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EIGENVALUES OF THE LAPLACIAN ON RIEMANNIAN MANIFOLDS

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For a bounded domain Ω with a piecewise smooth boundary in a complete Riemannian manifold M, we study eigenvalues of the Dirichlet eigenvalue problem of the Laplacian. By making use of a fact that eigenfunctions form an orthonormal basis of $L^2(\Omega)$ in place of the Rayleigh–Ritz formula, we obtain inequalities for eigenvalues of the Laplacian. In particular, for lower order eigenvalues, our results extend the results of Chen and Cheng [D. Chen and Q.-M. Cheng, Extrinsic estimates for eigenvalues of the Laplace operator, *J. Math. Soc. Japan* **60** (2008) 325–339].

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1. Introduction

Let $\Omega \subset M$ be a bounded domain with a piecewise smooth boundary $\partial\Omega$ in an n-dimensional complete Riemannian manifold M. We consider the following Dirichlet eigenvalue problem of the Laplacian:

$$\begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

It is well known that the spectrum of this problem is real and discrete:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \nearrow \infty,$$

where each λ_i has finite multiplicity which is repeated according to its multiplicity.

When M is an n-dimensional Euclidean space $\mathbf{R}^n$, Payne, Pólya and Weinberger [18] proved

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{kn} \sum_{i=1}^{k} \lambda_i. \tag{1.2}$$

Hile and Protter [16] generalized the above result to

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{kn}{4}. \tag{1.3}$$

In 1991, a much sharper inequality was obtained by Yang [20] (cf. [11]):

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\lambda_i. \tag{1.4}$$

When M is an n-dimensional unit sphere $S^n(1)$, Cheng and Yang [9] have proved an optimal universal inequality:

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\left(\lambda_i + \frac{n^2}{4}\right). \tag{1.5}$$

For the Dirichlet eigenvalue problem of the Laplacian on a bounded domain in an n-dimensional complete Riemannian manifold M, Chen and Cheng [6] and El Soufi, Harrell and Ilias [15] have proved, independently,

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\left(\lambda_i + \frac{n^2}{4} H_0^2\right), \tag{1.6}$$

where H_0^2 is a nonnegative constant which only depends on M and Ω. When M is the unit sphere, $H_0^2 = 1$, the above inequality is best possible, which becomes the result of Cheng and Yang [9]. For the Dirichlet eigenvalue problem of the Laplacian on a bounded domain in a hyperbolic space, universal inequalities for eigenvalues have been obtained by Cheng and Yang [12]. For complex projective spaces and so on, see [9, 10].

For lower order eigenvalues of the eigenvalue problem (1.1), when M is the Euclidean space $\mathbf{R}^n$, the following conjecture of Payne, Pólya and Weinberger is well known:

Conjecture of PPW. *For a bounded domain* Ω *in* $\mathbf{R}^n$, *eigenvalues of the eigenvalue problem* (1.1) *satisfy*

(1) $\dfrac{\lambda_2}{\lambda_1} \leq \left.\dfrac{\lambda_2}{\lambda_1}\right|_{\mathbf{B}^n} = \dfrac{j_{n/2,1}^2}{j_{n/2-1,1}^2}$,

(2) $\dfrac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n\dfrac{j_{n/2,1}^2}{j_{n/2-1,1}^2}$,

where $\mathbf{B}^n$ is the n-dimensional unit ball in $\mathbf{R}^n$, $j_{p,k}$ denotes the kth positive zero of the standard Bessel function $J_p(x)$ of the first kind of order p.

For the conjecture (1) of Payne, Pólya and Weinberger, many mathematicians studied it. For examples, Payne, Pólya and Weinberger [18], Brands [5], de Vries [14], Chiti [13], Hile and Protter [16], Marcellini [17] and so on. Finally, Ashbaugh and Benguria [2] (cf. [1, 3]) solved this conjecture.

