)}{\partial t} = -\alpha(t)N(t) + \boldsymbol{H}(t), \tag{1.1}$$

where

$$\alpha(t) = \frac{\int_M H(t)\langle N(t), N\rangle e^{-\frac{|X|^2}{2}} d\mu}{\int_M \langle N(t), N\rangle e^{-\frac{|X|^2}{2}} d\mu},$$

$\boldsymbol{H}(t) = \boldsymbol{H}(\cdot, t)$ and $N(t)$ denote the mean curvature vector and the unit normal vector of hypersurface $M_t = X(M^n, t)$ at point $X(\cdot, t)$, respectively, and N is the unit normal vector of $X : M \to \mathbb{R}^{n+1}$.

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One can prove that the flow (1.1) preserves the weighted volume $V(t)$ defined by

$$V(t) = \int_M \langle X(t), N\rangle \mathrm{e}^{-\frac{|X|^2}{2}} d\mu.$$

The weighted area functional $A : (-\varepsilon, \varepsilon) \to \mathbb{R}$ is defined by $A(t) = \int_M \mathrm{e}^{-\frac{|X(t)|^2}{2}} d\mu_t$, where $d\mu_t$ is the area element of M in the metric induced by $X(t)$.

Let $X(t) : M \to \mathbb{R}^{n+1}$ with $X(0) = X$ be a variation of X. If $V(t)$ is constant for any t, we call $X(t) : M \to \mathbb{R}^{n+1}$ *a weighted volume-preserving variation of* X. Cheng and Wei [5] proved that $X : M \to \mathbb{R}^{n+1}$ is a critical point of the weighted area functional $A(t)$ for all weighted volume-preserving variations if and only if there exists a constant λ such that

$$\langle X, N\rangle + H = \lambda. \tag{1.2}$$

An immersed hypersurface $X(t) : M \to \mathbb{R}^{n+1}$ is called *a λ-hypersurface of weighted volume-preserving mean curvature flow* if the equation (1.2) is satisfied.

View two. λ-hypersurfaces can be viewed as stationary solutions to the isoperimetric problem on the Gaussian space with metric given by

$$\mathrm{e}^{-\frac{|X|^2}{n}} (dx_1^2 + \cdots + dx_{n+1}^2) \tag{1.3}$$

on $\mathbb{R}^{n+1}$.

Examples of λ-hypersurfaces in $\mathbb{R}^{n+1}$ include the n-dimensional sphere $S^n(\frac{-\lambda+\sqrt{\lambda^2+4n}}{2})$ of radius $\frac{-\lambda+\sqrt{\lambda^2+4n}}{2}$ centered at the origin, the n-dimensional cylinder $S^k(\frac{-\lambda+\sqrt{\lambda^2+4k}}{2}) \times \mathbb{R}^{n-k}$ for $1 \leqslant k \leqslant n-1$, and the n-dimensional Euclidean space $\mathbb{R}^n$.

When $\lambda = 0$, λ-hypersurfaces are just self-shrinkers of mean curvature flow. Hence, λ-hypersurfaces are a generalization of self-shrinkers. For research on self-shrinkers, see [3, 4, 6, 7, 11].

In 1989, Angenent [1] constructed compact embedded self-shrinker $X : S^1 \times S^{n-1} \to \mathbb{R}^{n+1}$ and provided the numerical evidence for the existence of an immersed sphere self-shrinker. Many interesting examples of complete self-shrinker are found recently. For example, complete embedded self-shrinkers with higher genus in $\mathbb{R}^3$ were constructed by Kapouleas et al. [12] (also see [13, 14]) and Nguyen [15–17]. Drugan [9] constructed an immersed self-shrinker, which is a topological sphere $X : S^2 \to \mathbb{R}^3$. In [10], Drugan and Kleene constructed many complete immersed self-shrinkers with rotational symmetry for each of following topological types: the sphere, the plane, the cylinder and the torus. Recently, Brendle [2] proved that the round sphere is the only compact embedded self-shrinker in $\mathbb{R}^3$ of genus zero.

Our purpose in this paper is to construct compact embedded λ-hypersurfaces and compact immersed λ-hypersurfaces in $\mathbb{R}^{n+1}$. Motivated by Angenent's construction in constructing "Angenent's torus" solution to mean curvature flow and using a "shooting method", we prove the following theorem.

Theorem 1.1. *For $n \geqslant 2$ and any $\lambda > 0$, there exists a compact embedded rotational λ-hypersurface*

$$X : S^1 \times S^{n-1} \to \mathbb{R}^{n+1}, \tag{1.4}$$

which is called λ-torus.

In the proof of Theorem 1.1, we also give the upper bound of profile curve of λ-torus, i.e., $x(s) \leqslant \frac{\pi}{2\lambda}$.

Theorem 1.2. *For $n \geqslant 2$ and small λ, there are many compact immersed λ-hypersurfaces in $\mathbb{R}^{n+1}$.*

When $\lambda = 0$, these λ-hypersurfaces are just self-shrinkers. Moreover, some examples of compact immersed self-shrinkers are discovered firstly.

2 Equations of rotational λ-hypersurfaces in $\mathbb{R}^{n+1}$

Let $\gamma(s) = (x(s), r(s))$, $s \in (a, b)$ be a curve with $r > 0$ in the upper half plane $\mathbb{H} = \{x + \mathrm{i}r \mid r > 0,\ x \in \mathbb{R},\ \mathrm{i} = \sqrt{-1}\}$, where s is arc length parameter of $\gamma(s)$. We consider a rotational hypersurface

$X : (a, b) \times S^{n-1}(1) \hookrightarrow \mathbb{R}^{n+1}$ in $\mathbb{R}^{n+1}$ defined by

$$X : (a, b) \times S^{n-1}(1) \hookrightarrow \mathbb{R}^{n+1}, \quad X(s, \alpha) = (x(s), r(s)\alpha) \in \mathbb{R}^{n+1} \tag{2.1}$$

where $S^{n-1}(1)$ is the $(n-1)$-dimensional unit sphere (see [8]).

