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UNIVERSAL BOUNDS FOR EIGENVALUES OF A BUCKLING PROBLEM II

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Abstract. In this paper, we investigate universal estimates for eigenvalues of a buckling problem. For a bounded domain in a Euclidean space, we give a positive contribution for obtaining a sharp universal inequality for eigenvalues of the buckling problem. For a domain in the unit sphere, we give an important improvement on the results of Wang and Xia.

1. Introduction

Let M be an n-dimensional complete Riemannian manifold and $\Omega \subset M$ a bounded domain in M with piecewise smooth boundary $\partial\Omega$. A Dirichlet eigenvalue problem for the Laplacian is given by

$$\begin{cases} \triangle u = -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

which is also called *a fixed membrane problem*, where Δ denotes the Laplacian on M. The spectrum of this eigenvalue problem is real and discrete.

The following eigenvalue problem of a biharmonic operator is called *a buckling problem*:

$$\begin{cases} \Delta^2 u = -\Lambda \Delta u \quad \text{in } \Omega, \\ u|_{\partial\Omega} = \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

which describes the critical buckling load of a clamped plate subjected to a uniform compressive force around its boundary, where ν is the outward unit normal vector field of the boundary $\partial\Omega$. It is known that the spectrum of the buckling problem is also real and discrete.

When $\Omega \subset \mathbf{R}^n$ is a bounded domain in an n-dimensional Euclidean space $\mathbf{R}^n$, Payne, Pólya and Weinberger [17] and [18] proved the following inequality for eigenvalues of the eigenvalue problem (1.1): for $k = 1, 2, \dots$,

$$\lambda_{k+1} - \lambda_k \le \frac{4}{kn} \sum_{i=1}^{k} \lambda_i. \tag{1.3}$$

One calls it a universal inequality since it does not depend on the domain Ω.

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On the other hand, Payne, Pólya and Weinberger [17] and [18] also studied eigenvalues of the buckling problem on a bounded domain Ω in $\mathbf{R}^n$ and intended to derive a universal inequality for eigenvalues of the buckling problem. But it is very hard to deal with this problem. They only proved, for $n = 2$,

$$\Lambda_2 \leq 3\Lambda_1.$$

As an open problem, Payne, Pólya and Weinberger [17] and [18] proposed the following:

Problem. To determine whether one can obtain a universal inequality for eigenvalues of the buckling problem (1.2) on a bounded domain in a Euclidean space, which is similar to the universal inequality (1.3) for the eigenvalues of the fixed membrane problem (1.1).

For lower order eigenvalues, Hile and Yeh [14] and so on improved the result of Payne, Pólya and Weinberger to

$$\Lambda_2 \leq \frac{n^2 + 8n + 20}{(n+2)^2}\Lambda_1.$$

Furthermore, Ashbaugh [3] (cf. [2]) has obtained

$$\sum_{i=1}^{n} \Lambda_{i+1} \leq (n+4)\Lambda_1$$

and he has commented that *to obtain a universal inequality for eigenvalues of the buckling problem remains a challenge for mathematicians since 1955.* Many mathematicians have intended to attack this problem, but it has remained open for almost 50 years.

As one knows, in order to obtain a universal inequality for eigenvalues of the buckling problem, it is a key to find appropriate trial functions. Cheng and Yang [8], by introducing a new method to construct trial functions for the buckling problem, have obtained the following universal inequality for eigenvalues of the buckling problem (1.2):

$$\sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4(n+2)}{n^2}\sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)\Lambda_i. \tag{1.4}$$

Thus, the problem proposed by Payne, Pólya and Weinberger has been solved affirmatively. By making use of the asymptotic formula of Weyl for eigenvalues of the Dirichlet eigenvalue problem of the Laplacian and one of Agmon [1] and Pleijel [19] for eigenvalues of the clamped plate problem, we can have the asymptotic formula of eigenvalues for the buckling problem according to the variational characterization for eigenvalues of the buckling problem:

$$\Lambda_k \sim \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to \infty, \tag{1.5}$$

where ω_n denotes the volume of the unit ball in $\mathbf{R}^n$. By the results of Li and Yau [16] and the variational characterization for eigenvalues, one can obtain a lower bound for eigenvalues of the buckling problem (cf. Levine and Protter [15]):

$$\frac{1}{k}\sum_{j=1}^{k}\Lambda_j \geq \frac{n}{n+2}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}. \tag{1.6}$$

On the other hand, by making use of the recursion formula in [9], one can obtain an upper bound for eigenvalues of the buckling problem, which is sharp in the sense of the order of k, if one can get a sharp universal inequality for eigenvalues of the buckling problem as the following (cf. [8]):

Conjecture. *Eigenvalues of the buckling problem on a bounded domain in a Euclidean space* $\mathbf{R}^n$ *satisfy the following universal inequality:*

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\Lambda_i.$$

Therefore, the next landmark goal for the study on eigenvalues of the buckling problem will be to prove the above sharp universal inequality.

In [8], we decompose $x^p\nabla u_i$ into

$$x^p\nabla u_i = \nabla h_{pi} + \mathbf{w}_{pi}, \tag{1.7}$$

where the notation used may be found in section 2. We make use of the function h_{ip} to construct appropriate trial functions. In order to get our universal inequality, we estimated the L^2-norm of $\mathbf{w}_{pi}$ in [8]. As one knows that to find new appropriate trial functions is very difficult, many years were spent constructing appropriate trial functions in [8]. In this paper, we shall also use the trial functions constructed in [8] and our main observation is to introduce new functions q_{pi} and a careful exploitation of $\nabla q_{pi} = \nabla(x^p u_i - h_{pi})$ and $\Delta\mathbf{w}_{pi}$. Furthermore, the estimate on the lower bound of the L^2-norm of ∇q_{pi} will play an important role in the proof of our Theorem 1.1. In this paper, we will prove

$$\Lambda_i \sum_{p=1}^{n}\|\nabla q_{pi}\|^2 \geq \frac{5}{3}.$$

If one can prove that the L^2-norm of ∇q_{pi} satisfies

$$\Lambda_i \sum_{p=1}^{n}\|\nabla q_{pi}\|^2 \geq 3, \tag{1.8}$$

then the sharp universal inequality in the above conjecture will be obtained (see Remark 2.1 in section 2). In order to prove inequality (1.8), we have spent several years. But we still cannot prove it. Hence, we hope to share our new ideas with mathematicians who are interested in this field such that the landmark goal in the study on eigenvalues of the buckling problem will be finally realized, which also is one of our main purposes to publish this paper.

Theorem 1.1. *Let* Λ_i *be the* i*-th eigenvalue of the buckling problem* (1.2) *for a bounded domain* $\Omega \subset \mathbf{R}^n$*. Then, we have*

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \leq \frac{4(n+\frac{4}{3})}{n^2}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\Lambda_i. \tag{1.9}$$

Remark 1.1. Since our universal inequality is a quadratic inequality of the eigenvalue Λ_{k+1}, we can obtain an upper bound of the gap between two consecutive eigenvalues as in [8] from (1.9). We will not give the details.