For the conjecture (2) of Payne, Pólya and Weinberger, when $n = 2$, Brands [5] improved the bound $\frac{\lambda_2+\lambda_3}{\lambda_1} \le 6$ of Payne, Pólya and Weinberger [18], he proved $\frac{\lambda_2+\lambda_3}{\lambda_1} \le 3+\sqrt{7}$. Furthermore, Hile and Protter [16] obtained $\frac{\lambda_2+\lambda_3}{\lambda_1} \le 5.622$. In [17], Marcellini proved $\frac{\lambda_2+\lambda_3}{\lambda_1} \le (15+\sqrt{345})/6$. Recently, Chen and Zheng [7] have proved $\frac{\lambda_2+\lambda_3}{\lambda_1} \le 5.3507$. For a general dimension $n \ge 2$, Ashbaugh and Benguria [4] proved

$$\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \le n + 4. \tag{1.7}$$

Furthermore, Ashbaugh and Benguria [4] (cf. Hile and Protter [16]) improved the above result to

$$\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \le n + 3 + \frac{\lambda_1}{\lambda_2}. \tag{1.8}$$

Very recently, Cheng and Qi [8] have proved that, for any $1 \le j \le n+2$, eigenvalues satisfy at least one of the following:

$$\frac{\lambda_2}{\lambda_1} < 2 - \frac{\lambda_1}{\lambda_j},$$

$$\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \le n + 3 + \frac{\lambda_1}{\lambda_j}.$$

When M is the n-dimensional unit sphere $S^n(1)$, that is, for a bounded domain Ω in $S^n(1)$, Sun, Cheng and Yang [19] have proved

$$\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \le n + 4 + \frac{n^2}{\lambda_1}. \tag{1.9}$$

For a general complete Riemannian manifold M, Chen and Cheng [6] have proved that there exists a nonnegative constant H_0 such that

$$\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \le n + 4 + \frac{n^2 H_0^2}{\lambda_1}. \tag{1.10}$$

In this paper, by making use of the fact that eigenfunctions form an orthonormal basis of $L^2(\Omega)$ in place of the Rayleigh–Ritz formula, we obtain inequalities for eigenvalues of the Laplacian. In particular, we improve the above result.

2. Estimates for Lower Order Eigenvalues

In this section, first of all, we will mainly focus our mind on the investigation for lower order eigenvalues of the Dirichlet eigenvalue problem of the Laplacian by making use of the fact that eigenfunctions form an orthonormal basis of $L^2(\Omega)$ in place of the Rayleigh–Ritz formula. We prove the following theorem.

Theorem 2.1. *Let M be an n-dimensional complete Riemannian manifold, $\Omega \subset M$ a bounded domain with a piecewise smooth boundary $\partial\Omega$. Then, the lower order eigenvalues of the Dirichlet eigenvalue problem of the Laplacian satisfy*

$$\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \le n + \sqrt{Q_0\left(\frac{n^2 H_0^2}{\lambda_1} + 4\right)},$$

where H_0 is a nonnegative constant depending only on M and Ω and Q_0 is given by

$$Q_0 = \frac{1}{2}\left[\left(2 - \frac{\lambda_1}{\lambda_2}\right)\frac{n^2H_0^2}{\lambda_1} + 3 + \frac{\lambda_1}{\lambda_2} + \sqrt{\left(3 + \frac{\lambda_1}{\lambda_2} + \frac{n^2H_0^2}{\lambda_2}\right)^2 + 4\left(1 - \frac{\lambda_1}{\lambda_2}\right)\frac{n^2H_0^2}{\lambda_2}}\,\right].$$

Remark 2.2. It is not hard to prove, from $\frac{\lambda_1}{\lambda_2} < 1$,

$$Q_0 < \frac{n^2H_0^2}{\lambda_1} + 4.$$

In particular, when M is an n-dimensional complete minimal submanifold in the Euclidean space $\mathbf{R}^N$, we have

Corollary 2.3. *Let Ω be a bounded domain in an n-dimensional complete minimal submanifold M in $\mathbf{R}^N$. Then, we have*

$$\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \le n + 2\sqrt{3 + \frac{\lambda_1}{\lambda_2}}.$$

