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On some rigidity results of hypersurfaces in a sphere

Qing-Ming Cheng
Department of Mathematics,
Faculty of Science and Engineering,
Saga University, 840-8502 Saga,
Japan (cheng@ms.saga-u.ac.jp)

Haizhong Li
Department of Mathematical Sciences,
Tsinghua University, 100084 Beijing,
People's Republic of China (hli@math.tsinghua.edu.cn)

Guoxin Wei
School of Mathematical Sciences,
South China Normal University,
510631 Guangzhou, People's Republic of China
and
Department of Mathematics,
Faculty of Science and Engineering,
Saga University, 840-8502 Saga, Japan
(weigx03@mails.tsinghua.edu.cn)

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We study the weak stability index of an immersion $\phi : M \to S^{n+1}(1) \subset R^{n+2}$ of an n-dimensional compact Riemannian manifold. We prove that the weak stability index of a compact hypersurface M with constant scalar curvature in $S^{n+1}(1)$, which is not totally umbilical, is greater than or equal to $n+2$ if the mean curvature H_1 and H_3 are constant, and that the equality holds if and only if M is $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$. As an application, we show that the weak stability index of an n-dimensional compact hypersurface with constant scalar curvature in $S^{n+1}(1)$, which is neither totally umbilical nor a Clifford hypersurface, is greater than or equal to $2n+4$ if the mean curvature H_1 and H_3 are constant.

1. Introduction

Let $\phi : M \to S^{n+1}(1) \subset R^{n+2}$ be an isometric immersion of an n-dimensional complete Riemannian manifold. For any point $x \in M$, we will denote by T_xM and N_xM the tangent space and normal space of M at x, respectively. Let us denote by $\nu : M \to S^{n+1}(1)$ a normal vector field along M. The shape operator $A_x : T_xM \to T_xM$ is given by $A_x(v) = -d\nu_x(v) = -\beta'(0)$, where $\beta(t) = \nu(\alpha(t))$ and $\alpha(t)$ is any smooth curve in M such that $\alpha(0) = x$ and $\alpha'(0) = v$. We know that the linear map A_x is symmetric and that its eigenvalues $k_1(x), \dots, k_n(x)$ are called principal curvatures of M at x.

We consider elementary symmetric functions $S_m(x)$ of the principal curvatures of M defined by

$$\det(tI - A_x) = \sum_{m=0}^{n} (-1)^m S_m(x) t^{n-m}.$$

Now, $H_m(x) = S_m(x)/C_n^m$ with $C_n^m = n!/m!(n-m)!$ is called the mth mean curvature of M, namely,

$$H_m(x) = \frac{1}{C_n^m} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} k_{i_1}(x) \cdots k_{i_m}(x).$$

Hence, the mean curvature $H(x)$ of M satisfies $H(x) = (k_1(x) + \cdots + k_n(x))/n = H_1(x)$, the scalar curvature

$$R(x) = n(n-1)r(x) = n(n-1) + 2S_2(x) = n(n-1) + n(n-1)H_2(x)$$

and the Gauss–Kronercker curvature $K(x)$ of M is

$$K(x) = k_1(x) \cdots k_n(x) = H_n(x) = S_n(x).$$

For any C^2 function f defined on M, let $(f_{,ij})$ denote its Hessian. A differential operator $\Box$ defined by

$$\Box f = \sum_{i,j=1}^{n} (nH\delta_{ij} - h_{ij}) f_{,ij},$$

where h_{ij} denotes components of the second fundamental form of M, was introduced by Cheng and Yau in [5] to study compact hypersurfaces with constant scalar curvature in $S^{n+1}(1)$. They proved that if M is an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, $r \geqslant 1$, and if the sectional curvature of M is non-negative, then M is a totally umbilical hypersurface $S^n(c)$ or a Clifford hypersurface $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \leqslant m \leqslant n-1$, where $S^k(c)$ denotes a sphere of radius c. Cheng [4] and Li [6] also used the differential operator $\Box$ to study complete hypersurfaces with constant scalar curvature.

In [1], Alencar *et al*. studied the stability of compact hypersurfaces with constant scalar curvature $n(n-1)r$ in $S^{n+1}(1)$. In this case, its Jacobi operator J_s is given by

$$J_s = \Box + \{n(n-1)H + nHS - f_3\} = \Box + \{(n-1)S_1 + (S_1S_2 - 3S_3)\},$$

where

$$S = \sum_{i=1}^{n} k_i^2, \qquad f_3 = \sum_{i=1}^{n} k_i^3.$$

It is not difficult to prove that if $r > 1$, then J_s is elliptic. The spectral behaviour of J_s is directly related to the instability of hypersurfaces with constant scalar curvature in $S^{n+1}(1)$.

DEFINITION 1.1 (cf. [2, 8]). Let M be an n-dimensional, compact, orientable hypersurface with constant scalar curvature $n(n-1)r$, $r > 1$, in $S^{n+1}(1)$. A weak stability index of M, $\mathrm{Ind}_T(M)$ is the maximal dimension of any subspace V of $C_T^\infty(M)$ on which the quadratic form Q is negative definite, where

$$C_T^\infty(M) = \left\{ u \in C^\infty(M) : \int_M u \,\mathrm{d}v = 0 \right\} \quad \text{and} \quad Q(u,u) = -\int_M u J_s(u) \,\mathrm{d}v.$$

We study compact hypersurfaces with constant scalar curvature in $S^{n+1}(1)$ and we will estimate the weak stability index.

THEOREM 1.2. *Let M be a compact hypersurface in $S^{n+1}(1)$ with constant scalar curvature $R = n(n-1)r > n(n-1)$. If H_1 and H_3 are constant, then*

(i) *the weak stability index $\mathrm{Ind}_T(M)$ of M is equal to zero: in this case, M is totally umbilical, or*

(ii) *the weak stability index $\mathrm{Ind}_T(M)$ of M is greater than or equal to $n+2$, and the equality holds if and only if M is $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, where c satisfies*

$$\frac{nm + \sqrt{m[(2-n)m + (n-1)(n+2)]}}{(n-1)(n+2)} \leqslant c^2 \leqslant \frac{(nm+n-2) + \sqrt{(n-m)(3n-2m+nm-2)}}{(n-1)(n+2)}.$$

Given an arbitrary vector $v \in R^{n+2}$, we define functions $l_v : M \to R$ and $f_v : M \to R$ by $l_v(x) = \langle \phi(x), v \rangle$ and $f_v(x) = \langle \nu(x), v \rangle$.