When M is an n-dimensional unit sphere $S^n(1)$, Wang and Xia [20] have studied the buckling problem on a domain Ω in $S^n(1)$. They have obtained a universal inequality for eigenvalues of the buckling problem; namely, they have proved that eigenvalues of the buckling problem (1.2) on a domain Ω in the unit sphere $S^n(1)$ satisfy

$$2\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \tag{1.10}$$

$$\leq \sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\{\delta\Lambda_i+\frac{\delta^2\big(\Lambda_i-(n-2)\big)}{4(\delta\Lambda_i+n-2)}\} \tag{1.11}$$

$$+\frac{1}{\delta}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)(\Lambda_i+\frac{(n-2)^2}{4}), \tag{1.12}$$

where δ is an arbitrary positive constant.

According to our knowledge, we think that eigenvalues of the buckling problem on a domain in $S^n(1)$ should satisfy

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)(\Lambda_i+\frac{n^2}{4}).$$

Since one needs to use covariant derivatives for the unit sphere, in order to exchange the orders of covariant derivatives, one must use the Bochner formula, which is different from the case of the Euclidean spaces. Thus, one needs to deal with the terms of Ricci curvature. Hence, it will be very hard work to obtain the above universal inequality. The second purpose in this paper is to give an important improvement for the result of Wang and Xia.

Theorem 1.2. *The eigenvalues Λ_i of the buckling problem* (1.2) *on a domain Ω in the unit sphere $S^n(1)$ satisfy*

$$2\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2+(n-2)\sum_{i=1}^{k}\frac{(\Lambda_{k+1}-\Lambda_i)^2}{\Lambda_i-(n-2)} \tag{1.13}$$

$$\leq \sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\{\Lambda_i-\frac{n-2}{\Lambda_i-(n-2)}\}\delta_i \tag{1.14}$$

$$+\sum_{i=1}^{k}\frac{(\Lambda_{k+1}-\Lambda_i)}{\delta_i}(\Lambda_i+\frac{(n-2)^2}{4}) \tag{1.15}$$

for an arbitrary positive non-increasing monotone sequence $\{\delta_i\}_{i=1}^{k}$.

Remark 1.2. It is obvious that our result is sharper than one of Wang and Xia [20] even if we take $\delta_i=\delta$ for any i. Since our universal inequality is a quadratic inequality of Λ_{k+1}, we can obtain an explicit upper bound for the eigenvalue Λ_{k+1} from (1.11).

In particular, when $n=2$, we have

Corollary 1.1. *The eigenvalues Λ_i of the buckling problem* (1.2) *on a domain Ω in the unit sphere $S^2(1)$ satisfy*

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \leq \sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\Lambda_i^2. \tag{1.16}$$

Proof. Since $n = 2$, from Theorem 1.2 and taking $\delta_i = \dfrac{1}{\Lambda_i}$, for $i = 1, 2, \ldots, k$, for which $\{\delta_i\}_{i=1}^{k}$ is a positive non-increasing monotone sequence, we finish the proof of Corollary 1.1. □

Remark 1.3. For recent developments in universal inequalities for eigenvalues of the Dirichlet eigenvalue problem of the Laplacian and the clamped plate problem, readers can see [4], [5], [6], [7], [9], [12], [13] and [21].

2. Proof of Theorem 1.1

For the convenience of the readers, we review the method for constructing trial functions introduced by Cheng and Yang [8]. In this section, Ω is assumed to be a bounded domain in $\mathbf{R}^n$. For functions f and h, we define the *Dirichlet inner product* $(f, h)_D$ of f and h by

$$(f, h)_D = \int_\Omega \langle \nabla f, \nabla h \rangle.$$

The *Dirichlet norm* of a function f is defined by

$$||f||_D = \{(f, f)_D\}^{1/2} = \left(\int_\Omega \sum_{\alpha=1}^{n} |\nabla_\alpha f|^2 \right)^{1/2}.$$

Let u_i be the i-th orthonormal eigenfunction of the buckling problem (1.2) corresponding to the eigenvalue Λ_i; namely, u_i satisfies

$$\begin{cases} \Delta^2 u_i = -\Lambda_i \Delta u_i \quad \text{in} \ \ \Omega, \\ u_i|_{\partial\Omega} = \frac{\partial u_i}{\partial \nu}\Big|_{\partial\Omega} = 0, \\ (u_i, u_j)_D = \int_\Omega \langle \nabla u_i, \nabla u_j \rangle = \delta_{ij}. \end{cases} \tag{2.1}$$

$H_2^2(\Omega)$ defined by

$$H_2^2(\Omega) = \{f : f, \nabla_\alpha f, \nabla_\alpha \nabla_\beta f \in L^2(\Omega), \quad \alpha, \beta = 1, \ldots, n\}$$

is a Hilbert space with norm $\| \cdot \|_2$:

$$\|f\|_2 = \left(\int_\Omega |f|^2 + \int_\Omega |\nabla f|^2 + \sum_{\beta,\alpha=1}^{n} (\nabla_\alpha \nabla_\beta f)^2 \right)^{1/2}.$$

Let $H_{2,D}^2(\Omega)$ be a subspace of $H_2^2(\Omega)$defined as

$$H_{2,D}^2(\Omega) = \left\{ f \in H_2^2(\Omega) : \ f|_{\partial M} = \frac{\partial}{\partial \nu} f \Big|_{\partial\Omega} = 0 \right\}.$$

The biharmonic operator Δ^2 defines a selfadjoint operator acting on $H_{2,D}^2(\Omega)$ with discrete eigenvalues $\{0 < \Lambda_1 \le \Lambda_2 \le \cdots \le \Lambda_k \le \cdots\}$ for the buckling problem (1.2), and the eigenfunctions defined in (2.1),

$$\{u_i\}_{i=1}^{\infty} = \{u_1, u_2, \ldots, u_k, \ldots\},$$

form a complete orthogonal basis for the Hilbert space $H^2_{2,D}(\Omega)$. We define an inner product $(\mathbf{f}, \mathbf{h})$ for vector-valued functions $\mathbf{f} = (f^1, f^2, \dots, f^n) \in \mathbf{R}^n$ and $\mathbf{h} = (h^1, h^2, \dots, h^n) \in \mathbf{R}^n$ by

$$(\mathbf{f}, \mathbf{h}) \equiv \int_\Omega \langle \mathbf{f}, \mathbf{h} \rangle = \int_\Omega \sum_{\alpha=1}^n f^\alpha h^\alpha.$$

The norm of $\mathbf{f}$ is defined by

$$\|\mathbf{f}\| = (\mathbf{f}, \mathbf{f})^{1/2} = \left\{ \int_\Omega \sum_{\alpha=1}^n (f^\alpha)^2 \right\}^{1/2}.$$

Denote a Hilbert space $\mathbf{H}^2_1(\Omega)$ of the vector-valued functions as

$$\mathbf{H}^2_1(\Omega) = \{\mathbf{f} : f^\alpha, \nabla_\beta f^\alpha \in L^2(\Omega), \text{ for } \alpha, \beta = 1, \dots, n\}$$

with norm $\| \cdot \|_1$:

$$\|\mathbf{f}\|_1 = \left(\|\mathbf{f}\|^2 + \int_\Omega \sum_{\alpha,\beta=1}^n |\nabla_\alpha f^\beta|^2 \right)^{1/2}.$$

Let $\mathbf{H}^2_{1,D}(\Omega) \subset \mathbf{H}^2_1(\Omega)$ be a subspace of $\mathbf{H}^2_1(\Omega)$ spanned by the vector-valued functions $\{\nabla u_i\}_{i=1}^\infty$, which form a complete orthonormal basis of $\mathbf{H}^2_{1,D}(\Omega)$.