Since M is a complete Riemannian manifold, from a theorem of Nash, there exists an isometric immersion $\varphi : M \to \mathbf{R}^N$ from M into a Euclidean space $\mathbf{R}^N$. Let $(x^1, \ldots, x^n)$ denote an arbitrary local coordinate system of M. For any point $p \in \Omega$, we can write $\varphi(p) = (y_1, y_2, \ldots, y_N)$ with

$$y_\alpha = y_\alpha(x^1, \ldots, x^n), \quad 1 \le \alpha \le N,$$

which is the position vector of p in $\mathbf{R}^N$. Thus, we have

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left\langle \sum_{\alpha=1}^N \frac{\partial y_\alpha}{\partial x^i}\frac{\partial}{\partial y_\alpha}, \sum_{\beta=1}^N \frac{\partial y_\beta}{\partial x^j}\frac{\partial}{\partial y_\beta}\right\rangle = \sum_{\alpha=1}^N \frac{\partial y_\alpha}{\partial x^i}\frac{\partial y_\alpha}{\partial x^j},$$

where g denotes the induced metric of M from $\mathbf{R}^N$, $\langle , \rangle$ is the standard inner product in $\mathbf{R}^N$. We denote the gradient of a function f by ∇f. Then, the following lemma holds, which is proved by Chen and Cheng [6].

Lemma 2.4.

$$\sum_{\alpha=1}^{N} g(\nabla y_\alpha, \nabla y_\alpha) = \sum_{\alpha=1}^{N} |\nabla y_\alpha|^2 = n,$$

$$\sum_{\alpha=1}^{N} (\Delta y_\alpha)^2 = n^2 |H|^2,$$

$$\sum_{\alpha=1}^{N} \Delta y_\alpha \nabla y_\alpha = 0,$$

and for any function $u \in C^\infty(M)$,

$$\sum_{\alpha=1}^{N} (g(\nabla y_\alpha, \nabla u))^2 = \sum_{\alpha=1}^{N} (\nabla y_\alpha \cdot \nabla u)^2 = |\nabla u|^2,$$

where $|H|$ *is the mean curvature of* M.

Proof of Theorem 2.1. Let u_j be the eigenfunction corresponding to the eigenvalue λ_j such that $\{u_j\}_{j=1}^\infty$ becomes an orthonormal basis of $L^2(\Omega)$. Hence, $\int_\Omega u_i u_j = \delta_{ij}$ for $\forall\, i, j = 1, 2, \ldots$. Defining

$$a_{\alpha j} = \int_\Omega y_\alpha u_1 u_{j+1},$$

since u_1 does not change sign in Ω, we can assume $u_1 > 0$ in Ω. We consider the $N \times N$-matrix $A = (a_{\alpha j})$. From the orthogonalization of Gram and Schmidt, there exist an upper triangle matrix $R = (R_{\alpha j})$ and an orthogonal matrix $Q = (q_{\alpha\beta})$ such that $R = QA$. Thus,

$$R_{\alpha j} = \sum_{\beta=1}^{N} q_{\alpha\beta} a_{\beta j} = \int_\Omega \sum_{\beta=1}^{N} q_{\alpha\beta} y_\beta u_1 u_{j+1} = 0, \quad \text{for } 1 \le j < \alpha \le N.$$

Defining $\bar{y}_\alpha = \sum_{\gamma=1}^{N} q_{\alpha\gamma} y_\gamma$, we have

$$\int_\Omega \bar{y}_\alpha u_1 u_{j+1} = \int_\Omega \sum_{\gamma=1}^{N} q_{\alpha\gamma} y_\gamma u_1 u_{j+1} = 0, \quad \text{for } 1 \le j < \alpha \le N.$$

Putting

$$z_\alpha = \bar{y}_\alpha - b_\alpha, \quad b_\alpha = \int_\Omega \bar{y}_\alpha u_1^2, \quad \text{for } 1 \le \alpha \le N$$

and

$$A_{\alpha j} = \int_\Omega z_\alpha u_1 u_j,$$

we have

$$A_{\alpha j} = 0, \quad \text{for} \quad 1 \le j \le \alpha \le N. \tag{2.1}$$

Defining

$$B_{\alpha j} = \int_\Omega u_j \nabla z_\alpha \cdot \nabla u_1$$

and

$$C_{\alpha j} = \int_\Omega u_j u_1 \Delta z_\alpha,$$

from the Stokes theorem, we obtain

$$\begin{aligned}-\lambda_j A_{\alpha j} &= \int_\Omega z_\alpha u_1 \Delta u_j = \int_\Omega \Delta(z_\alpha u_1) u_j \\ &= \int_\Omega (2\nabla z_\alpha \cdot \nabla u_1 - \lambda_1 z_\alpha u_1 + u_1 \Delta z_\alpha) u_j \\ &= -\lambda_1 A_{\alpha j} + 2B_{\alpha j} + C_{\alpha j},\end{aligned}$$

namely,

$$2B_{\alpha j} = (\lambda_1 - \lambda_j) A_{\alpha j} - C_{\alpha j}. \tag{2.2}$$

