PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

FIRST EIGENVALUE OF JACOBI OPERATOR OF HYPERSURFACES WITH CONSTANT SCALAR CURVATURE*

QING-MING CHENG

ABSTRACT. Let M be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, $r > 1$, in a unit sphere $S^{n+1}(1)$. We know that such hypersurfaces can be characterized as critical points for a variational problem of the integral $\int_M H dM$ of the mean curvature H. In this paper, we study first eigenvalue of the Jacobi operator J_s of M. We derive an optimal upper bound for the first eigenvalue of J_s and this bound is attained if and only if M is a totally umbilical and non-totally geodesic hypersurface or M is a Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \le m \le n-1$.

1. INTRODUCTION

Let M be an n-dimensional hypersurface in a unit sphere $S^{n+1}(1)$ of dimension $n+1$. We denote the second fundamental form of M and its squared norm by α and S, respectively. Then, a Schrödinger operator

$$J_m = -\Delta - S - n,$$

where Δ stands for the Laplace-Beltrami operator, arose naturally in the study of the stability of both minimal hypersurfaces in $S^{n+1}(1)$ and hypersurfaces with constant mean curvature in $S^{n+1}(1)$. The J_m is called Jacobi operator or a stability operator, which represents the second variation of the volume. Its spectral behavior is directly related to the instability of such hypersurfaces (cf. [18] and [5]).

On the other hand, for any C^2-function f, denoting its Hessian by (f_{ij}), we define a differential operator

$$\Box f = \sum_{i,j=1}^{n} (nH\delta_{ij} - h_{ij}) f_{ij},$$

where H and h_{ij} denote the mean curvature and components of the second fundamental form of M. The differential operator $\Box$ was introduced and used by S. Y. Cheng and Yau in [11] to study compact hypersurfaces with constant scalar curvature in $S^{n+1}(1)$. They proved that if M is an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, $r \ge 1$, and if the sectional curvature of M is non-negative, then M is a totally umbilical hypersurface $S^n(c)$ or a Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \le m \le n-1$, where $S^k(c)$ denotes a sphere

Received by the editors November 29, 2006 and, in revised form, June 25, 2007.
2000 *Mathematics Subject Classification.* Primary 53C42; Secondary 58J50.
Key words and phrases. hypersurface with constant scalar curvature, Jacobi operator, mean curvature, first eigenvalue and principal curvatures.
* The author's research was partially supported by a Grant-in-Aid for Scientific Research from JSPS.

of radius c. We should notice that the differential operator $\Box$ is self-adjoint. By making use of the similar method which has been used by Nakagawa and the author in [9] and the differential operator $\Box$ introduced by S.Y. Cheng and Yau, Li [14] has proved that if M is an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, $r \geq 1$, and if $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, then M is a totally umbilical hypersurface or a Riemannian product $S^{n-1}(c) \times S^1(\sqrt{1-c^2})$ with $c^2 = \frac{n-2}{nr} \leq \frac{n-2}{n}$. Furthermore, the Riemannian product $S^{n-1}(c) \times S^1(\sqrt{1-c^2})$ has been characterized in [6], [7], [8] and [10].

In [1], Alencar, do Carmo and Colares have studied the stability of hypersurfaces with constant scalar curvature in $S^{n+1}(1)$. In this case, the Jacobi operator J_s is given by

$$J_s = -\Box - \{n(n-1)H + nHS - f_3\},$$

which is associated to the variational characterization of hypersurfaces with constant scalar curvature in $S^{n+1}(1)$, where $f_3 = \sum_{j=1}^n k_j^3$ and k_j's are the principal curvatures of M (cf. [16] and [17]). The spectral behavior of J_s is also directly related to the instability of hypersurfaces with constant scalar curvature.

The first eigenvalue of the Jacobi operator J_m of minimal hypersurfaces in $S^{n+1}(1)$ was studied by Simons [18], Wu [19] and Perdomo [15]. A characterization of Clifford torus is given by the first eigenvalue of the Jacobi operator J_m, that is, they proved that if M is an n-dimensional compact orientable minimal hypersurface in $S^{n+1}(1)$, then, the first eigenvalue $\lambda_1^{J_m}$ of the Jacobi operator J_m satisfies

(1) $\lambda_1^{J_m} = -n$ and M is totally geodesic.
(2) $\lambda_1^{J_m} \leq -2n$ and $\lambda_1^{J_m} = -2n$ if and only if M is a Clifford torus $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$, $1 \leq m \leq n-1$.