It is easy to see that for any $f \in H^2_{2,D}(\Omega)$, $\nabla f \in \mathbf{H}^2_{1,D}(\Omega)$ and for any $\mathbf{h} \in \mathbf{H}^2_{1,D}(\Omega)$, there exists a function $f \in H^2_{2,D}(\Omega)$ such that $\mathbf{h} = \nabla f$.

Let x^p for $p = 1, 2, \dots, n$ be the p-th coordinate function of $\mathbf{R}^n$. For the vector-valued function $x^p \nabla u_i, i = 1, \dots, k$, we decompose it into

$$x^p \nabla u_i = \nabla h_{pi} + \mathbf{w}_{pi}, \tag{2.2}$$

where $h_{pi} \in H^2_{2,D}(\Omega)$ and ∇h_{pi} is the projection of $x^p \nabla u_i$ onto $\mathbf{H}^2_{1,D}(\Omega)$ and $\mathbf{w}_{pi} \perp H^2_{1,D}(\Omega)$. Thus,

$$(\mathbf{w}_{pi}, \nabla u) = \int_\Omega \sum_{j=1}^n w^j_{pi} \nabla_j u = 0, \text{ for any } u \in H^2_{2,D}(\Omega). \tag{2.3}$$

Therefore, since $H^2_{2,D}(\Omega)$ is dense in $L^2(\Omega)$ and $C^1(\Omega)$ is dense in $L^2(\Omega)$, we have, for any function $h \in C^1(\Omega) \cap L^2(\Omega)$,

$$(\mathbf{w}_{pi}, \nabla h) = 0. \tag{2.4}$$

Hence, from the definition of $\mathbf{w}_{pi}$ and (2.4), we have

$$\begin{cases} \mathbf{w}_{pi}|_{\partial\Omega} = 0, \\ \|\mathrm{div} \mathbf{w}_{pi}\|^2 = 0 \quad (\mathrm{div} \mathbf{w}_{pi} \equiv \sum_{j=1}^n \nabla_j w^j_{pi}). \end{cases} \tag{2.5}$$

We define the function φ_{pi} by

$$\varphi_{pi} = h_{pi} - \sum_{j=1}^k b_{pij} u_j, \tag{2.6}$$

where

$$b_{pij} = \int x^p \langle \nabla u_i, \nabla u_j \rangle = b_{pji}.$$

It is easy to check, from definition (2.2) of h_{pi}, that φ_{pi} satisfies

$$\varphi_{pi}|_{\partial\Omega} = \frac{\partial\varphi_{pi}}{\partial\nu}|_{\partial\Omega} = 0 \quad \text{and} \quad (\varphi_{pi}, u_j)_D = (\nabla\varphi_{pi}, \nabla u_j) = 0, \tag{2.7}$$

for any $j = 1, 2, \ldots, k$. Hence, we know that φ_{pi} is a trial function.

In order to prove our Theorem 1.1, we prepare three lemmas.

Lemma 2.1. *For any p and i, we have*

$$1 + 2\|\langle\nabla x^p, \nabla u_i\rangle\|^2 = 2\int x^p u_i \langle\nabla x^p, \nabla(\Delta u_i)\rangle. \tag{2.8}$$

Proof. From the Stokes' formula, we have

$$\begin{aligned}
&\int \langle x^p u_i \nabla x^p, \nabla(\Delta u_i)\rangle \\
&\qquad = -\int \operatorname{div}(x^p u_i \nabla x^p)\Delta u_i \\
&\qquad = -\int u_i \Delta u_i - \int x^p \Delta u_i \langle\nabla x^p, \nabla u_i\rangle, \\
&\int x^p \Delta u_i \langle\nabla x^p, \nabla u_i\rangle \\
&\qquad = -\int \langle\nabla x^p, \nabla u_i\rangle^2 - \int x^p \langle\nabla u_i, \nabla\langle\nabla x^p, \nabla u_i\rangle\rangle \\
&\qquad = -\|\langle\nabla x^p, \nabla u_i\rangle\|^2 + \int \operatorname{div}(x^p \nabla\langle\nabla x^p, \nabla u_i\rangle) u_i \\
&\qquad = -\|\langle\nabla x^p, \nabla u_i\rangle\|^2 + \int \langle\nabla x^p, \nabla\langle\nabla x^p, \nabla u_i\rangle\rangle u_i + \int x^p u_i \Delta\langle\nabla x^p, \nabla u_i\rangle \\
&\qquad = -\|\langle\nabla x^p, \nabla u_i\rangle\|^2 - \int \langle\nabla x^p, \nabla u_i\rangle^2 + \int x^p u_i \langle\nabla x^p, \nabla(\Delta u_i)\rangle.
\end{aligned}$$

Since $\|\nabla u_i\|^2 = 1$, we have

$$1 + 2\|\langle\nabla x^p, \nabla u_i\rangle\|^2 = 2\int x^p u_i \langle\nabla x^p, \nabla(\Delta u_i)\rangle.$$

□

According to $x^p \nabla u_i = \nabla h_{pi} + \mathbf{w}_{pi}$ and $\nabla(x^p u_i) \in H^2_{1,D}(\Omega)$, we have

$$u_i \nabla x^p = \nabla(x^p u_i) - \nabla h_{pi} - \mathbf{w}_{pi} = \nabla q_{pi} - \mathbf{w}_{pi} \tag{2.9}$$

with $\nabla q_{pi} = \nabla(x^p u_i) - \nabla h_{pi}$ and $q_{pi} \in H^2_{2,D}(\Omega)$. Hence, we derive

$$\|u_i\|^2 = \|\nabla q_{pi}\|^2 + \|\mathbf{w}_{pi}\|^2. \tag{2.10}$$

Lemma 2.2. *For any p and i,*

$$3\|\langle\nabla x^p, \nabla u_i\rangle\|^2 - 2\Lambda_i \|\nabla q_{pi}\|^2 = \frac{1}{2} - \frac{1}{2}\Lambda_i \|u_i\|^2. \tag{2.11}$$