By a direct calculation, one has the unit normal vector

$$N = (-r', x'\alpha) \tag{2.2}$$

and the mean curvature

$$H = -x''r' + x'r'' - \frac{n-1}{r}x'. \tag{2.3}$$

Therefore, we know from (2.1) and (2.2)

$$\langle X, N\rangle = -xr' + rx'. \tag{2.4}$$

Hence, $X : (a, b) \times S^{n-1}(1) \hookrightarrow \mathbb{R}^{n+1}$ is a λ-hypersurface in $\mathbb{R}^{n+1}$, if and only if, from (1.2), (2.3) and (2.4),

$$-x''r' + x'r'' - \frac{n-1}{r}x' - xr' + rx' = \lambda. \tag{2.5}$$

Since s is arc length parameter of the profile curve $\gamma(s) = (x(s), r(s))$, we have

$$(x')^2 + (r')^2 = 1. \tag{2.6}$$

Thus, it follows that

$$x'x'' + r'r'' = 0. \tag{2.7}$$

The signed curvature $\kappa(s)$ of the profile curve $\gamma(s) = (x(s), r(s))$ is given by

$$\kappa(s) = -\frac{x''}{r'}, \tag{2.8}$$

and it is known that the integral of the signed curvature $\kappa(s)$ measures the total rotation of the tangent vector of $\gamma(s)$. From (2.5) and (2.7), one has

$$x'' = -r'\left[xr' + \left(\frac{n-1}{r} - r\right)x' + \lambda\right]. \tag{2.9}$$

Hence, we have

$$\begin{cases} (x')^2 + (r')^2 = 1 \\ -\dfrac{x''}{r'} = xr' + \left(\dfrac{n-1}{r} - r\right)x' + \lambda. \end{cases} \tag{2.10}$$

First of all, we consider several special solutions of (2.10).

1. $(x, r) = (0, s)$ is a solution. This curve corresponds to the hyperplane through $(0, 0)$ and $\lambda = 0$.
2. $(x, r) = (a\cos\frac{s}{a}, a\sin\frac{s}{a})$ is a solution, where $a = \frac{\sqrt{\lambda^2+4n}-\lambda}{2}$. This circle $x^2 + r^2 = a^2$ corresponds to a sphere $S^n(a)$ with radius a.
3. $(x, r) = (-s, a)$ is a solution, where $a = \frac{\sqrt{\lambda^2+4(n-1)}-\lambda}{2}$. This straight line corresponds to a cylinder $S^{n-1}(a) \times \mathbb{R}$.

3 Construction of λ-torus

We next consider to find several other solutions of (2.10) besides the above three special solutions. In fact, our purposes are to study properties of the profile curve and to find a simple and closed profile curve γ in the upper half plane $\mathbb{H} = \{x + \mathrm{i}r \mid r > 0,\ x \in \mathbb{R},\ \mathrm{i} = \sqrt{-1}\}$.

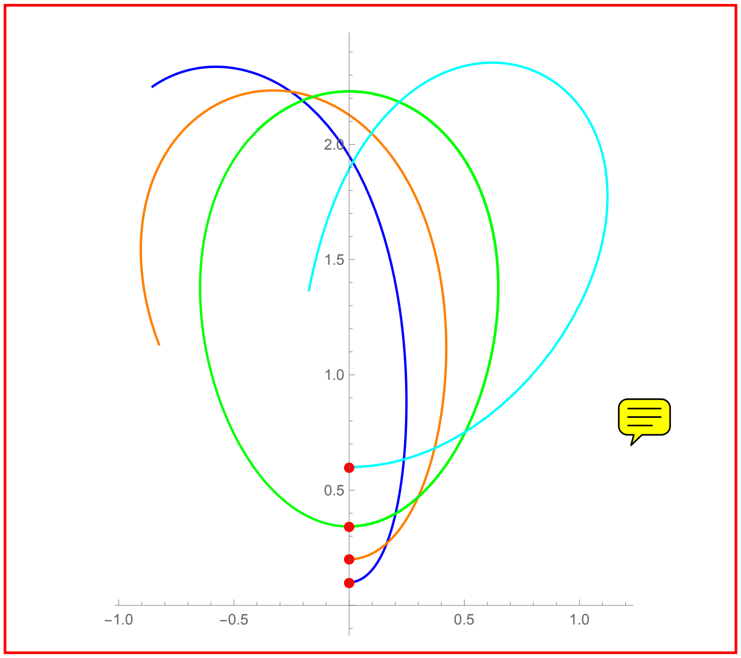

Figure 1 (Color online) The profile curves of λ-hypersurfaces, here $n = 2$, $\lambda = 0.1$

From now on, we consider general behavior of profile curve γ. As long as $x > 0$, $x' > 0$ and $r < \frac{\sqrt{\lambda^2+4(n-1)}+\lambda}{2}$, one has from (2.8) and (2.10) that

$$\kappa = -\frac{x''}{r'} = xr' + \left(\frac{n-1}{r} - r\right)x' + \lambda > 0,$$

i.e., the curve γ bends upwards, so that the curve will be convex.

At $s = 0$, the profile curve $\gamma_\delta(s) = (x_\delta(s), r_\delta(s))$ with $(x_\delta(0), r_\delta(0)) = (0, \delta)$ and the initial unit tangent vector $(x'_\delta(0), r'_\delta(0)) = (1, 0)$ will initially bend upwards, where $\delta < \frac{\sqrt{\lambda^2+4(n-1)}+\lambda}{2}$ (see Figure 1, where the vertical axis is r-axis and the horizontal axis is x-axis).

We next give the following definition of s_1.

Definition 3.1. Let $s_1 = s_1(\delta) > 0$ be the arc length of the first time, if any, at which either $x_\delta = 0$ or the unit tangent vector is $(1, 0)$, or $(-1, 0)$, i.e., either the curve γ_δ hits r-axis, or the curve γ_δ has a horizontal tangent. If this never happens, we take $s_1(\delta) = S(\delta)$, where $\gamma_\delta = (x_\delta, r_\delta) : [0, S(\delta)) \to \mathbb{R}^2$ is the maximal solution of (2.10) with initial value $(x_\delta(0), r_\delta(0), x'_\delta(0)) = (0, \delta, 1)$.