THEOREM 1.3. *Let $\phi : M \to S^{n+1}$ be an isometric immersion of an n-dimensional complete Riemannian manifold M with constant ratio of the Gauss–Kronercker curvature and the $(n-1)$th mean curvature, that is, $S_n(x) = cS_{n-1}(x)$, where c is a constant. If $l_v = \lambda f_v$, for some non-zero vector v and some real number λ, then $\phi(M)$ is either a totally umbilical sphere or a Clifford hypersurface.*

THEOREM 1.4. *Let $\phi : M \to S^{n+1}$ be an isometric immersion of an n-dimensional complete Riemannian manifold M with constant scalar curvature $n(n-1)r$, where r satisfies*

$$r \neq 2\frac{(2k+m)n^2 - (2k^2+4k+2km+m)n + 2k(m+k+1)}{n(2k+m)(2(n-1)-(2k+m))}$$

for $0 \leqslant m \leqslant n-2$ and $1 \leqslant k \leqslant n-1-m$. If $l_v = \lambda f_v$, for some non-zero vector v and some real number λ, then $\phi(M)$ is either a totally umbilical sphere or a Clifford hypersurface.

We now have the following corollary of theorem 1.2 and theorem 1.4.

COROLLARY 1.5. *Let M be a compact hypersurface in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$, with $r > 1$ and*

$$r \neq 2\frac{(2k+m)n^2 - (2k^2+4k+2km+m)n + 2k(m+k+1)}{n(2k+m)(2(n-1)-(2k+m))}$$

for $0 \leqslant m \leqslant n-2, 1 \leqslant k \leqslant n-1-m$. *If* H_1 *and* H_3 *are constants, then either*

(i) M *is totally umbilical,*

(ii) M *is a Clifford hypersurface or*

(iii) *the weak stability index of* M *is greater than or equal to* $2n+4$.

THEOREM 1.6. *Let* $\phi : M \to S^{n+1}$ *be an isometric immersion with constant Gauss–Kronercker curvature* c, $c \neq \pm 1$, *of an* n*-dimensional complete Riemannian manifold. If* $l_v = \lambda f_v$ *for some non-zero vector* v *and some real number* λ, *then* $\phi(M)$ *is either a totally umbilical sphere or a Clifford hypersurface.*

2. The weak stability index of Clifford hypersurfaces

In this section we will compute the weak stability index of the Clifford hypersurface $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \leqslant m \leqslant n-1$.

Since $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \leqslant m \leqslant n-1$, is an isoparametric hypersurface in $S^{n+1}(1)$, its principal curvatures are given by

$$k_1 = \cdots = k_m = -\frac{\sqrt{1-c^2}}{c}, \qquad k_{m+1} = \cdots = k_n = \frac{c}{\sqrt{1-c^2}}. \tag{2.1}$$

Hence, its mean curvature H, the squared norm $S = |A|^2$ of the second fundamental form and f_3 are given by

$$H = \frac{nc^2 - m}{nc\sqrt{1-c^2}}, \tag{2.2}$$

$$S = |A|^2 = \frac{nc^4 - 2mc^2 + m}{c^2(1-c^2)}, \tag{2.3}$$

$$f_3 = \frac{-m(1-c^2)^{3/2}}{c^3} + \frac{(n-m)c^3}{(1-c^2)^{3/2}}. \tag{2.4}$$

From the Gauss equation, we have

$$\begin{aligned} R - n(n-1) &= n(n-1)(r-1) \\ &= n^2H^2 - S \\ &= \frac{n(n-1)c^4 + 2m(1-n)c^2 + m(m-1)}{c^2(1-c^2)}, \end{aligned} \tag{2.5}$$

where R is the scalar curvature. Thus, we infer that $r > 1$ if and only if

$$c^2 > \frac{m(n-1) + \sqrt{m(n-1)(n-m)}}{n(n-1)} \quad \text{or} \quad c^2 < \frac{m(n-1) - \sqrt{m(n-1)(n-m)}}{n(n-1)}. \tag{2.6}$$

If the scalar curvature $R = n(n-1)r > n(n-1)$, we know from the Gauss equation $n^2H^2 = S + n(n-1)(r-1)$ that the mean curvature H does not vanish. Without loss of generality, assume the mean curvature $H > 0$, that is,

$$c^2 > \frac{m}{n}. \tag{2.7}$$

From (2.6) and (2.7), we have that

$$1 > c^2 > \frac{m(n-1) + \sqrt{m(n-1)(n-m)}}{n(n-1)}. \tag{2.8}$$

Therefore, we have

$$n(n-1)H + nHS - f_3 = \frac{(n-2m)(n-1)c^4 + 2m(m-1)c^2 - m(m-1)}{c^3(1-c^2)^{3/2}}, \tag{2.9}$$

and the Jacobi operator $J_s = \Box + \{n(n-1)H + nHS - f_3\}$ becomes

$$J_s = \Box + \frac{(n-2m)(n-1)c^4 + 2m(m-1)c^2 - m(m-1)}{c^3(1-c^2)^{3/2}}. \tag{2.10}$$

Thus, the eigenvalues of J_s are given by

$$\lambda_i^{J_s} = \lambda_i^{\Box} + \frac{(n-2m)(1-n)c^4 + 2m(1-m)c^2 + m(m-1)}{c^3(1-c^2)^{3/2}}, \tag{2.11}$$

where $\lambda_i^{\Box}$ denotes the eigenvalues of the differential operator $\Box$.

Since the differential operator $\Box$ is self-adjoint and the Clifford hypersurface is closed, we have $\lambda_1^{\Box} = 0$, and its corresponding eigenfunctions are non-zero constant functions. Hence,

$$\lambda_1^{J_s} = \frac{(n-2m)(1-n)c^4 + 2m(1-m)c^2 + m(m-1)}{c^3(1-c^2)^{3/2}}$$

with multiplicity one and its corresponding eigenfunctions are non-zero constant functions. Hence, $\lambda_1^{J_s}$ does not contribute to $\mathrm{Ind}_T(M)$. Since the other eigenfunctions u of J_s other than the first eigenfunctions are orthogonal to the constant functions, namely, $\int_M u = 0$, we know that the other eigenvalues of J_s contribute to $\mathrm{Ind}_T(M)$ if they are negative.