Very recently, Alías, Barros and Brasil [3] have extended the above results to compact hypersurfaces with constant mean curvature in $S^{n+1}(1)$. They have obtained that if M is an n-dimensional compact orientable hypersurface with constant mean curvature in $S^{n+1}(1)$, then, the first eigenvalue $\lambda_1^{J_m}$ of the Jacobi operator J_m satisfies

(1) $\lambda_1^{J_m} = -n(1+H^2)$ and M is totally umbilical or
(2) $\lambda_1^{J_m} \leq -2n(1+H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}}|H| \max\sqrt{S-nH^2}$ and equality holds if and only if M is a Riemannian product $S^{n-1}(c) \times S^1(\sqrt{1-c^2})$.

We should notice that the first eigenvalue of Jacobi operator J_m of the totally umbilical hypersurface $S^{n+1}(1)$ is different from the one of the Riemannian product $S^{n-1}(c) \times S^1(\sqrt{1-c^2})$ in $S^{n+1}(1)$.

Since the Laplace-Beltrami operator is always elliptic, the Jacobi operator J_m is always elliptic. But, in general, the operator $\Box$, and hence the Jacobi operator J_s are not elliptic. When $r > 1$, the differential operator $\Box$ is elliptic. In fact, from Gauss equation (2.4) in section 2, we have that the mean curvature H satisfies $n^2H^2 > S$. Hence, we can assume $H > 0$. Thus, the differential operator $\Box$ is elliptic if and only if $nH - k_j > 0$ for $j = 1, 2, \cdots, n$, where k_j's are the principal curvatures of M. If, for some j, $nH \leq k_j$ holds, then $n^2H^2 \leq k_j^2 \leq S$. This is impossible.

When $r=1$, let k_i, $i=1,2,\cdots,n$, denote the principal curvatures of M. We consider the elementary symmetric functions S_r of the principal curvatures:

$$S_0=1,\quad S_r=\sum_{i_1<\cdots<i_r} k_{i_1}\cdots k_{i_r}\quad (1\le r\le n).$$

$\square$ is elliptic if and only if $n\ge 3$ and the $S_3\neq 0$ on M (cf. [4], [2] and [12]).

In [4], Alías, Brasil and Sousa have studied the first eigenvalue of the Jacobi operator J_s of an n-dimensional hypersurface with constant scalar curvature $n(n-1)$. They have proved that if M is an n-dimensional compact orientable hypersurface with constant scalar curvature $n(n-1)$, in $S^{n+1}(1)$ and if $n\ge 3$ and S_3 does not vanish on M, then, the first eigenvalue $\lambda_1^{J_s}$ of the Jacobi operator J_s satisfies

$$\lambda_1^{J_s}\le -2n(n-1)\min|H|$$

Further, the equality holds if and only if M is a Riemannian product $S^m(c)\times S^{n-m}(\sqrt{1-c^2})$, $1\le m\le n-1$, with scalar curvature $n(n-1)$.

In this paper, we investigate the first eigenvalue of the Jacobi operator J_s of an n-dimensional hypersurface with constant scalar curvature $n(n-1)r$, $r>1$ and give a characterization of the totally umbilical and non-totally geodesic hypersurface and the Riemannian product $S^m(c)\times S^{n-m}(\sqrt{1-c^2})$, $1\le m\le n-1$ by the first eigenvalue of J_s. Namely, we will prove the following:

Theorem 1.1. *Let M be an n-dimensional compact orientable hypersurface with constant scalar curvature $n(n-1)r$, $r>1$, in $S^{n+1}(1)$. Then, the Jacobi operator J_s is elliptic, the mean curvature H does not vanish on M and the first eigenvalue $\lambda_1^{J_s}$ of the Jacobi operator J_s satisfies*

$$\lambda_1^{J_s}\le -\{2n(n-1)+n^2(n-1)(r-1)\}\min|H|+n(n-1)(r-1)\{(n-1)(r-1)+1\}\frac{1}{\min|H|}$$

and the equality holds if and only if either M is totally umbilical and non-totally geodesic, or M is a Riemannian product $S^m(c)\times S^{n-m}(\sqrt{1-c^2})$, $1\le m\le n-1$, with $r>1$.