From Definition 3.1, we have $r'(s) > 0$ in $(0, s_1)$. Hence, this curve can be written as a graph of $x = f_\delta(r)$, where $\delta < r < r_\delta(s_1)$. If $f'_\delta(r) = \frac{dx}{dr} = 0$, i.e., the profile curve γ_δ has a vertical tangent, then it follows from (2.10) that

$$f''_\delta(r) = \frac{d^2 f_\delta(r)}{dr^2} = -\frac{1}{(r')^3}\left[xr' + \left(\frac{n-1}{r} - r\right)x' + \lambda\right] = -\frac{1}{(r')^3}(xr' + \lambda) < 0. \tag{3.1}$$

This means that $f_\delta(r)$ can only have local maximum. Thus, $f_\delta(r)$ can have at most one critical point, which must be a maximum point.

Next, we shall prove that there exists a $\delta > 0$ such that $\gamma_\delta([0, s_1(\delta)])$ is a simple curve in the first quadrant which begins and ends on the r-axis, and whose tangent vectors on the r-axis are horizontal. From (2.10), one can get that the profile curve γ obtained by reflecting $\gamma_\delta([0, s_1(\delta)])$ in the r-axis is a simple and closed curve in the upper half plane. λ-torus is given by rotating the simple closed curve γ in the plane around r-axis and, thus, has the topology of a torus (see Figure 2, where the horizontal axis is the axis of rotation and the vertical axis is a line of reflection symmetry).

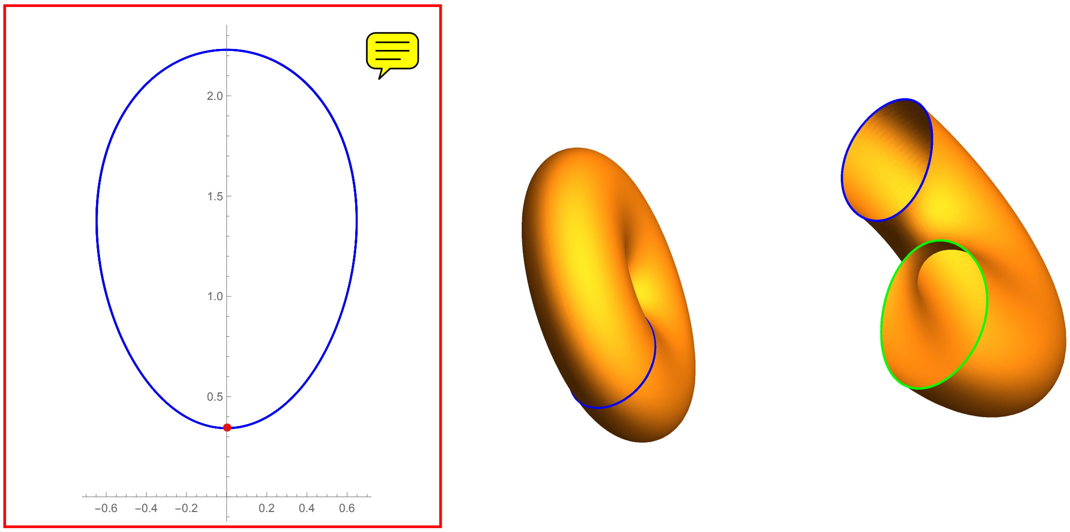

Figure 2 (Color online) The graph of profile curve of λ-torus, λ-torus and half of λ-torus, here $n = 2$, $\lambda = 0.1$ and $r_0 \approx 0.343$

3.1 An estimate on upper bounds of $r_\delta(s_1)$

We will consider behavior of profile curve γ as $\delta > 0$ is small enough in order to estimate supremum of $r_\delta(s_1)$. Since δ is very small, we define

$$\begin{cases} \xi(t) = \dfrac{1}{\delta} x(\delta t), \\ \rho(t) = \dfrac{1}{\delta}(r(\delta t) - \delta). \end{cases} \tag{3.2}$$

From (2.10), we have

$$\begin{cases} (\xi')^2 + (\rho')^2 = 1, \\ \dfrac{\xi''}{-\rho'} = \dfrac{\xi''}{-\sqrt{1-(\xi')^2}} \\ \quad = \delta^2 \xi \rho' + \left(\dfrac{n-1}{1+\rho} - \delta^2(1+\rho)\right)\xi' + \lambda\delta \\ \quad = \dfrac{n-1}{1+\rho}\xi' + \lambda\delta + O(\delta^2) \end{cases} \tag{3.3}$$

and

$$\xi(0) = 0, \quad \rho(0) = 0, \quad \xi'(0) = 1. \tag{3.4}$$

We consider equations

$$\begin{cases} (\xi')^2 + (\rho')^2 = 1, \\ \dfrac{\xi''}{-\rho'} = \dfrac{\xi''}{-\sqrt{1-(\xi')^2}} = \dfrac{n-1}{1+\rho}\xi', \end{cases} \tag{3.5}$$

with

$$\xi(0) = 0, \quad \rho(0) = 0, \quad \xi'(0) = 1. \tag{3.6}$$

From (3.5), one gets

$$1 - (\rho')^2 = \frac{1}{(1+\rho)^{2(n-1)}}.$$

If $\rho(t) \to +\infty$ as $t \to +\infty$, we have

$$\rho' = \frac{d\rho}{dt} = \frac{dr}{ds} \to 1.$$

If $\rho(t)$ is bounded, we have

$$c_1 \leqslant \frac{\xi''}{-\xi'\sqrt{1-(\xi')^2}} = \frac{n-1}{1+\rho} \leqslant n-1, \tag{3.7}$$

where $c_1 > 0$ is a constant. By a direct calculation, we get

$$\tanh(c_1 t) \leqslant \sqrt{1-(\xi')^2} = \rho'. \tag{3.8}$$

Hence, $\rho' \to 1$ as $t \to +\infty$.