Let Δ_1 and Δ_2 denote the Laplacians on $S^m(c)$ and on $S^{n-m}(\sqrt{1-c^2})$, respectively. We can derive

$$\Box f = (nH\delta_{i,j} - h_{i,j})f_{i,j} = (nH - k_1)\Delta_1 f + (nH - k_n)\Delta_2 f.$$

Hence, the eigenvalues $\lambda_l^{\Box}$ are given by

$$\lambda_l^{\Box} = (nH - k_1)\lambda_i^{\Delta_1} + (nH - k_n)\lambda_j^{\Delta_2}, \tag{2.12}$$

the multiplicity of $\lambda_l^{\Box}$ is the sum of the products $m_{\lambda_i^{\Delta_1}} m_{\lambda_j^{\Delta_2}}$ for all possible values of $\lambda_i^{\Delta_1}$ and $\lambda_j^{\Delta_2}$ which satisfy

$$\lambda_l^{\Box} = (nH - k_1)\lambda_i^{\Delta_1} + (nH - k_n)\lambda_j^{\Delta_2},$$

where $m_{\lambda_i^{\Delta_j}}$ denotes the multiplicity of $\lambda_i^{\Delta_j}$.

We recall that the eigenvalues of the Laplacian Δ_1 on $S^m(c)$ are given by

$$\lambda_i^{\Delta_1} = \frac{(i-1)(m+i-2)}{c^2}, \quad i = 1, 2, 3, \ldots,$$

with multiplicities

$$m_{\lambda_1^{\Delta_1}} = 1, \qquad m_{\lambda_2^{\Delta_1}} = m+1 \quad \text{and} \quad m_{\lambda_i^{\Delta_1}} = C_{m+i-1}^{i-1} - C_{m+i-3}^{i-3}, \quad i = 3, 4, \dots,$$

and the eigenvalues of the Laplacian Δ_2 on $S^{n-m}(\sqrt{1-c^2})$ are given by

$$\lambda_j^{\Delta_2} = \frac{(j-1)(n-m+j-2)}{1-c^2}, \quad j = 1, 2, 3, \dots$$

with multiplicities

$$m_{\lambda_1^{\Delta_2}} = 1, \qquad m_{\lambda_2^{\Delta_2}} = n-m+1$$

and

$$m_{\lambda_j^{\Delta_2}} = C_{n-m+j-1}^{j-1} - C_{n-m+j-3}^{j-3}, \quad j = 3, 4, \dots.$$

Therefore, we infer that

$$\begin{aligned}
\lambda_l^{J_s} &= \lambda_l^{\square} + \frac{(n-2m)(1-n)c^4 + 2m(1-m)c^2 + m(m-1)}{c^3(1-c^2)^{3/2}} \\
&= (nH - k_1)\lambda_i^{\Delta_1} + (nH - k_n)\lambda_j^{\Delta_2} \\
&\quad + \frac{(n-2m)(1-n)c^4 + 2m(1-m)c^2 + m(m-1)}{c^3(1-c^2)^{3/2}} \\
&= \left(\frac{nc^2 - m}{c\sqrt{1-c^2}} + \frac{\sqrt{1-c^2}}{c}\right)\frac{(i-1)(m+i-2)}{c^2} \\
&\quad + \left(\frac{nc^2 - m}{c\sqrt{1-c^2}} - \frac{c}{\sqrt{1-c^2}}\right)\frac{(j-1)(n-m+j-2)}{1-c^2} \\
&\quad + \frac{(n-2m)(1-n)c^4 + 2m(1-m)c^2 + m(m-1)}{c^3(1-c^2)^{3/2}}.
\end{aligned} \tag{2.13}$$

It is not difficult to prove that

$$\begin{aligned}
&(nH - k_1)\lambda_2^{\Delta_1} + (nH - k_n)\lambda_2^{\Delta_2} \\
&\qquad + \frac{(n-2m)(1-n)c^4 + 2m(1-m)c^2 + m(m-1)}{c^3(1-c^2)^{3/2}} = 0.
\end{aligned}$$

Thus, in order to calculate the weak stability index, it suffices to estimate when

$$(nH - k_1)\lambda_i^{\Delta_1} + (nH - k_n)\lambda_j^{\Delta_2} < (nH - k_1)\lambda_2^{\Delta_1} + (nH - k_n)\lambda_2^{\Delta_2} \tag{2.14}$$

for $i = 1$, $j > 1$ and $i > 1$, $j = 1$. By a direct calculation, we obtain, from (2.8),

$$\begin{aligned}
&(nH - k_1)\lambda_1^{\Delta_1} + (nH - k_n)\lambda_2^{\Delta_2} \\
&\qquad + \frac{(n-2m)(1-n)c^4 + 2m(1-m)c^2 + m(m-1)}{c^3(1-c^2)^{3/2}} \\
&\qquad\quad = \frac{m(c^2-1)[(n-1)c^2 - (m-1)]}{c^3(1-c^2)^{3/2}} \\
&\qquad\quad < 0
\end{aligned} \tag{2.15}$$

with multiplicity $n-m+1$, and

$$\begin{aligned}(nH-k_1)\lambda_2^{\Delta_1}+(nH-k_n)\lambda_1^{\Delta_2}&\\ +\frac{(n-2m)(1-n)c^4+2m(1-m)c^2+m(m-1)}{c^3(1-c^2)^{3/2}}&\\ =\frac{(n-m)[(1-n)c^4+mc^2]}{c^3(1-c^2)^{3/2}}&\\ <0&\end{aligned}\tag{2.16}$$

with multiplicity $m+1$. Therefore, the weak stability index $\mathrm{Ind}_T(M)\geqslant n+2$ for $M=S^m(c)\times S^{n-m}(\sqrt{1-c^2})$ with constant scalar curvature $n(n-1)r$, $r>1$. Moreover, $\mathrm{Ind}_T(M)=n+2$ if and only if

$$(nH-k_1)\lambda_1^{\Delta_1}+(nH-k_n)\lambda_3^{\Delta_2}\geqslant(nH-k_1)\lambda_2^{\Delta_1}+(nH-k_n)\lambda_2^{\Delta_2}\tag{2.17}$$

and

$$(nH-k_1)\lambda_3^{\Delta_1}+(nH-k_n)\lambda_1^{\Delta_2}\geqslant(nH-k_1)\lambda_2^{\Delta_1}+(nH-k_n)\lambda_2^{\Delta_2}.\tag{2.18}$$