Corollary 1.2. *Let M be an n-dimensional compact orientable hypersurface with constant scalar curvature $n(n-1)r$, $r>1$, in $S^{n+1}(1)$. Then, the Jacobi operator J_s is elliptic and the first eigenvalue $\lambda_1^{J_s}$ of the Jacobi operator J_s satisfies*

$$\lambda_1^{J_s}\le -n(n-1)r\sqrt{r-1}$$

and the equality holds if and only if M is totally umbilical and non-totally geodesic.

Remark 1.3. We should notice that the totally umbilical hypersurfaces do not appear in the result of [4]. From the proof of theorem 1.1 in section 3, we shall see that our results do hold for the case where $r=1$, if we assume that the Jacobi operator J_s is elliptic. Hence, the result in [4] can be seen as a direct consequence of the theorem 1.1.

Acknowledgements. We would like to express our gratitude to the referee for him/her to tell us the paper [4] and the valuable suggestions.

2. Prelimenary

Throughout this paper, all manifolds are assumed to be smooth and connected without boundary. Let M be an n-dimensional hypersurface in a unit sphere $S^{n+1}(1)$. We choose a local orthonormal frame $\{\mathbf{e}_1, \cdots, \mathbf{e}_n, \mathbf{e}_{n+1}\}$ and the dual coframe $\{\omega_1, \cdots, \omega_n, \omega_{n+1}\}$ in such a way that $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is a local orthonormal frame on M. Hence, we have

$$\omega_{n+1} = 0$$

on M. From Cartan's lemma, we have

$$\omega_{in+1} = \sum_{j=1}^{n} h_{ij}\omega_j, \ h_{ij} = h_{ji}. \tag{2.1}$$

The mean curvature H and the second fundamental form α of M are defined, respectively, by

$$H = \frac{1}{n}\sum_{i=1}^{n} h_{ii}, \ \alpha = \sum_{i,j=1}^{n} h_{ij}\omega_i \otimes \omega_j \mathbf{e}_{n+1}.$$

When the mean curvature H of M is identically zero, we recall that M is by definition a minimal hypersurface.

From the structure equations of M, Gauss equation and Codazzi equation are given by

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.2}$$

$$h_{ijk} = h_{ikj}. \tag{2.3}$$

From (2.2), we have

$$n(n-1)r = n(n-1) + n^2H^2 - S, \tag{2.4}$$

where $n(n-1)r$ and S denote the scalar curvature and the squared norm of the second fundamental form of M, respectively.

For any C^2-function f on M, we define its gradient and Hessian by

$$df = \sum_{i=1}^{n} f_i\omega_i,$$

$$\sum_{j=1}^{n} f_{ij}\omega_j = df_i + \sum_{j=1}^{n} f_j\omega_{ji}.$$

Thus, the differential operator $\Box$ is defined by

$$\Box f = \sum_{i,j=1}^{n} (nH\delta_{ij} - h_{ij})f_{ij}.$$

3. Proofs of results

First of all, we will consider the first eigenvalue of the Jacobi operator J_s of both the totally umbilical and non-totally geodesic hypersurface and the Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-1$.

Example 3.1. Let M be a totally umbilical and non-totally geodesic hypersurface in $S^{n+1}(1)$. In this case, $\square = (n-1)H\Delta$ and

$$J_s = -\square - \{n(n-1)H + nHS - f_3\} = -\{(n-1)H\Delta + n(n-1)H(1+H^2)\}.$$

Hence,

$$\lambda_1^{J_s} = -n(n-1)H(1+H^2) = -n(n-1)r\sqrt{r-1},$$

from Gauss equation (2.4). Since

$$\begin{aligned}&n(n-1)r\sqrt{r-1}\\ =&\{2n(n-1) + n^2(n-1)(r-1)\}H - n(n-1)(r-1)\{(n-1)(r-1)+1\}\frac{1}{H},\end{aligned}$$

we know that

$$\begin{aligned}\lambda_1^{J_s} &= n(n-1)r\sqrt{r-1}\\ &= -\{2n(n-1) + n^2(n-1)(r-1)\}H\\ &\quad + n(n-1)(r-1)\{(n-1)(r-1)+1\}\frac{1}{H}.\end{aligned}$$