Since the solution of (3.3) depends smoothly on the parameter δ, we may obtain from (3.8) that there is a $T > 0$ such that for all sufficiently small $\delta > 0$, one has $T\delta < S(\delta)$ and at $s = T\delta$,

$$r'_\delta(T\delta) \geqslant \begin{cases} \dfrac{\sqrt{3}}{2}, & \text{if } \lambda > \dfrac{\pi}{3\sqrt{n-1}}, \\ \sin\left(\dfrac{\pi}{2} - \lambda\dfrac{\sqrt{n-1}}{2}\right), & \text{if } \lambda \leqslant \dfrac{\pi}{3\sqrt{n-1}}, \end{cases} \tag{3.9}$$

and $x_\delta = O(\delta)$, $r_\delta = \delta + O(\delta)$ from (3.2).

Lemma 3.2. *For $0 < s < s_1$, we have*

$$x_\delta(s) \leqslant C_1\delta, \quad r_\delta(s) \leqslant \sqrt{n-1} + \frac{\pi}{2\lambda} \quad \textit{and} \quad s_1(\delta) < \infty \tag{3.10}$$

if δ is small enough, where constant C_1 does not depend on δ.

Proof. For $0 < s < s_1 = s_1(\delta)$, we have $r' > 0$, so that γ_δ can be written as a graph of $x = f_\delta(r)$. From (2.10), we have

$$-\frac{x''}{r'} = xr' + \left(\frac{n-1}{r} - r\right)x' + \lambda \geqslant \left(\frac{n-1}{r} - r\right)x' \tag{3.11}$$

because of $x > 0$, $\lambda > 0$ and $r' > 0$.

Letting $r_T = r_\delta(T\delta)$ and for r satisfying $x'(\widetilde{r}) > 0$ if $r_T < \widetilde{r} < r$, integrating (3.11) from r_T to $r > r_T$, we have

$$\frac{x'(s(r_T))}{x'(s(r))} \geqslant \mathrm{e}^{\frac{r_T^2 - r^2}{2}}\left(\frac{r}{r_T}\right)^{n-1}, \tag{3.12}$$

i.e., for all r such that $x' > 0$ on (r_T, r), we have

$$x'(s(r)) \leqslant \left(\frac{r_T}{r}\right)^{n-1} \mathrm{e}^{\frac{r^2 - r_T^2}{2}} x'(s(r_T)). \tag{3.13}$$

We will prove that the above r satisfies the following claim.

Claim. $r \leqslant \sqrt{n-1}$ if δ is small enough.

If $r(s_1) \leqslant \sqrt{n-1}$, then we know that this result is obvious because of $r'(s) > 0$.

We only need to consider the case of $r(s_1) > \sqrt{n-1}$. In this case, there exists an s_3 such that $0 < s_3 < s_1$ and $r(s_3) = \sqrt{n-1}$.

If $x'(s_3) \leqslant 0$, we have $x'(s) < 0$ for $s_3 < s < s_1$. Hence, $r < r(s_3) = \sqrt{n-1}$ holds because of $x'(r) > 0$.

If $x'(s_3) > 0$, for $s \in (T\delta, s_3)$, $r'(s) > 0$ holds. One has $0 < r(s) < \sqrt{n-1}$ and

$$\kappa = -\frac{x''}{r'} = xr' + \left(\frac{n-1}{r} - r\right)x' + \lambda \geqslant \lambda \tag{3.14}$$

because of $r(s_3) = \sqrt{n-1}$. By integrating (3.14) from $s = T\delta$ to s_3, we obtain

$$\int_{T\delta}^{s_3} \kappa ds \geqslant \lambda(s_3 - T\delta) \geqslant \lambda(r(s_3) - r(T\delta)) \geqslant \lambda(\sqrt{n-1} - r_T). \tag{3.15}$$

Here we use that the length of the profile curve $\gamma(s)$ from the point $\gamma(T\delta)$ to the point $\gamma(s_3)$ is not less than the Euclidean distance between these two points. Therefore, we have

$$r_T \geqslant \sqrt{n-1} - \frac{1}{\lambda}\int_{T\delta}^{s_3} \kappa ds. \tag{3.16}$$

If $\lambda > \frac{\pi}{3\sqrt{n-1}}$, we have $\frac{\pi}{2} - \lambda\frac{\sqrt{n-1}}{2} < \frac{\pi}{3}$. If $\lambda \leqslant \frac{\pi}{3\sqrt{n-1}}$, then we have $\frac{\pi}{2} - \lambda\frac{\sqrt{n-1}}{2} \geqslant \frac{\pi}{3}$. Since the integral $\int_{T\delta}^{s_3}\kappa ds$ measures the total rotation of the tangent vector of γ_δ from $T\delta$ to s_3, from (3.9), we know

$$\int_{T\delta}^{s_3}\kappa ds \leqslant \frac{\pi}{2} - \max\left\{\frac{\pi}{3},\ \frac{\pi}{2} - \lambda\frac{\sqrt{n-1}}{2}\right\}.$$

Thus, for $\lambda > \frac{\pi}{3\sqrt{n-1}}$, we obtain from (3.16)

$$r_T \geqslant \sqrt{n-1} - \frac{1}{\lambda}\int_{T\delta}^{s_3}\kappa ds \geqslant \sqrt{n-1} - \frac{1}{\lambda}\left(\frac{\pi}{2} - \frac{\pi}{3}\right) \geqslant \frac{\sqrt{n-1}}{2}. \tag{3.17}$$

It is impossible because $r_T = r(T\delta) = O(\delta)$ is very small.