Since

$$\begin{aligned}(nH-k_1)\lambda_1^{\Delta_1}+(nH-k_n)\lambda_3^{\Delta_2}&\\ +\frac{(n-2m)(1-n)c^4+2m(1-m)c^2+m(m-1)}{c^3(1-c^2)^{3/2}}&\\ =\frac{(n-1)(n+2)c^4-2nmc^2+m(m-1)}{c^3(1-c^2)^{3/2}},&\end{aligned}\tag{2.19}$$

$$\begin{aligned}(nH-k_1)\lambda_3^{\Delta_1}+(nH-k_n)\lambda_1^{\Delta_2}&\\ +\frac{(n-2m)(1-n)c^4+2m(1-m)c^2+m(m-1)}{c^3(1-c^2)^{3/2}}&\\ =\frac{(n+2)(1-n)c^4+(2nm+2n-4)c^2+(m+2)(1-m)}{c^3(1-c^2)^{3/2}}&\end{aligned}\tag{2.20}$$

and

$$c^2>\frac{m(n-1)+\sqrt{m(n-1)(n-m)}}{n(n-1)},$$

we obtain that $\mathrm{Ind}_T(M)=n+2$ if and only if

$$\frac{nm+\sqrt{m[(2-n)m+(n-1)(n+2)]}}{(n-1)(n+2)}\leqslant c^2\leqslant\frac{(nm+n-2)+\sqrt{(n-m)(3n-2m+nm-2)}}{(n-1)(n+2)}.$$

In addition, it is interesting to point out that the weak stability index of Clifford hypersurfaces $S^m(c)\times S^{n-m}(\sqrt{1-c^2})$ converge to infinity as c^2 converges to 1.

In fact, we can obtain that, for every $j \geqslant 3$,

$$(nH-k_1)\lambda_1^{\Delta_1}+(nH-k_n)\lambda_j^{\Delta_2}+\frac{(n-2m)(1-n)c^4+2m(1-m)c^2+m(m-1)}{c^3(1-c^2)^{3/2}}<0$$

if and only if

$$\frac{m(n-1)+\sqrt{m(n-1)(n-m)}}{n(n-1)}<c^2<p_j,$$

where

$$p_j=\frac{m((j-1)(n-m+j-2)+2(m-1))+\sqrt{D}}{2(n-1)((j-1)(n-m+j-2)-(n-2m))},$$

$$D=m^2((j-1)(n-m+j-2)+2(m-1))^2 - 4m(m-1)(n-1)((j-1)(n-m+j-2)-(n-2m)).$$

For every $i \geqslant 3$, we have

$$(nH-k_1)\lambda_i^{\Delta_1}+(nH-k_n)\lambda_1^{\Delta_2} + \frac{(n-2m)(1-n)c^4+2m(1-m)c^2+m(m-1)}{c^3(1-c^2)^{3/2}}<0$$

if and only if

$$q_i<c^2<1,$$

where

$$q_i=\frac{-((n+m-2)(i-1)(m+i-2)+2m(1-m))-\sqrt{E}}{2(1-n)((i-1)(m+i-2)+(n-2m))},$$

$$E=((n+m-2)(i-1)(m+i-2)+2m(1-m))^2 - 4(1-n)(m-1)((i-1)(m+i-2)+(n-2m))((1-i)(m+i-2)+m).$$

Hence, we know that if

$$\frac{m(n-1)+\sqrt{m(n-1)(n-m)}}{n(n-1)}<p_{j+1}\leqslant c^2<p_j,$$

then

$$\mathrm{Ind}_T(M)=n+2+\sum_{l=3}^{j} m_{\lambda_l^{\Delta_2}}=m+C_{n-m+j-2}^{j-2}+C_{n-m+j-1}^{j-1}.$$

If $q_i<c^2\leqslant q_{i+1}<1$, then

$$\mathrm{Ind}_T(M)=n+2+\sum_{l=3}^{i} m_{\lambda_l^{\Delta_1}}=n-m+1+C_{m+i-1}^{i-1}+C_{m+i-2}^{i-2}.$$

Moreover, $\{q_i\}\nearrow 1$ and $\mathrm{Ind}_T(M)\nearrow\infty$ as $i\nearrow\infty$.

3. Proofs of theorems

In this section, we will prove our theorems.

Proof of theorem 1.2. If M is a totally umbilical hypersurface, then $\mathrm{Ind}_T(M) = 0$. Hence, we can assume that M is not totally umbilical.

For a fixed vector $v \in R^{n+2}$, gradients of the functions $l_v = \langle \phi, v \rangle$ and $f_v = \langle \nu, v \rangle$ are given by

$$\nabla l_v = v^{\mathrm{T}}, \qquad \nabla f_v = -A(v^{\mathrm{T}}), \tag{3.1}$$

where v^{T} denotes the tangent component of v along the immersion ϕ. By a direct calculation, we have

$$\Box l_v = (n^2H^2 - |A|^2) f_v + n(1-n)H l_v = 2S_2 f_v - (n-1)S_1 l_v, \tag{3.2}$$

$$\Box f_v = (f_3 - nH|A|^2) f_v + (n^2H^2 - |A|^2) l_v = (3S_3 - S_1S_2) f_v + 2S_2 l_v. \tag{3.3}$$

Hence, we derive

$$J_s l_v = (n^2H^2 - |A|^2) f_v + (nH|A|^2 - f_3) l_v = 2S_2 f_v + (S_1S_2 - 3S_3) l_v, \tag{3.4}$$

$$J_s f_v = n(n-1)H f_v + (n^2H^2 - |A|^2) l_v = (n-1)S_1 f_v + 2S_2 l_v. \tag{3.5}$$

We consider a function $f_v + \alpha l_v$, where $\alpha \in R$ is a real number. Since

$$J_s(f_v + \alpha l_v) = [(n-1)S_1 + 2\alpha S_2] f_v + [2S_2 + \alpha(S_1S_2 - 3S_3)] l_v \tag{3.6}$$

and S_1, S_2 and S_3 are constant, we can derive that functions $f_v + \alpha l_v$ are eigenfunctions of J_s if α is a solution of the following quadratic equation:

$$2S_2\alpha^2 + [(n-1)S_1 - S_1S_2 + 3S_3]\alpha - 2S_2 = 0 \tag{3.7}$$

and $-(n-1)S_1 - 2\alpha S_2$ is an eigenvalue of J_s.