Proof. Since, from the Stokes' formula,

$$\begin{aligned}
&\int x^p u_i \langle \nabla x^p, \nabla(\Delta u_i)\rangle \\
&\quad = \int \Delta(x^p u_i)\langle \nabla x^p, \nabla u_i\rangle \\
&\quad = -\int \langle u_i \nabla x^p, \nabla\big(\Delta(x^p u_i)\big)\rangle \\
&\quad = -\int \langle \nabla q_{pi}, \nabla\big(\Delta(x^p u_i)\big)\rangle \qquad \text{(from (2.4) and (2.9))} \\
&\quad = \int q_{pi}\Delta^2(x^p u_i) \\
&\quad = \int q_{pi}\big(4\langle \nabla x^p, \nabla(\Delta u_i)\rangle - \Lambda_i x^p \Delta u_i\big) \\
&\quad = -4\int \Delta u_i \langle \nabla q_{pi}, \nabla x^p\rangle - \Lambda_i \int q_{pi} x^p \Delta u_i
\end{aligned}$$

and

$$\begin{aligned}
-\Lambda_i \int q_{pi} x^p \Delta u_i &= \Lambda_i \int \langle \nabla q_{pi}, x^p \nabla u_i\rangle + \Lambda_i \int q_{pi}\langle \nabla x^p, \nabla u_i\rangle \\
&= \Lambda_i \int \langle \nabla q_{pi}, x^p \nabla u_i\rangle - \Lambda_i \int \langle \nabla q_{pi}, u_i \nabla x^p\rangle \\
&= \Lambda_i \int \langle \nabla q_{pi}, x^p \nabla u_i\rangle - \Lambda_i \|\nabla q_{pi}\|^2, \\
&\quad -4\int \Delta u_i \langle \nabla q_{pi}, \nabla x^p\rangle \\
&= -4\int \langle \nabla(\Delta q_{pi}), u_i \nabla x^p\rangle \\
&= 4\int \Delta q_{pi}\langle \nabla x^p, \nabla u_i\rangle \\
&= -4\int \langle \nabla q_{pi}, \nabla\langle \nabla x^p, \nabla u_i\rangle\rangle \\
&= -4\int \langle u_i \nabla x^p, \nabla\langle \nabla x^p, \nabla u_i\rangle\rangle \\
&= 4\|\langle \nabla x^p, \nabla u_i\rangle\|^2,
\end{aligned}$$

we obtain

$$\int x^p u_i \langle \nabla x^p, \nabla(\Delta u_i)\rangle = 4\|\langle \nabla x^p, \nabla u_i\rangle\|^2 + \Lambda_i \int \langle \nabla q_{pi}, x^p \nabla u_i\rangle - \Lambda_i \|\nabla q_{pi}\|^2. \tag{2.12}$$

From Lemma 2.1 and the above equality, we have

$$6\|\langle \nabla x^p, \nabla u_i\rangle\|^2 - 2\Lambda_i \|\nabla q_{pi}\|^2 - 1 = -2\Lambda_i \int \langle \nabla q_{pi}, x^p \nabla u_i\rangle. \tag{2.13}$$

Furthermore, from (2.4), $x^p\nabla u_i = \nabla h_{pi} + \mathbf{w}_{pi}$ and $\nabla q_{pi} = \nabla(x^p u_i) - \nabla h_{pi}$, we have

$$\begin{aligned}
&\int \langle \nabla q_{pi}, x^p \nabla u_i\rangle \\
&\quad = \int \langle \nabla q_{pi}, \nabla h_{pi}\rangle \\
&\quad = \int \langle \nabla q_{pi}, \nabla(x^p u_i) - \nabla q_{pi}\rangle \\
&\quad = \int \langle \nabla q_{pi}, \nabla(x^p u_i)\rangle - \|\nabla q_{pi}\|^2 \\
&\quad = \int \langle u_i \nabla x^p, \nabla(x^p u_i)\rangle - \|\nabla q_{pi}\|^2 \\
&\quad = \|u_i\|^2 + \int \langle u_i \nabla x^p, x^p \nabla u_i\rangle - \|\nabla q_{pi}\|^2.
\end{aligned} \tag{2.14}$$

Since

$$\int \langle u_i \nabla x^p, x^p \nabla u_i\rangle = -\|u_i\|^2 - \int \langle u_i \nabla x^p, x^p \nabla u_i\rangle,$$

we obtain

$$\int \langle u_i \nabla x^p, x^p \nabla u_i\rangle = -\frac{1}{2}\|u_i\|^2.$$

According to (2.13) and (2.14), we have

$$3\|\langle \nabla x^p, \nabla u_i\rangle\|^2 - 2\Lambda_i \|\nabla q_{pi}\|^2 = \frac{1}{2} - \frac{1}{2}\Lambda_i \|u_i\|^2.$$

This finishes the proof of Lemma 2.2. □

Lemma 2.3. *For any i,*

$$\Lambda_i \sum_{p=1}^{n} \|\mathbf{w}_{pi}\|^2 \geq (n-1) \tag{2.15}$$

holds.

Proof. Since

$$\nabla_\beta(x^p \nabla_\alpha u_i) - \nabla_\alpha(x^p \nabla_\beta u_i) = \nabla_\beta w_{pi}^\alpha - \nabla_\alpha w_{pi}^\beta, \tag{2.16}$$

where $w_{pi}^\alpha = x^p \nabla_\alpha u_i - \nabla_\alpha h_{pi}$ denotes the α-th component of $\mathbf{w}_{pi}$, we infer, from $\mathrm{div}(\mathbf{w}_{pi}) = 0$, that

$$\begin{aligned}
\|\nabla \mathbf{w}_{pi}\|^2 &= \sum_{\alpha,\beta=1}^{n} \|\nabla_\alpha w_{pi}^\beta\|^2 \\
&= \frac{1}{2} \sum_{\alpha,\beta=1}^{n} \|\nabla_\beta w_{pi}^\alpha - \nabla_\alpha w_{pi}^\beta\|^2 + \|\mathrm{div}(\mathbf{w}_{pi})\|^2 \\
&= \frac{1}{2} \sum_{\alpha,\beta=1}^{n} \|\nabla_\beta(x^p \nabla_\alpha u_i) - \nabla_\alpha(x^p \nabla_\beta u_i)\|^2 \\
&= 1 - \|\nabla_p u_i\|^2.
\end{aligned} \tag{2.17}$$

Furthermore, we have

$$\begin{aligned}
\Delta w_{pi}^{\alpha} &= \Delta(x^p \nabla_\alpha u_i - \nabla_\alpha h_{pi}) \\
&= \Delta(x^p \nabla_\alpha u_i) - \nabla_\alpha \Big(\mathrm{div}(\nabla h_{pi}) \Big) \\
&= \Delta(x^p \nabla_\alpha u_i) - \nabla_\alpha \Big(\mathrm{div}(x^p \nabla u_i) \Big) \\
&= \nabla_p \nabla_\alpha u_i - \nabla_\alpha x^p \Delta u_i.
\end{aligned}$$

Thus, we obtain

$$\Delta \mathbf{w}_{pi} = \nabla \langle \nabla x^p, \nabla u_i \rangle - \Delta u_i \nabla x^p. \tag{2.18}$$

For any positive constant ϵ_i, we have

$$\begin{aligned}
\|\nabla \mathbf{w}_{pi}\|^2 &= -\int \langle \mathbf{w}_{pi}, \Delta \mathbf{w}_{pi} \rangle \\
&= -\int \langle \mathbf{w}_{pi}, \nabla \langle \nabla x^p, \nabla u_i \rangle - \Delta u_i \nabla x^p \rangle \\
&\le \frac{\epsilon_i}{2} \|\mathbf{w}_{pi}\|^2 + \frac{1}{2\epsilon_i} \|\nabla \langle \nabla x^p, \nabla u_i \rangle - \Delta u_i \nabla x^p \|^2.
\end{aligned} \tag{2.19}$$