For $\lambda \leqslant \frac{\pi}{3\sqrt{n-1}}$, we have from (3.16) that

$$r_T \geqslant \sqrt{n-1} - \frac{1}{\lambda}\int_{T\delta}^{s_3}\kappa ds \geqslant \sqrt{n-1} - \frac{1}{\lambda}\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - \lambda\frac{\sqrt{n-1}}{2}\right)\right) = \frac{\sqrt{n-1}}{2}. \tag{3.18}$$

It is also impossible because of $r_T = r(T\delta) = O(\delta)$.

From the above arguments, we complete the proof of the claim.

By a direct calculation, we have from (3.13) and (3.9) that

$$x'(s(r)) \leqslant x'(s(\delta_T)) \leqslant \frac{1}{2} \tag{3.19}$$

for $r \in (r_T, \sqrt{n-1})$ with $x' > 0$ on (r_T, r) since

$$\left(\frac{r_T}{r}\right)^{n-1} \mathrm{e}^{\frac{r^2-r_T^2}{2}}$$

is a decreasing function of r in $(r_T, \sqrt{n-1})$. Hence, for $r \in (r_T, \sqrt{n-1})$ one has

$$\frac{x'}{r'} \leqslant 2x'. \tag{3.20}$$

Since $\frac{dx}{dr} = f'_\delta(r) = \frac{x'}{r'}$, we have from (3.13) and (3.20) that

$$\int_{r_T}^{r} f'_\delta(r)dr = \int_{r_T}^{r}\frac{x'}{r'}dr \leqslant 2\int_{r_T}^{r} x'dr \leqslant 2\int_{r_T}^{r}\left(\frac{r_T}{r}\right)^{n-1}\mathrm{e}^{\frac{r^2-r_T^2}{2}}x'(s(r_T))dr. \tag{3.21}$$

Hence, we obtain

$$f_\delta(r) \leqslant f_\delta(r_T) + 2\int_{r_T}^{r}\left(\frac{r_T}{r}\right)^{n-1}\mathrm{e}^{\frac{r^2-r_T^2}{2}}dr \leqslant f_\delta(r_T) + c_2\int_{r_T}^{r}\left(\frac{r_T}{r}\right)^{n-1}dr \leqslant c_3 r_T \leqslant C_1\delta \tag{3.22}$$

if $\delta > 0$ is small enough, where c_2, c_3 and C_1 are constants.

Since $x_\delta(s)$ gets its maximum at $x'(s) = 0$, we conclude from (3.22) that

$$0 \leqslant x_\delta(s) \leqslant C_1\delta, \quad \text{for } 0 < s < s_1. \tag{3.23}$$

If $x'(s_3) = 0$ at $s = s_3$, then we have $x'(s) < 0$ for $s_3 < s < s_1$. According to the argument in the above claim, we know $r(s_3) \leqslant \sqrt{n-1}$ as long as $\delta > 0$ is small enough. If there exists an s_2 with $s_3 < s_2 < s_1$ such that $r(s_2) = \sqrt{n-1}$, then $r(s) > \sqrt{n-1}$ and $x'(s) < 0$ for $s_2 < s < s_1$. Hence, we have

$$\left(\frac{n-1}{r(s)} - r(s)\right)x'(s) > 0$$

and

$$\kappa = -\frac{x''}{r'} = xr' + \left(\frac{n-1}{r} - r\right)x' + \lambda > \lambda, \tag{3.24}$$

for $s_2 < s < s_1$. By integrating (3.24) from $s = s_2$ to s_1, we have

$$\frac{\pi}{2} \geqslant \int_{s_2}^{s_1} \kappa ds \geqslant \lambda(s_1 - s_2) \geqslant \lambda(r(s_1) - r(s_2)) = \lambda(r(s_1) - \sqrt{n-1}), \tag{3.25}$$

i.e.,

$$r_\delta(s_1) \leqslant \sqrt{n-1} + \frac{\pi}{2\lambda}. \tag{3.26}$$

Hence, we get that $x_\delta(s)$ and $r_\delta(s)$ are bounded. We know $s_1(\delta) < \infty$. This finishes the proof of the lemma. $\square$

Lemma 3.3. *For $\delta > 0$ small enough, one has $x_\delta(s_1) = 0$.*

Proof. If the statement is not true, then there exists a sequence $\delta_m \to 0^+$ for which $x_{\delta_m}(s) > 0$. Putting $f_m(r) = f_{\delta_m}(r)$, the function $x_{\delta_m} = f_m(r)$ is defined for $\delta_m < r < \sqrt{n-1} + \frac{\pi}{2\lambda}$ and satisfies

$$\frac{f_m''(r)}{1 + (f_m'(r))^2} + \left(\frac{n-1}{r} - r\right) f_m'(r) + f_m(r) + \frac{\lambda}{\sqrt{1 + (f_m'(r))^2}} = 0. \tag{3.27}$$

From Lemma 3.2, we get that $f_m(r)$ satisfies $0 < f_m(r) = x_{\delta_m}(r) \leqslant C_1\delta_m \to 0$. Thus, we know that $\gamma_{\delta_m} = (x_{\delta_m}, r_{\delta_m})$ gets close to the r-axis, and its tangents also must converge to the r-axis. Hence, we have $x_{\delta_m}(r) = f_m(r)$ and $f_m'(r)$ converge to zero on compact intervals.

On the other hand, from (3.27), we have

$$f_m''(r) \to -\lambda < 0. \tag{3.28}$$

Therefore, for $\delta_m > 0$ small enough,

$$f_m''(r) < -\frac{\lambda}{2} < 0, \tag{3.29}$$

and $f_m'(r)$ is a monotone decreasing function. This is impossible. Hence, there exists a $\delta > 0$ small enough such that $x_\delta(s_1) = 0$. $\square$

According to Lemmas 3.2 and 3.3, we know that there exists a $\delta_0 > 0$ small enough such that $x_{\delta_0}(s_1) = 0$ and $s_1(\delta_0) < \infty$. Since the solutions of (2.10) depend smoothly on the initial value, we define δ^* as the following definition.