Since the equation (3.7) has two different real roots,

$$\alpha_{\pm} = \frac{S_1S_2 - (n-1)S_1 - 3S_3 \mp \sqrt{D}}{4S_2},$$

where $D = [(n-1)S_1 - S_1S_2 + 3S_3]^2 + 16S_2^2 > 0$, the corresponding eigenvalues λ of J_s are given by

$$\lambda_{\pm} = -(n-1)S_1 - 2\alpha_{\pm}S_2 = \frac{-S_1S_2 - (n-1)S_1 + 3S_3 \pm \sqrt{D}}{2}. \tag{3.8}$$

According to $H_2 = r-1 > 0$ and the Gauss equation, we can choose the orientation such that $H = H_1 > 0$. Then we have the following inequalities [7]:

$$H_1^2 \geqslant H_2, \qquad H_1H_2 \geqslant H_3, \qquad H_2^2 \geqslant H_1H_3. \tag{3.9}$$

From (3.8) and (3.9) we infer that

$$\begin{aligned}\lambda_- &= \frac{-S_1S_2-(n-1)S_1+3S_3-\sqrt{D}}{2}\\ &< \lambda_+\\ &= \frac{-S_1S_2-(n-1)S_1+3S_3+\sqrt{D}}{2}\\ &< 0.\end{aligned} \tag{3.10}$$

In fact,

$$\begin{aligned}&-S_1S_2-(n-1)S_1+3S_3\\ &\qquad= -\frac{n^2(n-1)}{2}H_1H_2-n(n-1)H_1+\frac{n(n-1)(n-2)}{2}H_3\\ &\qquad\leqslant -\frac{n^2(n-1)}{2}H_1H_2-n(n-1)H_1+\frac{n(n-1)(n-2)}{2}H_1H_2\\ &\qquad= -n(n-1)H_1H_2-n(n-1)H_1\\ &\qquad< 0\end{aligned}$$

and

$$\begin{aligned}&[-S_1S_2-(n-1)S_1+3S_3]^2-D\\ &\qquad= 4[-3(n-1)S_1S_3+(n-1)S_1^2S_2-4S_2^2]\\ &\qquad= -2n^2(n-1)^2(n-2)H_1H_3\\ &\qquad\qquad+2n^3(n-1)^2H_1^2H_2-4n^2(n-1)^2H_2^2\\ &\qquad\geqslant 2n^2(n-1)^2[-(n-2)H_2^2+nH_2^2-2H_2^2]\\ &\qquad= 0.\end{aligned}$$

Therefore, λ_- and λ_+ are negative eigenvalues of J_s. Putting

$$U_\pm = \{f_v + \alpha_\pm l_v : v \in R^{n+2}\}, \tag{3.11}$$

we have $J_s u + \lambda_\pm u = 0$ for any $u \in U_\pm$.

On the other hand, if $u \in U_\pm$, then

$$\Box u + (S_1S_2 - 3S_3)u + (n-1)S_1 u + \lambda_\pm u = 0. \tag{3.12}$$

Set $\mu_\pm = (S_1S_2 - 3S_3) + (n-1)S_1 + \lambda_\pm$. Then

$$\mu_+ = -\lambda_- > -\lambda_+ = \mu_- > 0, \quad \int_M u\,\mathrm{d}v = 0. \tag{3.13}$$

Hence, functions belonging to $U_\pm$ are non-constant eigenfunctions of the $\Box$ and they satisfy the condition $\int_M u = 0$.

Hence,

$$\mathrm{Ind}_T(M) \geqslant \dim(U_- \oplus U_+) = \dim U_- + \dim U_+, \tag{3.14}$$

since U_- and U_+ are eigenspaces of $\square$ associated to different eigenvalues. Define $\varphi_\pm : R^{n+2} \to U_\pm$ by

$$\varphi_\pm(v) = f_v + \alpha_\pm l_v.$$

Claim 3.1. $\ker\varphi_- \cap \ker\varphi_+ = \varnothing$.

Assume that there exists a unit vector $v \in \ker\varphi_- \cap \ker\varphi_+$. Then we have

$$f_v + \alpha_+ l_v = 0 = f_v + \alpha_- l_v.$$

It follows that $l_v = 0 = f_v$. This means that M is a totally geodesic equator of $S^{n+1}(1)$, which is impossible. Thus, $\ker\varphi_- \cap \ker\varphi_+ = \varnothing$. Therefore,

$$\dim\ker\varphi_- + \dim\ker\varphi_+ = \dim(\ker\varphi_- \oplus \ker\varphi_+) \leqslant n+2.$$

Because of $\dim U_\pm = n+2-\dim\ker\varphi_\pm$, we obtain

$$\begin{aligned}\mathrm{Ind}_T(M) &\geqslant \dim U_- + \dim U_+\\ &= 2(n+2) - (\dim\ker\varphi_- + \dim\ker\varphi_+)\\ &\geqslant n+2.\end{aligned} \tag{3.15}$$

If $\mathrm{Ind}_T(M) = n+2$, then we have $\dim(\ker\varphi_- \oplus \ker\varphi_+) = n+2$, that is, $R^{n+2} = \ker\varphi_- \oplus \ker\varphi_+$. Then we have, for any point $p \in M$,

$$T_pM = T_pM \cap R^{n+2} = (T_pM \cap \ker\varphi_-) \oplus (T_pM \cap \ker\varphi_+).$$

Let $T_pM^\pm = T_pM \cap \ker\varphi_\pm$. Assume that $0 \neq v \in T_pM^-$, then $f_v + \alpha_- l_v = 0$ on M. It follows that $\alpha_- \neq 0$. Otherwise, $f_v = 0$ and M is totally geodesic. This is impossible. Since $f_v + \alpha_- l_v \equiv 0$, we have $\nabla(f_v + \alpha_- l_v) = -A(v^{\mathrm{T}}) + \alpha_- v^{\mathrm{T}} = 0$ on M. From $v \in T_pM^-$, we know that $v^{\mathrm{T}}(p) = v$ and $A_p(v) = \alpha_- v$, that is, T_pM^- is a subspace of T_pM with constant principal curvature α_-. By the same assertion, we can show that T_pM^+ is a subspace of T_pM with constant principal curvature α_+. If $T_{p_0}M^- = \varnothing$ at some point p_0, then we have $T_{p_0}M = T_{p_0}M^+$, that is, p_0 is an umbilical point; it follows that $|A|^2(p_0) - nH^2(p_0) = 0$. Since $H_1 = H$ and $|A|^2 = n^2H^2 - n(n-1)H_2$ are constants, we know that $|A|^2 - nH^2 = 0$ on M, this means that M is totally umbilical. This is a contradiction. Therefore, we derive that M has two different constant principal curvatures. From Cartan theorem, we know that M is a Clifford hypersurface $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$ and

$$\frac{nm + \sqrt{m[(2-n)m + (n-1)(n+2)]}}{(n-1)(n+2)} \leqslant c^2 \leqslant \frac{(nm+n-2) + \sqrt{(n-m)(3n-2m+nm-2)}}{(n-1)(n+2)}$$

since $\mathrm{Ind}_T(M) = n+2$. This completes the proof of theorem 1.2. $\square$