Since, from (2.17),

$$\sum_{p=1}^{n} \|\nabla \mathbf{w}_{pi}\|^2 = n-1, \quad \sum_{p=1}^{n} \|\nabla \langle \nabla x^p, \nabla u_i \rangle \|^2 = \Lambda_i,$$

by taking the sum on p from 1 to n for (2.19), we have

$$(n-1) \le \frac{\epsilon_i}{2} \sum_{p=1}^{n} \|\mathbf{w}_{pi}\|^2 + \frac{n-1}{2\epsilon_i} \Lambda_i.$$

Putting

$$\epsilon_i = \sqrt{\frac{(n-1)\Lambda_i}{\sum_{p=1}^{n} \|\mathbf{w}_{pi}\|^2}},$$

we obtain

$$\Lambda_i \sum_{p=1}^{n} \|\mathbf{w}_{pi}\|^2 \ge (n-1).$$

This completes the proof of Lemma 2.3. $\square$

Proof of Theorem 1.1. Since φ_{pi} is a trial function, from the Rayleigh-Ritz inequality, we have

$$\Lambda_{k+1} \|\nabla \varphi_{pi}\|^2 \le \int \varphi_{pi} \Delta^2 \varphi_{pi} = -\int \nabla \varphi_{pi} \cdot \nabla (\Delta \varphi)_{pi}. \tag{2.20}$$

By making use of the same arguments as in Cheng and Yang [8], we have, for any p and i,

(2.21)
$$(\Lambda_{k+1}-\Lambda_i)\|\nabla\varphi_{pi}\|^2 \le 1+3\|\nabla_p u_i\|^2-\Lambda_i(\|u_i\|^2-\|\mathbf{w}_{pi}\|^2)+\sum_{j=1}^k(\Lambda_i-\Lambda_j)b_{pij}^2,$$

$$1+2\sum_{j=1}^k b_{pij}c_{pij}=-2\int_\Omega\langle\nabla\varphi_{pi},\nabla\langle\nabla x^p,\nabla u_i\rangle\rangle, \tag{2.22}$$

where
$$c_{pij}=\int\langle\nabla\langle\nabla x^p,\nabla u_i\rangle,\nabla u_j\rangle=-c_{pji}.$$

Hence, we have, for any positive constant δ_i,

$$\begin{aligned}&(\Lambda_{k+1}-\Lambda_i)^2(1+2\sum_{j=1}^k b_{pij}c_{pij})\\ &=(\Lambda_{k+1}-\Lambda_i)^2\int_\Omega -2\langle\nabla\varphi_{pi},\nabla\langle\nabla x^p,\nabla u_i\rangle-\sum_{j=1}^k c_{pij}\nabla u_j\rangle\\ &\le\delta_i(\Lambda_{k+1}-\Lambda_i)^3\|\nabla\varphi_{pi}\|^2+\frac{1}{\delta_i}(\Lambda_{k+1}-\Lambda_i)\left(\|\nabla\langle\nabla x^p,\nabla u_i\rangle\|^2-\sum_{j=1}^k c_{pij}^2\right).\end{aligned}$$

From (2.21) and $\|u_i\|^2=\|\nabla q_{pi}\|^2+\|\mathbf{w}_{pi}\|^2$, we obtain

$$\begin{aligned}&(\Lambda_{k+1}-\Lambda_i)^2(1+2\sum_{j=1}^k b_{pij}c_{pij})\\ &\le\delta_i(\Lambda_{k+1}-\Lambda_i)^2\Big(1+3\|\nabla_p u_i\|^2-\Lambda_i\|\nabla q_{pi}\|^2+\sum_{j=1}^k(\Lambda_i-\Lambda_j)b_{pij}^2\Big)\\ &\quad+\frac{1}{\delta_i}(\Lambda_{k+1}-\Lambda_i)\left(\|\nabla\langle\nabla x^p,\nabla u_i\rangle\|^2-\sum_{j=1}^k c_{pij}^2\right).\end{aligned}\tag{2.23}$$

By taking the sum on p from 1 to n, we derive

$$\begin{aligned}&(\Lambda_{k+1}-\Lambda_i)^2(n+2\sum_{p=1}^n\sum_{j=1}^k b_{pij}c_{pij})\\ &\le\delta_i(\Lambda_{k+1}-\Lambda_i)^2\Big(n+3-\Lambda_i\sum_{p=1}^n\|\nabla q_{pi}\|^2+\sum_{p=1}^n\sum_{j=1}^k(\Lambda_i-\Lambda_j)b_{pij}^2\Big)\\ &\quad+\frac{1}{\delta_i}(\Lambda_{k+1}-\Lambda_i)\left(\Lambda_i-\sum_{p=1}^n\sum_{j=1}^k c_{pij}^2\right).\end{aligned}\tag{2.24}$$

From Lemma 2.2, Lemma 2.3 and
$$\|u_i\|^2=\|\nabla q_{pi}\|^2+\|\mathbf{w}_{pi}\|^2,$$

we infer that

$$\Lambda_i \sum_{p=1}^{n} \|\nabla q_{pi}\|^2 \geq \frac{5}{3}.$$

Thus, we obtain, for any i,

$$\begin{aligned}
&(\Lambda_{k+1}-\Lambda_i)^2(n+2\sum_{p=1}^{n}\sum_{j=1}^{k} b_{pij}c_{pij}) \\
&\leq \delta_i(\Lambda_{k+1}-\Lambda_i)^2\left(n+\frac{4}{3}+\sum_{p=1}^{n}\sum_{j=1}^{k}(\Lambda_i-\Lambda_j)b_{pij}^2\right) \\
&\quad + \frac{1}{\delta_i}(\Lambda_{k+1}-\Lambda_i)\left(\Lambda_i-\sum_{p=1}^{n}\sum_{j=1}^{k}c_{pij}^2\right).
\end{aligned} \tag{2.25}$$

By taking the sum for i from 1 to k and noticing that b_{pij} is symmetric and c_{pij} is antisymmetric on i,j, we have

$$\begin{aligned}
&n\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 - 2\sum_{p=1}^{n}\sum_{i,j=1}^{k}(\Lambda_{k+1}-\Lambda_i)(\Lambda_i-\Lambda_j)b_{pij}c_{pij} \\
&\leq (n+\frac{4}{3})\sum_{i=1}^{k}\delta_i(\Lambda_{k+1}-\Lambda_i)^2 + \sum_{i=1}^{k}\frac{1}{\delta_i}(\Lambda_{k+1}-\Lambda_i)\Lambda_i \\
&\quad -\sum_{p=1}^{n}\sum_{i,j=1}^{k}\delta_i(\Lambda_{k+1}-\Lambda_i)(\Lambda_i-\Lambda_j)^2 b_{pij}^2 - \sum_{i,j=1}^{k}\frac{1}{\delta_i}(\Lambda_{k+1}-\Lambda_i)c_{pij}^2 \\
&\quad +\sum_{p=1}^{n}\sum_{i,j=1}^{k}\delta_i(\Lambda_{k+1}-\Lambda_i)(\Lambda_i-\Lambda_j)^2 b_{pij}^2 \\
&\quad +\sum_{p=1}^{n}\sum_{i,j=1}^{k}\delta_i(\Lambda_{k+1}-\Lambda_i)^2(\Lambda_i-\Lambda_j) b_{pij}^2.
\end{aligned} \tag{2.26}$$