Definition 3.4. δ^* is defined as

$$\delta^* = \sup\{\delta > 0 : x_\delta(s_1) = 0 \text{ and } s_1 = s_1(\delta) < \infty\}. \tag{3.30}$$

Lemma 3.5. *The following holds*:

$$\sup_{\delta_0 < \delta < \delta^*} r_\delta(s_1) < +\infty. \tag{3.31}$$

Proof. From Lemma 3.3, we know $r_\delta(s_1)$ is bounded if $\delta > 0$ is small enough. If $r_{\delta_m}(s_1) \to +\infty$ for some sequence $\delta_m \to \delta^*$, we will prove that it is impossible. In fact, according to the mean value theorem, there exists an s_3 such that $0 < s_3 < s_1$ and $x_{\delta_m}'(s_3) = 0$ because of $x_{\delta_m}(s_1) = 0$. We should remark that s_1 and s_3 depend on δ_m. Furthermore, $x_{\delta_m}'(s) < 0$ for $s_3 < s < s_1$ and we have

$$\frac{n-1}{r_{\delta_m}} - r_{\delta_m} < 0,$$

for $s_3 < s < s_1$. Otherwise, there exists an s_4 such that $r_{\delta_m}(s_4) = \sqrt{n-1}$ with $s_4 > s_3$. Thus, $x_{\delta_m}'(s) < 0$ and $\frac{n-1}{r_{\delta_m}} - r_{\delta_m} < 0$ for $s_4 < s < s_1$. From the equations (2.10), we have

$$\kappa = -\frac{x_{\delta_m}''}{r_{\delta_m}'} = x_{\delta_m} r_{\delta_m}' + \left(\frac{n-1}{r_{\delta_m}} - r_{\delta_m}\right) x_{\delta_m}' + \lambda > \lambda. \tag{3.32}$$

By integrating (3.32) from s_4 to s_1, we get

$$\frac{\pi}{2} \geqslant \int_{s_4}^{s_1} \kappa ds \geqslant \lambda(s_1 - s_4) \geqslant \lambda(r_{\delta_m}(s_1) - r_{\delta_m}(s_4)) = \lambda(r_{\delta_m}(s_1) - \sqrt{n-1}). \tag{3.33}$$

Hence, we have

$$r_{\delta_m}(s_1) \leqslant \frac{\pi}{2\lambda} + \sqrt{n-1}. \tag{3.34}$$

This is impossible since $r_{\delta_m}(s_1) \to +\infty$. Hence, $\frac{n-1}{r_{\delta_m}} - r_{\delta_m} < 0$ for $s_3 < s < s_1$. For $s_3 < s < s_1$, we have

$$\kappa = -\frac{x''_{\delta_m}}{r'_{\delta_m}} = x_{\delta_m} r'_{\delta_m} + \left(\frac{n-1}{r_{\delta_m}} - r_{\delta_m}\right) x'_{\delta_m} + \lambda > \lambda. \tag{3.35}$$

Integrating (3.35) from $s = s_3$ to s_1, we get

$$\frac{\pi}{2} \geqslant \int_{s_3}^{s_1} \kappa ds \geqslant \lambda(s_1 - s_3) \geqslant \lambda(r_{\delta_m}(s_1) - r_{\delta_m}(s_3)). \tag{3.36}$$

Hence,

$$r_{\delta_m}(s_1) \leqslant \frac{\pi}{2\lambda} + r_{\delta_m}(s_3). \tag{3.37}$$

We have

$$r_{\delta_m}(s_3) \to +\infty \tag{3.38}$$

because of $r_{\delta_m}(s_1) \to +\infty$.

Since $x'_{\delta_m}(s_3) = 0$ and $x_{\delta_m}(s_1) = 0$ hold, from $r_{\delta_m}(s_3) \to +\infty$, for some δ_m, which is very near δ^*, we know that there exists an s_5 with $s_3 < s_5 < s_1$ such that

$$x'_{\delta_m}(s_5) = -\sin\left(\frac{1}{\sqrt{r_{\delta_m}(s_3)}}\right). \tag{3.39}$$

If we integrate (3.35) from $s = s_3$ to s_5, we obtain

$$\frac{1}{\sqrt{r_{\delta_m}(s_3)}} = \int_{s_3}^{s_5} \kappa ds \geqslant \lambda(s_5 - s_3). \tag{3.40}$$

Since $r_{\delta_m}(s_3) \to +\infty$ holds,

$$|x'(s_5)| = \sin\frac{1}{\sqrt{r_{\delta_m}(s_3)}} > \frac{1}{2\sqrt{r_{\delta_m}(s_3)}} > \frac{1}{2\sqrt{r_{\delta_m}(s_5)}} \tag{3.41}$$

yields

$$\begin{aligned}\left(\frac{n-1}{r_{\delta_m}(s)} - r_{\delta_m}(s)\right) x'_{\delta_m}(s) &\geqslant \left(\frac{n-1}{r_{\delta_m}(s_5)} - r_{\delta_m}(s_5)\right) x'_{\delta_m}(s_5) \\ &> \frac{n-1}{r_{\delta_m}(s_5)} x'_{\delta_m}(s_5) + \frac{1}{2}\sqrt{r_{\delta_m}(s_5)} \\ &> \frac{1}{4}\sqrt{r_{\delta_m}(s_5)}\end{aligned} \tag{3.42}$$

for $s_5 \leqslant s \leqslant s_1$ since $r_{\delta_m}(s_5) \to +\infty$.