$\square$

Example 3.2. Let $S^m(c)\times S^{n-m}(\sqrt{1-c^2})$, $1\le m\le n-1$, be a hypersurface with $r>1$ in $S^{n+1}(1)$. In this case, the position vector is $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in S^m(c)\times S^{n-m}(\sqrt{1-c^2})$ and the unit normal vector at the point $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in S^m(c)\times S^{n-m}(\sqrt{1-c^2})$ is given by $\mathbf{e}_{n+1} = (-\frac{\sqrt{1-c^2}}{c}\mathbf{x}_1, \frac{c}{\sqrt{1-c^2}}\mathbf{x}_2)$. Its principal curvatures are given by

$$k_1 = \cdots = k_m = \frac{\sqrt{1-c^2}}{c}, \quad k_{m+1} = \cdots = k_n = -\frac{c}{\sqrt{1-c^2}}.$$

Thus, the mean curvature H of $S^m(c)\times S^{n-m}(\sqrt{1-c^2})$ satisfies

$$nH = \frac{m-nc^2}{c\sqrt{1-c^2}}. \tag{3.1}$$

Since

$$\begin{aligned}n(n-1)(r-1) &= n^2H^2 - S\\ &= \frac{(m-nc^2)^2}{c^2(1-c^2)} - m\frac{1-c^2}{c^2} - (n-m)\frac{c^2}{1-c^2}\\ &= \frac{n(n-1)c^4 - 2m(n-1)c^2 + m(m-1)}{c^2(1-c^2)}\\ &= \frac{(n-1)(nc^2+n-2m)(c^2-1) + (n-m)(n-1-m)}{c^2(1-c^2)},\end{aligned} \tag{3.2}$$

we know that $r>1$ if and only if either

$$0 < c^2 < \frac{m}{n} - \frac{\sqrt{m(n-m)}}{n\sqrt{n-1}}$$

or

$$\frac{m}{n} + \frac{\sqrt{m(n-m)}}{n\sqrt{n-1}} < c^2 < 1.$$

Since the principal curvatures of $S^m(c)\times S^{n-m}(\sqrt{1-c^2})$ are constant, we know that S and f_3 are constant. Because the differential operator $\Box$ is self-adjoint and elliptic, we have that the first eigenvalue of the Jacobi operator J_s is given by

$$\lambda_1^{J_s} = -\{n(n-1)H + nHS - f_3\}. \tag{3.3}$$

Furthermore, from (3.1), we have

$$\begin{aligned}
&n(n-1)H + nHS - f_3\\
=&n(n-1)H + nH\{m\frac{1-c^2}{c^2} + (n-m)\frac{c^2}{1-c^2}\}\\
&- m\frac{(1-c^2)^{\frac{3}{2}}}{c^3} + (n-m)\frac{c^3}{(1-c^2)^{\frac{3}{2}}}\\
=&nH\Big\{n-1+m\frac{1-c^2}{c^2} + (n-m)\frac{c^2}{1-c^2}\\
&-\frac{m}{m-nc^2}\frac{(1-c^2)^2}{c^2} + \frac{(n-m)}{m-nc^2}\frac{c^4}{(1-c^2)}\Big\}\\
=&\frac{nH}{m-nc^2}\Big\{(n-1)(m-nc^2) + \frac{m(1-c^2)\{m-1-(n-1)c^2\}}{c^2}\\
&+\frac{(n-m)c^2\{m-(n-1)c^2\}}{1-c^2}\Big\}.
\end{aligned}\tag{3.4}$$

On the other hand, from (3.1) and (3.2), we have

$$\begin{aligned}
&n(n-1)(r-1)\frac{1}{n^2H^2}\\
&=\frac{(n-1)(nc^2+n-2m)(c^2-1)+(n-m)(n-1-m)}{(m-nc^2)^2},
\end{aligned}\tag{3.5}$$

$$\begin{aligned}
&\{2n(n-1)+n^2(n-1)(r-1)\}H\\
&=nH\frac{(n-1)\{(n-2)c^2+n-2m\}(c^2-1)+(n-m)(n-1-m)}{c^2(1-c^2)}.
\end{aligned}\tag{3.6}$$