Since, for a non-increasing monotone sequence $\{\delta_i\}_{i=1}^{k}$,

$$\begin{aligned}
&\sum_{p=1}^{n}\sum_{i,j=1}^{k}\delta_i(\Lambda_{k+1}-\Lambda_i)(\Lambda_i-\Lambda_j)^2 b_{pij}^2 + \sum_{p=1}^{n}\sum_{i,j=1}^{k}\delta_i(\Lambda_{k+1}-\Lambda_i)^2(\Lambda_i-\Lambda_j) b_{pij}^2 \\
&= \frac{1}{2}\sum_{p=1}^{n}\sum_{i,j=1}^{k}(\Lambda_{k+1}-\Lambda_i)(\Lambda_{k+1}-\Lambda_j)(\Lambda_i-\Lambda_j)(\delta_i-\delta_j)b_{pij}^2 \leq 0,
\end{aligned}$$

we conclude from (2.26) and the above formula, for a non-increasing monotone sequence $\{\delta_i\}_{i=1}^{k}$, that

$$n\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \leq (n+\frac{4}{3})\sum_{i=1}^{k}\delta_i(\Lambda_{k+1}-\Lambda_i)^2 + \sum_{i=1}^{k}\frac{1}{\delta_i}(\Lambda_{k+1}-\Lambda_i)\Lambda_i.$$

In particular, putting

$$\delta_i = \frac{n}{2(n+\frac{4}{3})}$$

for any i, we obtain

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \leq \frac{4(n+\frac{4}{3})}{n^2}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\Lambda_i.$$

This finishes the proof of Theorem 1.1. □

Remark 2.1. If one can prove, for any i,

$$\Lambda_i \sum_{p=1}^{n}\|\nabla q_{pi}\|^2 \geq 3,$$

one will infer that

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\Lambda_i,$$

which solves the conjecture.

3. Proof of Theorem 1.2

For the unit sphere

$$S^n(1)=\left\{(x^1,x^2,\ldots,x^{n+1})\in \mathbf{R}^{n+1}\ :\ \sum_{i=1}^{n+1}(x^p)^2=1\right\},$$

we denote the induced metric on $S^n(1)$ by the canonical metric $\langle\cdot,\cdot\rangle$ on $\mathbf{R}^{n+1}$ also. For any p, we have

$$\nabla_i\nabla_j x^p = -g_{ij}x^p, \qquad \Delta x^p = -nx^p, \tag{3.1}$$

where g_{ij} denotes the components of the metric tensor of $S^n(1)$. Let u_i be the i-th orthonormal eigenfunction of the buckling problem (1.2) corresponding to the eigenvalue Λ_i; namely, u_i satisfies

$$\begin{cases} \Delta^2 u_i = -\Lambda_i \Delta u_i \quad \text{in} \quad \Omega, \\ u_i|_{\partial\Omega} = \frac{\partial u_i}{\partial \nu}\big|_{\partial\Omega} = 0, \\ (u_i,u_j)_D = \int_\Omega \langle \nabla u_i, \nabla u_j\rangle = \delta_{ij}. \end{cases} \tag{3.2}$$

For constructing trial functions, we use the same notation as in section 2. We would like to remark that vector-valued functions in this section have $n+1$ components. Although the orders of differentiations of functions in the Euclidean space can be exchanged freely, we must do it very carefully for the covariant differentiations of functions in the case of the unit sphere.

Since x^p for $p=1,2,\ldots,n+1$ is a coordinate function of $\mathbf{R}^{n+1}$, for the vector-valued function $x^p\nabla u_i, i=1,\ldots,k$, we decompose it into

$$x^p\nabla u_i = \nabla h_{pi} + \mathbf{w}_{pi}, \tag{3.3}$$

where $h_{pi}\in H^2_{2,D}(\Omega)$ and ∇h_{pi} is the projection of $x^p\nabla u_i$ onto $\mathbf{H}^2_{1,D}(\Omega)$ and $\mathbf{w}_{pi}\perp H^2_{1,D}(\Omega)$. Thus, we have, for any function $h\in C^1(\Omega)\cap L^2(\Omega)$,

$$(\mathbf{w}_{pi}, \nabla h) = 0. \tag{3.4}$$

Hence, $\mathbf{w}_{pi}$ satisfies

$$\begin{cases} \mathbf{w}_{pi}|_{\partial\Omega} = 0, \\ \|\text{div}\mathbf{w}_{pi}\|^2 = 0. \end{cases} \tag{3.5}$$

We define the function φ_{pi} by

$$\varphi_{pi} = h_{pi} - \sum_{j=1}^{k} b_{pij} u_j, \tag{3.6}$$

where

$$b_{pij} = \int x^p \langle \nabla u_i, \nabla u_j \rangle = b_{pji}.$$

It is easy to check that φ_{pi} satisfies

$$\varphi_{pi}|_{\partial\Omega} = \frac{\partial \varphi_{pi}}{\partial \nu}|_{\partial\Omega} = 0 \quad \text{and} \quad (\varphi_{pi}, u_j)_D = (\nabla \varphi_{pi}, \nabla u_j) = 0,$$

for any $j = 1, 2, \ldots, k$, that is, φ_{pi} is a trial function. Since $\sum_{p=1}^{n+1} (x^p)^2 = 1$, from (3.3), we have, for any i,

$$1 = \sum_{p=1}^{n+1} \|\nabla h_{pi}\|^2 + \sum_{p=1}^{n+1} \|\mathbf{w}_{pi}\|^2. \tag{3.7}$$

Lemma 3.1. *For any i, we have*

$$\sum_{p=1}^{n+1} \|\mathbf{w}_{pi}\|^2 \leq \frac{\Lambda_i - (n-1)}{\Lambda_i - (n-2)}. \tag{3.8}$$

Proof. From $\sum_{p=1}^{n+1} (x^p)^2 = 1$, we have

$$\begin{aligned} 1 &= \sum_{p=1}^{n+1} \|\langle \nabla x^p, \nabla u_i \rangle\|^2 \\ &= -\sum_{p=1}^{n+1} \int x^p \mathrm{div}\{\langle \nabla x^p, \nabla u_i \rangle \nabla u_i\} \\ &= -\sum_{p=1}^{n+1} \int x^p \langle \nabla x^p, \nabla u_i \rangle \Delta u_i - \sum_{p=1}^{n+1} \int \langle x^p \nabla u_i, \nabla \langle \nabla x^p, \nabla u_i \rangle \rangle \\ &= -\sum_{p=1}^{n+1} \int \langle \nabla h_{pi}, \nabla \langle \nabla x^p, \nabla u_i \rangle \rangle. \end{aligned}$$

For any positive constant ϵ_i, we have

$$1 \leq \epsilon_i \sum_{p=1}^{n+1} \|\nabla h_{pi}\|^2 + \frac{1}{4\epsilon_i} \sum_{p=1}^{n+1} \|\nabla \langle \nabla x^p, \nabla u_i \rangle\|^2. \tag{3.9}$$