From the equations (2.10) and (3.42), we have

$$\kappa = -\frac{x''_{\delta_m}}{r'_{\delta_m}} = x_{\delta_m} r'_{\delta_m} + \left(\frac{n-1}{r_{\delta_m}} - r_{\delta_m}\right) x'_{\delta_m} + \lambda > \frac{1}{4}\sqrt{r_{\delta_m}(s_5)} \tag{3.43}$$

as $r_{\delta_m} \to \infty$. Integrating (3.43) from $s = s_5$ to s_1, we have

$$\frac{\pi}{2} > \int_{s_5}^{s_1} \kappa ds > \frac{1}{4}\sqrt{r_{\delta_m}(s_5)}(s_1 - s_5). \tag{3.44}$$

Thus, one obtains from (3.40) and (3.44) that

$$\begin{aligned}\max(x_{\delta_m}) &= x_{\delta_m}(s_3) = x_{\delta_m}(s_3) - x_{\delta_m}(s_1)\\ &\leqslant s_1 - s_5 + s_5 - s_3\\ &\leqslant \frac{2\pi}{\sqrt{r_{\delta_m}(s_5)}} + \frac{1}{\lambda\sqrt{r_{\delta_m}(s_3)}}\\ &\leqslant \left(2\pi + \frac{1}{\lambda}\right)\frac{1}{\sqrt{r_{\delta_m}(s_3)}}.\end{aligned} \tag{3.45}$$

We conclude from (3.38) and (3.45) that $\max(x_{\delta_m}) = x_{\delta_m}(s_3) \to 0$ if $r_{\delta_m}(s_1) \to +\infty$. Hence, $\gamma_{\delta_m} = (x_{\delta_m}, r_{\delta_m})$ gets close to the r-axis, and its tangents also must converge to the r-axis. It is impossible since $\gamma_{\delta_m} = (x_{\delta_m}, r_{\delta_m})$ converges to γ_{δ^*} which is not r-axis. This finishes our proof. □

3.2 An upper bound of $x_\delta(s)$

Lemma 3.6. *The following holds*:

$$\sup_{\delta_0<\delta<\delta^*}\sup_{0<s<s_1} x_\delta(s) \leqslant \frac{\pi}{2\lambda}. \tag{3.46}$$

Proof. For $\delta > 0$, letting $x'(s_3) = 0$ with $0 < s_3 < s_1$, we have

$$x'(s) > 0 \quad \text{for } 0 < s < s_3, \tag{3.47}$$

$$x'(s) < 0 \quad \text{for } s_3 < s < s_1. \tag{3.48}$$

If $r(s_3) \leqslant \sqrt{n-1}$, we see from $r'(s) > 0$ that $r(s) < \sqrt{n-1}$ for $0 < s < s_3$. Thus, it follows from (2.10) and (3.47) that

$$\kappa = -\frac{x''}{r'} = xr' + \left(\frac{n-1}{r} - r\right)x' + \lambda > \lambda \tag{3.49}$$

for $0 < s < s_3$.

By integrating (3.49) from $s = 0$ to s_3, we have

$$\frac{\pi}{2} = \int_0^{s_3} \kappa ds \geqslant s_3\lambda \geqslant \lambda x(s_3) = \lambda \sup_{0<s<s_1} x_\delta(s), \tag{3.50}$$

$$\sup_{0<s<s_1} x_\delta(s) = x(s_3) \leqslant \frac{\pi}{2\lambda}. \tag{3.51}$$

If $r(s_3) > \sqrt{n-1}$ for some $\delta > 0$, we have from $r'(s) > 0$ that $r(s) > \sqrt{n-1}$ for $s_3 < s < s_1$. Then it follows from (2.10) and (3.48) that

$$\kappa = -\frac{x''}{r'} = xr' + \left(\frac{n-1}{r} - r\right)x' + \lambda > \lambda \tag{3.52}$$

for $s_3 < s < s_1$.

By integrating (3.52) from $s = s_3$ to s_1, we have, from the mean value theorem,

$$\frac{\pi}{2} \geqslant \int_{s_3}^{s_1} \kappa ds \geqslant \lambda(s_1 - s_3) \geqslant \lambda x(s_3) = \lambda \sup_{0<s<s_1} x_\delta(s). \tag{3.53}$$

From (3.51) and (3.53), we get, for any $\delta > 0$,

$$\sup_{0<s<s_1} x_\delta(s) = x(s_3) \leqslant \frac{\pi}{2\lambda},$$

i.e.,

$$\sup_{\delta_0<\delta<\delta^*}\sup_{0<s<s_1} x_\delta(s) \leqslant \frac{\pi}{2\lambda}.$$

This completes the proof of the lemma. □

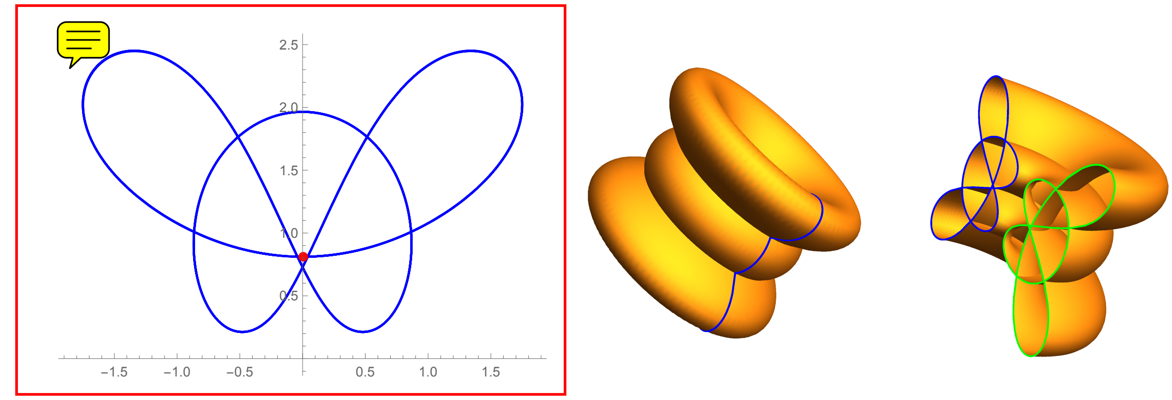

Figure 3 (Color online) The graph of profile curve of compact λ-hypersurface, λ-hypersurface and half of λ-hypersurface, here $n = 2$, $\lambda = 0.1$ and $r_0 \approx 0.811$