Therefore, we obtain, from (3.2), (3.5) and (3.6), by making use of a direct computation

$$
\begin{aligned}
&\{2n(n-1)+n^2(n-1)(r-1)\}H\\
&-n(n-1)(r-1)\{(n-1)(r-1)+1\}\frac{1}{H}\\
=&nH\frac{(n-1)\{(n-2)c^2+n-2m\}(c^2-1)+(n-m)(n-1-m)}{c^2(1-c^2)}\\
&-nH\frac{(n-1)(nc^2+n-2m)(c^2-1)+(n-m)(n-1-m)}{(m-nc^2)^2}\\
&\times\frac{\{n(n-2)c^2+(n-1)(n-2m)\}(c^2-1)+(n-m)(n-1-m)}{c^2(1-c^2)}\\
=&\frac{nH}{(m-nc^2)^2c^2(1-c^2)}\bigg\{2(n-1)c^2(1-c^2)(m-nc^2)^2\\
&+\{(n-1)(nc^2+n-2m)(c^2-1)+(n-m)(n-1-m)\}\\
&\times\bigg[(m-nc^2)^2-\big[\{n(n-2)c^2+(n-1)(n-2m)\}(c^2-1)\\
&+(n-m)(n-1-m)\big]\bigg]\bigg\}\\
=&\frac{nH}{(m-nc^2)^2c^2(1-c^2)}\bigg\{2(n-1)c^2(1-c^2)(m-nc^2)^2\\
&+\big[(n-1)(nc^2+n-2m)(c^2-1)+(n-m)(n-1-m)\big](m-nc^2)(1-2c^2)\bigg\}\\
=&\frac{nH}{(m-nc^2)c^2(1-c^2)}\bigg\{2(n-1)c^2(1-c^2)(m-nc^2)\\
&+(n-1)(nc^2-m)(c^2-1)(1-2c^2)+(n-m)\{(n-1)c^2-m\}(1-2c^2)\bigg\}\\
=&\frac{nH}{(m-nc^2)}\bigg\{\frac{(n-1)(m-nc^2)}{c^2}\\
&+\frac{(n-m)\{(n-1)c^2-m\}}{c^2}+\frac{(n-m)\{m-(n-1)c^2\}}{1-c^2}\bigg\}\\
=&\frac{nH}{m-nc^2}\bigg\{(n-1)(m-nc^2)+\frac{m(1-c^2)\{m-1-(n-1)c^2\}}{c^2}\\
&+\frac{(n-m)c^2\{m-(n-1)c^2\}}{1-c^2}\bigg\}.
\end{aligned}
$$

Hence, we infer, from (3.4) and the above equality, that

$$
\begin{aligned}
\lambda_1^{J_s}=&-\{n(n-1)H+nHS-f_3\}\\
=&-\{2n(n-1)+n^2(n-1)(r-1)\}H\\
&+n(n-1)(r-1)\{(n-1)(r-1)+1\}\frac{1}{H}.
\end{aligned}
$$

□

Remark 3.3. We must notice that the first eigenvalue of the Jacobi operator J_s of both the totally umbilical and non-totally geodesic hypersurface and the Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$ has the same representation formula. But, as we have seen in the introduction, the first eigenvalue of the Jacobi operator J_m of the totally umbilical hypersurface is different from one of the Riemannian product $S^{n-1}(c) \times S^1(\sqrt{1-c^2})$. It is a very interesting fact.