According to the following Bochner formula for a smooth function f:

$$\begin{aligned} \frac{1}{2} \Delta |\nabla f|^2 &= |\nabla^2 f|^2 + \langle \nabla f, \nabla(\Delta f) \rangle + \mathrm{Ric}(\nabla f, \nabla f) \\ &= |\nabla^2 f|^2 + \langle \nabla f, \nabla(\Delta f) \rangle + (n-1)|\nabla f|^2, \end{aligned}$$

where Ric and $\nabla^2 f$ denote the Ricci tensor of $S^n(1)$ and the Hessian of f, respectively, we can derive, from (3.1) and by making use of a direct computation,

$$\Delta\langle\nabla x^p, \nabla u_i\rangle = -2x^p\Delta u_i + \langle\nabla x^p, \nabla(\Delta u_i)\rangle + (n-2)\langle\nabla x^p, \nabla u_i\rangle. \tag{3.10}$$

Hence, we have

$$\begin{aligned}
&\sum_{p=1}^{n+1}\|\nabla\langle\nabla x^p, \nabla u_i\rangle\|^2\\
&\quad= -\sum_{p=1}^{n+1}\int\langle\nabla x^p, \nabla u_i\rangle\Delta\langle x^p, \nabla u_i\rangle\\
&\quad= -\sum_{p=1}^{n+1}\int\langle\nabla x^p, \nabla u_i\rangle\Big\{-2x^p\Delta u_i + \langle\nabla x^p, \nabla(\Delta u_i)\rangle + (n-2)\langle\nabla x^p, \nabla u_i\rangle\Big\}\\
&\quad= -\sum_{p=1}^{n+1}\Big\{\int\langle\nabla x^p, \nabla u_i\rangle\langle\nabla x^p, \nabla(\Delta u_i)\rangle + (n-2)\langle\nabla x^p, \nabla u_i\rangle^2\Big\}\\
&\quad= -\int\langle\nabla u_i, \nabla(\Delta u_i)\rangle - (n-2)\|\nabla u_i\|2\\
&\quad= \Lambda_i - (n-2),
\end{aligned}$$

that is,

$$\sum_{p=1}^{n+1}\|\nabla\langle\nabla x^p, \nabla u_i\rangle\|^2 = \Lambda_i - (n-2). \tag{3.11}$$

Here we have used

$$\sum_{p=1}^{n+1}\int\langle\nabla x^p, \nabla u_i\rangle\langle\nabla x^p, \nabla(\Delta u_i)\rangle = \int\langle\nabla u_i, \nabla(\Delta u_i)\rangle.$$

Therefore, from (3.9), we obtain

$$1 \le \epsilon_i\sum_{p=1}^{n+1}\|\nabla h_{pi}\|^2 + \frac{1}{4\epsilon_i}\Big(\Lambda_i - (n-2)\Big).$$

From (3.7), we have

$$1 + \epsilon_i\sum_{p=1}^{n+1}\|\mathbf{w}_{pi}\|^2 \le \epsilon_i + \frac{1}{4\epsilon_i}\Big(\Lambda_i - (n-2)\Big).$$

Taking

$$\epsilon_i = \frac{\Lambda_i - (n-2)}{2},$$

we complete the proof of Lemma 3.1. □

Proof of Theorem 1.2. By making use of the trial function φ_{pi} and the same argumants as in Wang and Xia [20], we have, for any p and i,

$$(\Lambda_{k+1} - \Lambda_i)\|\nabla\varphi_{pi}\|^2 \le P_{pi} + \|\langle\nabla x^p, \nabla u_i\rangle\|^2 + \Lambda_i\|\mathbf{w}_{pi}\|^2 + \sum_{j=1}^{k}(\Lambda_i - \Lambda_j)b_{pij}^2, \tag{3.12}$$

where

$$P_{pi} = \int \langle \nabla (x^p)^2, u_i \nabla(\Delta u_i) + \Lambda_i u_i \nabla u_i \rangle.$$

Defining

$$Z_{pi} = \nabla \langle \nabla x^p, \nabla u_i \rangle - \frac{n-2}{2} x^p \nabla u_i,$$
$$c_{pij} = \int \langle \nabla u_j, Z_{pi} \rangle = -c_{pji}$$

has been proved in Wang and Xia [20]. Since

$$\begin{aligned}\gamma_{pi} &= -2 \int \langle x^p \nabla u_i, Z_{pi} \rangle \\ &= -2 \int \langle \nabla h_{pi} + \mathbf{w}_{pi}, Z_{pi} \rangle \\ &= -2 \int \langle \nabla \varphi_{pi} + \sum_{j=1}^{k} b_{pij} \nabla u_j + \mathbf{w}_{pi}, Z_{pi} \rangle \\ &= -2 \int \langle \nabla \varphi_{pi}, Z_{pi} - \sum_{j=1}^{k} c_{pij} \nabla u_j \rangle - 2 \sum_{j=1}^{k} b_{pij} c_{pij} + (n-2) \|\mathbf{w}_{pi}\|^2,\end{aligned}$$

we have

$$\gamma_{pi} + 2 \sum_{j=1}^{k} b_{pij} c_{pij} = -2 \int \langle \nabla \varphi_{pi}, Z_{pi} - \sum_{j=1}^{k} c_{pij} \nabla u_j \rangle + (n-2) \|\mathbf{w}_{pi}\|^2.$$

Hence, for any positive constant δ_i, we have, according to (3.12),

(3.13)

$$\begin{aligned}&(\Lambda_{k+1} - \Lambda_i)^2 \left(\gamma_{pi} + 2 \sum_{j=1}^{k} b_{pij} c_{pij} \right) - (n-2)(\Lambda_{k+1} - \Lambda_i)^2 \|\mathbf{w}_{pi}\|^2 \\ &\quad \leq \delta_i (\Lambda_{k+1} - \Lambda_i)^3 \|\nabla \varphi_{pi}\|^2 + \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \left(\|Z_{pi}\|^2 - \sum_{j=1}^{k} c_{pij}^2 \right) \\ &\quad \leq \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \left\{ P_{pi} + \|\langle \nabla x^p, \nabla u_i \rangle\|^2 + \Lambda_i \|\mathbf{w}_{pi}\|^2 + \sum_{j=1}^{k} (\Lambda_i - \Lambda_j) b_{pij}^2 \right\} \\ &\qquad + \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \left(\|Z_{pi}\|^2 - \sum_{j=1}^{k} c_{pij}^2 \right).\end{aligned}$$

By taking the sum on p from 1 to n, we derive

$$
\begin{aligned}
(\Lambda_{k+1}-\Lambda_i)^2\sum_{p=1}^{n+1}\Big(\gamma_{pi}+2\sum_{j=1}^{k}b_{pij}c_{pij}\Big)&-(n-2)(\Lambda_{k+1}-\Lambda_i)^2\sum_{p=1}^{n+1}\|\mathbf{w}_{pi}\|^2\\
&\le \delta_i(\Lambda_{k+1}-\Lambda_i)^2\sum_{p=1}^{n+1}\Big\{P_{pi}+\|\langle\nabla x^p,\nabla u_i\rangle\|^2\\
&\quad+\Lambda_i\|\mathbf{w}_{pi}\|^2+\sum_{j=1}^{k}(\Lambda_i-\Lambda_j)b_{pij}^2\Big\}\\
&\quad+\frac{1}{\delta_i}(\Lambda_{k+1}-\Lambda_i)\sum_{p=1}^{n+1}\left(\|Z_{pi}\|^2-\sum_{j=1}^{k}c_{pij}^2\right).
\end{aligned}
\tag{3.14}
$$