Proof of theorem 1.3. Without loss of generality we will assume that M is not totally umbilical. For any fixed vector v in R^{n+2}, $v^{\mathrm{T}} : M \to R^{n+2}$ defined by

$$v^{\mathrm{T}}(x) = v - l_v(x)x - f_v(x)\nu(x) \quad \text{for all } x \in M$$

is a tangent vector field on M because $\langle v^{\mathrm{T}}(x), x\rangle = 0$ and $\langle v^{\mathrm{T}}(x), \nu(x)\rangle = 0$ for every point $x \in M$. By multiplying the equation $l_v = \lambda f_v$ by an appropriated constant, we may assume that $|v| = 1$. We will also assume that l_v is not constant. Otherwise, $M \subset S^n(c)$ for some c. According to the completeness of M we have $M = S^n(c)$, that is, M is totally umbilical.

Since l_v is not constant, then $\lambda \neq 0$. From [3], we know that principal curvatures of M along the integral curve of v^{T} are

$$\lambda_1(\beta_x(s)) = -\frac{1}{\lambda},$$
$$\lambda_i(\beta_x(s)) = -\frac{1}{\lambda} + \frac{(1+\lambda^2)(\lambda^{-1}+\lambda_i(x))}{\lambda(\lambda-\lambda_i(x))\cos(ws) + (1+\lambda\lambda_i(x))}, \quad 2 \leqslant i \leqslant n.$$

For every $x \in N = S^n(1) \cap M$ (see [3]), let

$$I_1(x) = \{i \in \{2, \dots, n\} : \lambda_i(x) = -\lambda^{-1}\},$$
$$I_2(x) = \{i \in \{2, \dots, n\} : \lambda_i(x) = \lambda\},$$
$$I_3(x) = \{2, \dots, n\} \setminus (I_1(x) \cup I_2(x)).$$

Letting us denote the number of elements in $I_i(x)$ by n_i, for $i = 1, 2, 3$, then, we have $n_1 + n_2 + n_3 = n - 1$. If $i \in I_1$ and $j \in I_2$, then $\lambda_i(\beta_x(s)) = -\lambda^{-1}$ and $\lambda_j(\beta_x(s)) = \lambda$.

Claim 3.2. $I_3(x) = \varnothing$.

In fact, for every $i \in I_3(x)$, $a_i(x) = \lambda^{-1} + \lambda_i(x) \neq 0$ and $b_i(x) = \lambda(\lambda - \lambda_i(x)) \neq 0$. For each point $x \in M$ and $s \in (-\pi/2w, \pi/2w)$, we infer that

$$\begin{aligned}
&S_n(\beta_x(s)) \\
&\quad = \prod_{i=1}^{n} \lambda_i(\beta_x(s)) \\
&\quad = \left(-\frac{1}{\lambda}\right)^{n_1+1} (\lambda)^{n_2} \prod_{j=1}^{n_3} \left(-\frac{1}{\lambda} + \frac{(1+\lambda^2)a_j(x)}{b_j(x)\cos(ws) + \lambda a_j(x)}\right) \\
&\quad = \left(-\frac{1}{\lambda}\right)^{n-n_2} (\lambda)^{n_2} + \left(-\frac{1}{\lambda}\right)^{n_1+1} (\lambda)^{n_2} \sum_{k=1}^{n_3} \left(-\frac{1}{\lambda}\right)^{n_3-k} (1+\lambda^2)^k \\
&\qquad\quad \times \left(\sum_{j_1<\cdots<j_k} \frac{a_{j_1}(x)\cdots a_{j_k}(x)}{(b_{j_1}(x)\cos(ws) + \lambda a_{j_1}(x)) \cdots (b_{j_k}(x)\cos(ws) + \lambda a_{j_k}(x))} \right),
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&S_{n-1}(\beta_x(s)) \\
&\quad = \sum_{i=1}^{n} \lambda_1(\beta_x(s)) \cdots \widehat{\lambda_i(\beta_x(s))} \cdots \lambda_n(\beta_x(s)) \\
&\quad = (n_1+1)\left(-\frac{1}{\lambda}\right)^{n_1} (\lambda)^{n_2} \prod_{j=1}^{n_3} \left(-\frac{1}{\lambda} + \frac{(1+\lambda^2)a_j(x)}{b_j(x)\cos(ws) + \lambda a_j(x)}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ n_2\Big(-\frac{1}{\lambda}\Big)^{n_1+1}(\lambda)^{n_2-1}\prod_{j=1}^{n_3}\Big(-\frac{1}{\lambda}+\frac{(1+\lambda^2)a_j(x)}{b_j(x)\cos(ws)+\lambda a_j(x)}\Big)\\
&+ \sum_{i=1}^{n_3}\Big(-\frac{1}{\lambda}\Big)^{n_1+1}(\lambda)^{n_2}\prod_{j=1,\, j\neq i}^{n_3}\Big(-\frac{1}{\lambda}+\frac{(1+\lambda^2)a_j(x)}{b_j(x)\cos(ws)+\lambda a_j(x)}\Big)\\
&= (n_1+1)\Big(-\frac{1}{\lambda}\Big)^{n_1}(\lambda)^{n_2}\Big(-\frac{1}{\lambda}\Big)^{n_3} + n_2\Big(-\frac{1}{\lambda}\Big)^{n_1+1}(\lambda)^{n_2-1}\Big(-\frac{1}{\lambda}\Big)^{n_3}\\
&+ n_3\Big(-\frac{1}{\lambda}\Big)^{n_1+1}(\lambda)^{n_2}\Big(-\frac{1}{\lambda}\Big)^{n_3-1}\\
&+ \sum_{k=1}^{n_3}\Big(-\frac{1}{\lambda}\Big)^{n_1+n_3-k}(\lambda)^{n_2-1}\Big\{(n_1+1)\lambda - n_2\frac{1}{\lambda}+(n_3-k)\lambda\Big\}(1+\lambda^2)^k\\
&\times\Big(\sum_{j_1<\cdots<j_k}\frac{a_{j_1}(x)\cdots a_{j_k}(x)}{(b_{j_1}(x)\cos(ws)+\lambda a_{j_1}(x))\cdots(b_{j_k}(x)\cos(ws)+\lambda a_{j_k}(x))}\Big),
\end{aligned}
\tag{3.17}
$$

where $\widehat{\cdot}$ means that this term is deleted.