Proof of Theorem 1.1. Since $r > 1$, from the assertion in the introduction, we know that $\Box$ is elliptic and $n^2H^2 - S = n(n-1)(r-1) > 0$. Hence, $H \neq 0$. Thus, we can assume $H > 0$. We choose a local field of orthonormal frames $\mathbf{e}_1, \cdots, \mathbf{e}_n$ on M such that, at the point that we consider,

$$h_{ij} = \begin{cases} k_i & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases}$$

where k_i's are the principal curvatures of M. Thus,

$$\Box(nH) = nH\Delta(nH) - \sum_{i=1}^{n} k_i (nH)_{ii}.$$

From the Gauss equation (2.4), we have

$$nH\Delta(nH) = \frac{1}{2}\Delta(nH)^2 - |\nabla(nH)|^2 = \frac{1}{2}\Delta S - |\nabla(nH)|^2.$$

By making use of a standard computation of Simons' type formula (cf. [11], [1] and [18]), we have

$$\frac{1}{2}\Delta S = \sum_{i,j,k=1}^{n} h_{ijk}^2 + \sum_{i=1}^{n} k_i (nH)_{ii} + nS - n^2H^2 + nHf_3 - S^2.$$

Hence, we infer

$$\Box(nH) = \sum_{i,j,k=1}^{n} h_{ijk}^2 - n^2|\nabla H|^2 + nS - n^2H^2 + nHf_3 - S^2.$$

Therefore, we have

$$\begin{aligned} J_s H = & -\Box H - \{n(n-1)H + nHS - f_3\}H \\ = & -\Bigg\{\frac{1}{n}\sum_{i,j,k=1}^{n} h_{ijk}^2 - n|\nabla H|^2 + S - nH^2 + Hf_3 \\ & - \frac{S^2}{n} + n(n-1)H^2 + nH^2S - Hf_3\Bigg\} \\ = & -\left\{\frac{1}{n}\sum_{i,j,k=1}^{n} h_{ijk}^2 - n|\nabla H|^2 + S + n(n-2)H^2 + nH^2S - \frac{S^2}{n}\right\} \\ = & -\Bigg[\frac{1}{n}\sum_{i,j,k=1}^{n} h_{ijk}^2 - n|\nabla H|^2 + 2n(n-1)H^2 \\ & + n^2(n-1)(r-1)H^2 - n(n-1)(r-1)\{(n-1)(r-1)+1\}\Bigg] \end{aligned}$$

From the min-max principle, we have

$$\begin{aligned}
\lambda_1^{J_s} \leq & \frac{\int_M HJ_sHdM}{\int_M H^2dM} \\
= & -\frac{\int_M H(\frac{1}{n}\sum_{i,j,k=1}^n h_{ijk}^2 - n|\nabla H|^2)dM}{\int_M H^2dM} \\
& -\frac{\int_M H\{2n(n-1)+n^2(n-1)(r-1)\}H^2dM}{\int_M H^2dM} \\
& +\frac{\int_M Hn(n-1)(r-1)\{(n-1)(r-1)+1\}dM}{\int_M H^2dM} \\
\leq & -\Big\{2n(n-1)+n^2(n-1)(r-1)\Big\}\min|H| \\
& +n(n-1)(r-1)\Big\{(n-1)(r-1)+1\Big\}\frac{1}{\min|H|}.
\end{aligned}$$

Here we have used the following inequality (cf. [1] and [14]):

$$\sum_{i,j,k=1}^n h_{ijk}^2 \geq n^2|\nabla H|^2,$$

which can be proved by using the Gauss equation (2.4) and $r > 1$. When the equality holds, we know that H is constant and $\sum_{i,j,k=1}^n h_{ijk}^2 = 0$ on M, that is, the second fundamental form of M is parallel. Hence, M is an isoparametric hypersurface with at most two distinct principal curvatures (cf. [13]). Thus, M is totally umbilical and non-totally geodesic or a Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-1$.

□

Proof of Corollary 1.2. From Gauss equation (2.4) and $r > 1$, we have $n(n-1)H^2 \geq n^2H^2 - S = n(n-1)(r-1) > 0$. Hence, $|H| \geq \sqrt{r-1}$ and the equality holds at umbilical points. Since $|H| \geq \sqrt{r-1} > 0$, we have $H \neq 0$. From $|H| \geq \sqrt{r-1}$, we infer

$$\begin{aligned}
& -\{2n(n-1)+n^2(n-1)(r-1)\}\min|H| \\
& +n(n-1)(r-1)\{(n-1)(r-1)+1\}\frac{1}{\min|H|} \\
& \leq -n(n-1)r\min|H| \leq -n(n-1)r\sqrt{r-1}.
\end{aligned}$$