Since

$$
\begin{aligned}
\gamma_{pi}&=-2\int\langle x^p\nabla u_i,Z_{pi}\rangle\\
&=-2\int\langle x^p\nabla u_i,\nabla\langle\nabla x^p,\nabla u_i\rangle-\frac{n-2}{2}x^p\nabla u_i\rangle\\
&=2\int\langle\nabla x^p,\nabla u_i\rangle^2+2\int\Delta u_i\langle x^p\nabla x^p,\nabla u_i\rangle+(n-2)\int(x^p)^2\langle\nabla u_i,\nabla u_i\rangle,
\end{aligned}
$$

we have

$$\sum_{p=1}^{n+1}\gamma_{pi}=n.$$

From the definition of Z_{pi}, we have

$$
\begin{aligned}
&\sum_{p=1}^{n+1}\|Z_{pi}\|^2\\
&=\sum_{p=1}^{n+1}\int|\nabla\langle\nabla x^p,\nabla u_i\rangle-\frac{n-2}{2}x^p\nabla u_i|^2\\
&=\sum_{p=1}^{n+1}\Big\{\|\nabla\langle\nabla x^p,\nabla u_i\rangle\|^2-(n-2)\int\langle\nabla\langle\nabla x^p,\nabla u_i\rangle,x^p\nabla u_i\rangle+\frac{(n-2)^2}{4}\|x^p\nabla u_i\|^2\Big\}\\
&=\Lambda_i+\frac{(n-2)^2}{4}\quad\text{(from (3.11))}.
\end{aligned}
$$

Since $P_{pi}=\int\langle\nabla(x^p)^2,u_i\nabla(\Delta u_i)+\Lambda_iu_i\nabla u_i\rangle$, we have

$$\sum_{p=1}^{n+1}P_{pi}=0.$$

From Lemma 3.1 and (3.14), we obtain

$$\begin{aligned}(\Lambda_{k+1}-\Lambda_i)^2\Big(n+2\sum_{p=1}^{n+1}\sum_{j=1}^{k}b_{pij}c_{pij}\Big)&-(n-2)(\Lambda_{k+1}-\Lambda_i)^2\frac{\Lambda_i-(n-1)}{\Lambda_i-(n-2)}\\ &\le \delta_i(\Lambda_{k+1}-\Lambda_i)^2\Big\{1+\Lambda_i\frac{\Lambda_i-(n-1)}{\Lambda_i-(n-2)}+\sum_{p=1}^{n+1}\sum_{j=1}^{k}(\Lambda_i-\Lambda_j)b_{pij}^2\Big\}\\ &\quad+\frac{1}{\delta_i}(\Lambda_{k+1}-\Lambda_i)\Big(\Lambda_i+\frac{(n-2)^2}{4}\Big)-\frac{1}{\delta_i}(\Lambda_{k+1}-\Lambda_i)\sum_{p=1}^{n+1}\sum_{j=1}^{k}c_{pij}^2,\end{aligned}$$

that is,

(3.15)

$$\begin{aligned}2(\Lambda_{k+1}-\Lambda_i)^2&+(n-2)\frac{(\Lambda_{k+1}-\Lambda_i)^2}{\Lambda_i-(n-2)}\\ &\le \delta_i(\Lambda_{k+1}-\Lambda_i)^2\Big\{\Lambda_i-\frac{(n-2)}{\Lambda_i-(n-2)}\Big\}+\frac{1}{\delta_i}(\Lambda_{k+1}-\Lambda_i)\Big(\Lambda_i+\frac{(n-2)^2}{4}\Big)\\ &\quad-2(\Lambda_{k+1}-\Lambda_i)^2\sum_{p=1}^{n+1}\sum_{j=1}^{k}b_{pij}c_{pij}+\delta_i(\Lambda_{k+1}-\Lambda_i)^2\sum_{p=1}^{n+1}\sum_{j=1}^{k}(\Lambda_i-\Lambda_j)b_{pij}^2\\ &\quad-\frac{1}{\delta_i}(\Lambda_{k+1}-\Lambda_i)\sum_{p=1}^{n+1}\sum_{j=1}^{k}c_{pij}^2.\end{aligned}$$

Since, for a non-increasing monotone sequence $\{\delta_i\}_{i=1}^k$,

$$\begin{aligned}\sum_{p=1}^{n}\sum_{i,j=1}^{k}&\delta_i(\Lambda_{k+1}-\Lambda_i)(\Lambda_i-\Lambda_j)^2b_{pij}^2+\sum_{p=1}^{n}\sum_{i,j=1}^{k}\delta_i(\Lambda_{k+1}-\Lambda_i)^2(\Lambda_i-\Lambda_j)b_{pij}^2\\ &=\frac{1}{2}\sum_{p=1}^{n}\sum_{i,j=1}^{k}(\Lambda_{k+1}-\Lambda_i)(\Lambda_{k+1}-\Lambda_j)(\Lambda_i-\Lambda_j)(\delta_i-\delta_j)b_{pij}^2\le 0\end{aligned}$$

and

$$\begin{aligned}&-2\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\sum_{p=1}^{n+1}\sum_{j=1}^{k}b_{pij}c_{pij}-\sum_{i=1}^{k}\delta_i(\Lambda_{k+1}-\Lambda_i)\sum_{p=1}^{n}\sum_{j=1}^{k}(\Lambda_i-\Lambda_j)^2b_{pij}^2\\ &-\sum_{i=1}^{k}\frac{1}{\delta_i}(\Lambda_{k+1}-\Lambda_i)\sum_{p=1}^{n+1}\sum_{j=1}^{k}c_{pij}^2\\ &=-\sum_{p=1}^{n}\sum_{i,j=1}^{k}\Big(\sqrt{\delta_i(\Lambda_{k+1}-\Lambda_i)}(\Lambda_i-\Lambda_j)b_{pij}-\frac{1}{\sqrt{\delta_i}}\sqrt{(\Lambda_{k+1}-\Lambda_i)}c_{pij}\Big)^2\le 0,\end{aligned}$$

by taking the sum on i from 1 to k for (3.15), we obtain

$$
\begin{aligned}
&2\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2+(n-2)\sum_{i=1}^{k}\frac{(\Lambda_{k+1}-\Lambda_i)^2}{\Lambda_i-(n-2)}\\
&\le \sum_{i=1}^{k}\delta_i(\Lambda_{k+1}-\Lambda_i)^2\left\{\Lambda_i-\frac{(n-2)}{\Lambda_i-(n-2)}\right\}\\
&+\sum_{i=1}^{k}\frac{1}{\delta_i}(\Lambda_{k+1}-\Lambda_i)\left(\Lambda_i+\frac{(n-2)^2}{4}\right).
\end{aligned}
\tag{3.16}
$$

This completes the proof of Theorem 1.2. □

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