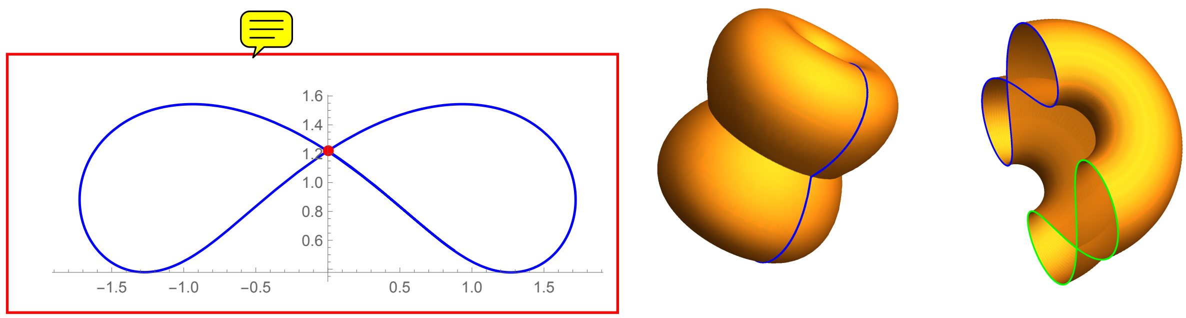

Figure 4 (Color online) The graph of profile curve of compact λ-hypersurface, λ-hypersurface and half of λ-hypersurface, here $n = 2$, $\lambda = 0$ and $r_0 \approx 1.22$

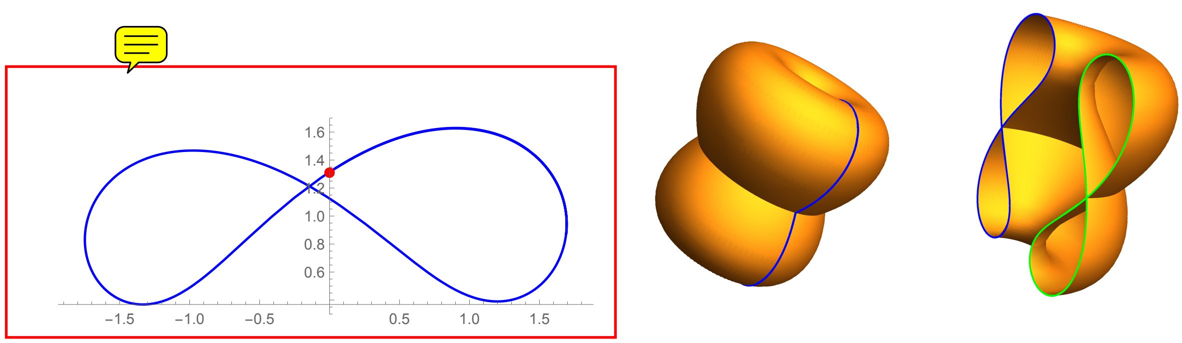

Figure 5 (Color online) The graph of profile curve of compact λ-hypersurface, λ-hypersurface and half of λ-hypersurface, here $n = 2$, $\lambda = -0.1$ and $r_0 \approx 1.31449$

3.3 Proof of Theorem 1.1

Proof of Theorem 1.1. From the above lemmas, we have found that the profile curve $\gamma_\delta(s) = (x_\delta, r_\delta)$ $(0 \leqslant s \leqslant s_1)$ stays away from the x-axis, and remain bounded as $\delta \to \delta^*$. Therefore, the limiting profile curve $\gamma^*(s) = \gamma_{\delta^*}(s)$ $(0 \leqslant s \leqslant s_1)$ begins and ends on the r-axis, i.e., from $(0, \delta^*)$ to $(0, r^*)$, where $r^* = r_{\delta^*}(s_1)$.

We now claim that the profile curve has the horizontal tangent, i.e., $x'_{\delta^*}(s_1) = -1$. From the definition of δ^*, we obtain that $x'_{\delta^*}(s_1) \geqslant -1$. If $x'_{\delta^*}(s_1) > -1$, one can choose δ such that $\delta > \delta^*$ and δ is near δ^*. Then one can still obtain a profile curve γ_δ, which, in the first quadrant, is a graph $x = f_\delta(r)$, and which hits the r-axis in finite time. This contradicts the definition of δ^*. Hence, the profile curve γ^* has the horizontal tangent, i.e., $x'_{\delta^*}(s_1) = -1$. We can get that the profile curve γ obtained by reflecting $\gamma_{\delta^*}([0, s_1])$ in the r-axis is a simple and closed curve in the upper half plane. This finishes our proof of

Theorem 1.1. □

4 Construction of compact immersed λ-hypersurfaces

Many examples of complete immersed self-shrinkers comes from hypersurfaces with rotational symmetry. In [10], Drugan and Kleene discovered many complete self-shrinkers for each of the rotational topological types: S^n, $S^1 \times S^{n-1}$, $\mathbb{R}^n$ and $S^1 \times \mathbb{R}^{n-1}$. The main idea for the construction is to study the behavior of profile curves near the profile curve of two known self-shrinkers among sphere, Euclidean space, cylinder and use continuity arguments to find complete self-shrinkers between them. By analyzing (2.10) and using the similar way to that of Drugan and Kleene, one can prove the following theorem.

Theorem 4.1. *For $n \geqslant 2$ and small λ, there are many compact immersed λ-hypersurfaces in $\mathbb{R}^{n+1}$.*

Additional details on the behavior of the profile curves are needed to be discussed and established. Here are some numerical approximation of profile curves and λ-hypersurfaces. The horizontal axis is the axis of rotation. For small λ, compact immersed λ-hypersurfaces can be given by rotating a closed curve in the upper half plane around the horizontal axis; see Figure 3.

Moreover, we also find many compact immersed rotational λ-hypersurfaces whose profile curves do not intersect r-axis perpendicularly; see Figure 4 and Figure 5.

Acknowledgements The first author was supported by JSPS Grant-in-Aid for Scientific Research (B) (Grant No. 16H03937) and Challenging Exploratory Research. The second author was supported by National Natural Science Foundation of China (Grant No. 11771154) and by Guangdong Province Universities and Colleges Pearl River Scholar Funded Scheme (2018).

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