Therefore, we know that the first eigenvalue of the Jacobi operator J_s satisifies $\lambda_1^{J_s} \leq -n(n-1)r\sqrt{r-1}$. When equality holds, we know

$$\begin{aligned}
\lambda_1^{J_s} = & -\{2n(n-1)+n^2(n-1)(r-1)\}\min|H| \\
& +n(n-1)(r-1)\{(n-1)(r-1)+1\}\frac{1}{\min|H|} \\
= & -n(n-1)r\sqrt{r-1}.
\end{aligned}$$

Hence, H is constant from the theorem 1.1 and $S = nH^2$. Namely, M is totally umbilical. If M is totally umbilical, from the example 3.1, we know that $\lambda_1^{J_s} = -n(n-1)r\sqrt{r-1}$.

□

References

1. Alencar, H., do Carmo, M. & Colares, A. G., Stable hypersurfaces with constant scalar curvature, Math. Z., **213**(1993), 117-131.
2. Alencar, H., do Carmo, M. & Santos, W., A gap theorem for hypersurfaces of the sphere with constant scalar curvature one, Comment. Math. Helv., **77**(2002), 549-562.
3. Alías, L. J., Barros, A. & Brasil, A. Jr, A spectral characterization of $H(r)$-torus by the first stability eigenvalue, Proc. Amer. Math. Soc., **133**(2005), 875-884.
4. Alías, L. J., Brasil, A. Jr & Sousa, L. Jr, A characterization of Clifford tori with constant scalar curvatrue one by the first stability eigenvalue, Bull. Braz. Math. Soc., **35**(2004), 165-175.
5. Barbosa, J. L., do Carmo, M. & Eschenburg, J., Stability of hypersurfaces with constant mean curvature in Riemannian Manifolds, Math. Z., **197**(1988), 123-138.
6. Cheng, Q. -M., The rigidity of Clifford torus $S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})$, Comment. Math. Helv., **71**(1996), 60-69.
7. Cheng, Q. -M., Hypersurfaces in a unit sphere $S^{n+1}(1)$ with constant scalar curvature, J. London Math. Soc., **64**(2001), 755-768.
8. Cheng, Q. -M., Compact hypersurfaces in a unit sphere with infinite fundamental group, Pacific J. Math., **212**(2003), 49-56.
9. Cheng, Q. -M. & Nakagawa, H., Totally umbilical hypersurfaces, Hiroshima Math. J., **20**(1990), 1-10.
10. Cheng, Q. -M., Shu, S. C., & Suh, Y. J., Compact hypersurfaces in a unit sphere, Proc. Royal Soc. Edinburgh, **135A**(2005), 1129-1137.
11. Cheng, S. Y., & Yau. S. T., Hypersurfaces with constant scalar curvature, Math. Ann., **225**(1997), 195-204.
12. Hounie, J.& Leite, M. L., Two-ended hypersurfaces with zero scalar curvature, Indiana Univ. Math. J., **48**(1999), 867-882.
13. Lawson H. B. Jr., Local rigidity theorems for minimal hypersurfaces, Ann. of Math., **89** (1969), 167-179.
14. Li, H., Hypersurfaces with constant scalar curvature in space forms, Math. Ann., **305**(1996), 665-672.
15. Perdomo, O., First stability eigenvalue characterization of Clifford hypersurfaces, Proc. Amer. Math. Soc., **130**(2002), 3379-3384
16. Reilly, R. C., Variational properties of functions of the mean curvatures for hypersurfaces in space forms, J. Diff. Geom., **8**(1973), 465-477.
17. Rosenberg, H., Hypersurfaces of constant curvatures in space forms, Bull. Sci. Math., **117**(1993), 211-239.
18. Simons, J., Minimal varieties in Riemannian manifolds, Ann. of Math., **88**(1968), 62-105.
19. Wu, C., New characterizations of the Clifford tori and the Veronese surface, Arch. Math. (Basel), **61**(1993), 277-284

Department of Mathematics, Faculty of Science and Engineering, Saga University, Saga 840-8502, Japan

E-mail address: `cheng@ms.saga-u.ac.jp`