For any point $x \in M$, we have $S_n(x) = cS_{n-1}(x)$. Since $\beta_x(s) \in M$, then $S_n(\beta_x(s)) = cS_{n-1}(\beta_x(s))$, it follows from (3.16) and (3.17) that

$$
\begin{aligned}
&\Big(-\frac{1}{\lambda}\Big)^{n-n_2-1}(\lambda)^{n_2-1}\Big\{-1+\frac{n_2}{\lambda}c-(n_1+1)\lambda c - n_3\lambda c\Big\}\\
&\quad+\sum_{k=1}^{n_3}\Big(-\frac{1}{\lambda}\Big)^{n_1+n_3-k}(\lambda)^{n_2-1}(1+\lambda^2)^k\Big\{-1+\frac{n_2}{\lambda}c-(n_1+1)\lambda c-(n_3-k)\lambda c\Big\}\\
&\quad\times\Big(\sum_{j_1<\cdots<j_k}\frac{a_{j_1}(x)\cdots a_{j_k}(x)}{(b_{j_1}(x)\cos(ws)+\lambda a_{j_1}(x))\cdots(b_{j_k}(x)\cos(ws)+\lambda a_{j_k}(x))}\Big)=0,
\end{aligned}
$$

which means that, for every $s \in (-\pi/2w, \pi/2w)$, $\cos(ws)$ is a root of the following polynomial equation on X,

$$
\begin{aligned}
&\Big(-\frac{1}{\lambda}\Big)^{n-n_2-1}(\lambda)^{n_2-1}\Big\{-1+\frac{n_2}{\lambda}c-(n_1+1)\lambda c - n_3\lambda c\Big\}\\
&\quad+\sum_{k=1}^{n_3}\Big(-\frac{1}{\lambda}\Big)^{n_1+n_3-k}(\lambda)^{n_2-1}(1+\lambda^2)^k\\
&\quad\times\Big\{-1+\frac{n_2}{\lambda}c-(n_1+1)\lambda c-(n_3-k)\lambda c\Big\}\\
&\quad\times\Big(\sum_{j_1<\cdots<j_k}\frac{a_{j_1}(x)\cdots a_{j_k}(x)}{(b_{j_1}(x)X+\lambda a_{j_1}(x))\cdots(b_{j_k}(x)X+\lambda a_{j_k}(x))}\Big)=0.
\end{aligned}
\tag{3.18}
$$

We know that the polynomial equation should have finite roots, but equation (3.18) has infinite roots. So we can deduce that the coefficients of X^q in equation (3.18) are zero for any integer $q \in \{0, 1, 2, \dots\}$. Hence, we obtain that the coefficients of X^{n_3} are zero, that is,

$$\left(-\frac{1}{\lambda}\right)^{n-n_2-1} (\lambda)^{n_2-1} \left\{ -1 + \frac{n_2}{\lambda} c - (n_1+1)\lambda c - n_3 \lambda c \right\} = 0. \tag{3.19}$$

From (3.19), we have

$$c = \frac{-1}{(n_1+1)\lambda - n_2\lambda^{-1} + n_3\lambda} = \frac{-\lambda}{(n-n_2)\lambda^2 - n_2},$$

so

$$-1 + \frac{n_2}{\lambda} c - (n_1+1)\lambda c = \frac{-n_3\lambda^2}{(n-n_2)\lambda^2 - n_2} \neq 0.$$

Substituting c into (3.18) and noting that the constant term of equation (3.18) equals zero, we obtain

$$\begin{aligned}
&\sum_{k=1}^{n_3} \left(-\frac{1}{\lambda}\right)^{n_1} (\lambda)^{n_2-1} (-1)^{n_3-k} C_{n_3}^k (1+\lambda^2)^k \\
&\quad \times \left\{ -1 + \frac{n_2}{\lambda} c - (n_1+1)\lambda c - (n_3-k)\lambda c \right\} \prod_{i=1}^{n_3} a_i(x) \\
&\qquad = (-1)^{n_1+1} \frac{n_3 \lambda^{2n_3+n_2-n_1-1}(\lambda^2+1)}{(n-n_2)\lambda^2 - n_2} \prod_{i=1}^{n_3} a_i(x) \\
&\qquad = 0,
\end{aligned}$$

then we have

$$\prod_{i=1}^{n_3} a_i(x) = 0.$$

This is a contradiction with $a_i(x) \neq 0$. Hence, $I_3(x) = \varnothing$ for every $x \in N$. Thus, we derive that all the principal curvatures of M at the points of N are constant and that they are equal to either $-\lambda^{-1}$ or λ. Using the same arguments as in [3] we can conclude that M is either a totally umbilical sphere or a Clifford hypersurface. This completes the proof of theorem 1.3. □

Proof of theorem 1.4. Without loss of generality, we will assume that M is not totally umbilical. For any fixed vector v in R^{n+2}, we know that $v^{\mathrm{T}} : M \to R^{n+2}$ is a tangent vector field on M. By making use of the same notation as in the proof of theorem 1.3, we may assume that $|v| = 1$ and l_v is not constant. Since l_v is not constant, then $\lambda \neq 0$. From [3], we know that principal curvatures of M along the integral curve of v^{T} are

$$\begin{aligned}
\lambda_1(\beta_x(s)) &= -\frac{1}{\lambda}, \\
\lambda_i(\beta_x(s)) &= -\frac{1}{\lambda} + \frac{(1+\lambda^2)(\lambda^{-1} + \lambda_i(x))}{\lambda(\lambda - \lambda_i(x))\cos(ws) + (1+\lambda\lambda_i(x))}, \quad 2 \leqslant i \leqslant n.
\end{aligned}$$

For each $x \in N$, by making use of the same notation as in the proof of theorem 1.3, we have the following claim.

Claim 3.3. $I_3(x) = \varnothing$.

In fact, for every $i \in I_3(x)$, $a_i(x) = \lambda^{-1} + \lambda_i(x) \neq 0$ and $b_i(x) = \lambda(\lambda - \lambda_i(x)) \neq 0$. For each $s \in (-\pi/2w, \pi/2w)$ we obtain

$$\begin{aligned}
&2S_2(\beta_x(s))\\
&\quad= \sum_{i\neq j} \lambda_i(\beta_x(s))\lambda_j(\beta_x(s))\\
&\quad= \Big(-\frac{1}{\lambda}\Big)\Big(-\frac{n_1}{\lambda} + n_2\lambda + \sum_{i=1}^{n_3}\Big(-\frac{1}{\lambda} + \frac{(1+\lambda^2)a_i(x)}{b_i(x)\cos(ws) + \lambda a_i(x)}\Big)\Big)\\
&\qquad - \frac{n_1}{\lambda}\Big(-\frac{n_1}{\lambda} + n_2\lambda + \sum_{i=1}^{n_3}\Big(-\frac{1}{\lambda} + \frac{(1+\lambda^2)a_i(x)}{b_i(x)\cos(ws) + \lambda a_i(x)}\Big)\Big)\\
&\qquad + n_2\lambda\Big(-\frac{n_1+1}{\lambda} + (n_2-1)\lambda + \sum_{i=1}^{n_3}\Big(-\frac{1}{\lambda} + \frac{(1+\lambda^2)a_i(x)}{b_i(x)\cos(ws) + \lambda a_i(x)}\Big)\Big)\\
&\qquad + \sum_{j=1}^{n_3}\Big(-\frac{1}{\lambda} + \frac{(1+\lambda^2)a_j(x)}{b_j(x)\cos(ws) + \lambda a_j(x)}\Big)\\
&\qquad \times \Big(-\frac{n_1+1}{\lambda} + n_2\lambda + \sum_{i=1, i\neq j}^{n_3}\Big(-\frac{1}{\lambda} + \frac{(1+\lambda^2)a_i(x)}{b_i(x)\cos(ws) + \lambda a_i(x)}\Big)\Big)\\
&\quad= \frac{1}{\lambda^2}(n_1+n_3)(n_1+n_3+1) - 2n_2(n_1+n_3+1) + n_2(n_2-1)\lambda^2\\
&\qquad + 2(1+\lambda^2)\Big(-\frac{n_1+n_3}{\lambda} + n_2\lambda\Big)\Big(\sum_{k=1}^{n_3}\frac{a_k(x)}{b_k(x)\cos(ws) + \lambda a_k(x)}\Big)\\
&\qquad + \sum_{i\neq j}(1+\lambda^2)^2\frac{a_i(x)}{b_i(x)\cos(ws) + \lambda a_i(x)} \times \frac{a_j(x)}{b_j(x)\cos(ws) + \lambda a_j(x)}.
\end{aligned}$$

This means that, for every $s \in (-\pi/2w, \pi/2w)$, $\cos(ws)$ is a root of the following polynomial equation on X:

$$\begin{aligned}
&\Big\{2S_2 - \frac{1}{\lambda^2}(n_1+n_3)(n_1+n_3+1) + 2n_2(n_1+n_3+1) - n_2(n_2-1)\lambda^2\Big\}\\
&\quad \times \prod_{i=1}^{n_3}(b_i(x)X + \lambda a_i(x))\\
&\qquad = 2(1+\lambda^2)(-\frac{n_1+n_3}{\lambda} + n_2\lambda)\Big(\sum_{k=1}^{n_3}\Big(a_k(x) \times \prod_{i=1,\, i\neq k}^{n_3}(b_i(x)X + \lambda a_i(x))\Big)\Big)\\
&\qquad\quad + \sum_{i\neq j}(1+\lambda^2)^2 a_i(x)a_j(x) \times \prod_{k=1,\, k\neq i,\, k\neq j}^{n_3}(b_k(x)X + \lambda a_k(x)).
\end{aligned}$$

Since the polynomial equation should only have finite roots, we derive that the coefficients of X^q are zero for any integer $q \in \{0, 1, 2, \dots\}$. It follows that

$$2S_2 = \frac{1}{\lambda^2}(n_1+n_3)(n_1+n_3+1) - 2n_2(n_1+n_3+1) + n_2(n_2-1)\lambda^2$$

and

$$2(1+\lambda^2)\Big(-\frac{n_1+n_3}{\lambda} + n_2\lambda\Big)n_3\lambda^{n_3-1}\prod_{i=1}^{n_3} a_i + (1+\lambda^2)^2 n_3(n_3-1)\lambda^{n_3-2}\prod_{i=1}^{n_3} a_i = 0. \tag{3.20}$$

Since

$$r \neq 2\frac{(2k+m)n^2 - (2k^2+4k+2km+m)n + 2k(m+k+1)}{n(2k+m)(2(n-1)-(2k+m))},$$

$$0 \leqslant m \leqslant n-2, \qquad 1 \leqslant k \leqslant n-1-m \quad \text{and} \quad 2S_2 = n(n-1)(r-1),$$

we obtain

$$\begin{aligned} 2S_2 &= \frac{1}{\lambda^2}(n_1+n_3)(n_1+n_3+1) - 2n_2(n_1+n_3+1) + n_2(n_2-1)\lambda^2 \\ &= n(n-1)(r-1) \\ &\neq (n-1)\frac{n(m^2-4k)+4k(m+k+1)}{(2k+m)(2(n-1)-(2k+m))}. \end{aligned} \tag{3.21}$$

Letting $m = n_3 - 1$ and $k = n_1 + 1$, we then have, from (3.21), that

$$2S_2 \neq (n-1)\frac{n[(n_3-1)^2 - 4(n_1+1)] + 4(n_1+1)(n_1+n_3+1)}{(2n_1+n_3+1)[2(n-1)-(2n_1+n_3+1)]}. \tag{3.22}$$

By using (3.21) and (3.22), we infer that

$$\lambda^2 \neq \frac{2n_1+n_3+1}{2n_2+n_3-1}. \tag{3.23}$$

Hence, we can deduce from (3.20) and (3.23) that

$$\prod_{i=1}^{n_3} a_i = 0.$$

This is in contradiction with $a_i \neq 0$. Therefore, $I_3(x) = \varnothing$. By using the same arguments as in the proof of theorem 1.3, we conclude that M is either a totally umbilical sphere or a Clifford hypersurface. This completes the proof of theorem 1.4. □

Proof of corollary 1.5. If M is neither a totally umbilical sphere nor a Clifford hypersurface, we obtain from theorem 1.4 that $f_v + \alpha_\pm l_v \not\equiv 0$ for any fixed vector $v \in R^{n+2}$. Then $\dim U_+ = \dim U_- = n+2$. It follows from equation (3.14) that $\mathrm{Ind}_T(M) \geqslant 2n+4$. □

Proof of theorem 1.6. We can prove theorem 1.6 using similar arguments to those used in the proof of theorem 1.4. □

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