Since $\{u_j\}_{j=1}^\infty$ is an orthonormal basis in $L^2(\Omega)$ and $A_{\alpha j} = 0$, for $1 \le j \le \alpha \le N$, we have

$$z_\alpha u_1 = \sum_{j=\alpha+1}^\infty A_{\alpha j} u_j \quad \text{and} \quad \|z_\alpha u_1\|^2 = \sum_{j=\alpha+1}^\infty A_{\alpha j}^2. \tag{2.3}$$

Furthermore,

$$\int_\Omega u_1^2 z_\alpha \Delta z_\alpha = \sum_{j=\alpha+1}^\infty A_{\alpha j} C_{\alpha j}, \tag{2.4}$$

$$2\int_\Omega z_\alpha u_1 \nabla z_\alpha \cdot \nabla u_1 = 2\sum_{j=\alpha+1}^\infty A_{\alpha j} B_{\alpha j} = \sum_{j=\alpha+1}^\infty (\lambda_1 - \lambda_j) A_{\alpha j}^2 - \sum_{j=\alpha+1}^\infty A_{\alpha j} C_{\alpha j}. \tag{2.5}$$

Since for any function $f \in C^2(\Omega) \cap C(\bar{\Omega})$,

$$-2\int_\Omega f u_1 \nabla f \cdot \nabla u_1 = \int_\Omega u_1^2 f \Delta f + \int_\Omega |\nabla f|^2 u_1^2, \tag{2.6}$$

we have

$$\int_\Omega |\nabla z_\alpha|^2 u_1^2 = -\int_\Omega z_\alpha u_1 (2\nabla z_\alpha \cdot \nabla u_1 + u_1 \Delta z_\alpha). \tag{2.7}$$

We obtain

$$\sum_{j=\alpha+1}^{\infty} (\lambda_j - \lambda_1) A_{\alpha j}^2 = \int_\Omega |\nabla z_\alpha|^2 u_1^2. \tag{2.8}$$

For any positive integer k, we have

$$\begin{aligned}\sum_{j=\alpha+1}^{\infty} (\lambda_j - \lambda_1) A_{\alpha j}^2 &= \sum_{j=\alpha+1}^{k} (\lambda_j - \lambda_1) A_{\alpha j}^2 + \sum_{j=k+1}^{\infty} (\lambda_j - \lambda_1) A_{\alpha j}^2 \\ &\geq \sum_{j=\alpha+1}^{k} (\lambda_j - \lambda_1) A_{\alpha j}^2 + (\lambda_{k+1} - \lambda_1) \sum_{j=k+1}^{\infty} A_{\alpha j}^2 \\ &= \sum_{j=\alpha+1}^{k} (\lambda_j - \lambda_1) A_{\alpha j}^2 + (\lambda_{k+1} - \lambda_1) \sum_{j=\alpha+1}^{\infty} A_{\alpha j}^2 \\ &\quad - (\lambda_{k+1} - \lambda_1) \sum_{j=\alpha+1}^{k} A_{\alpha j}^2 \\ &= \sum_{j=\alpha+1}^{k} (\lambda_j - \lambda_{k+1}) A_{\alpha j}^2 + (\lambda_{k+1} - \lambda_1) \sum_{j=\alpha+1}^{\infty} A_{\alpha j}^2.\end{aligned}$$

Thus, we infer

$$(\lambda_{k+1} - \lambda_1) \|z_\alpha u_1\|^2 \leq \sum_{j=\alpha+1}^{k} (\lambda_{k+1} - \lambda_j) A_{\alpha j}^2 + \int_\Omega |\nabla z_\alpha|^2 u_1^2, \tag{2.9}$$

and, in particular,

$$(\lambda_{\alpha+1} - \lambda_1) \|z_\alpha u_1\|^2 \leq \int_\Omega |\nabla z_\alpha|^2 u_1^2. \tag{2.10}$$

For any α, we have

$$|\nabla z_\alpha|^2 \leq 1. \tag{2.11}$$

In fact, for any fixed point $p_0 \in \Omega$, we can choose a new coordinate system $\widetilde{y} = (\widetilde{y}_1, \ldots, \widetilde{y}_N)$ of $\mathbf{R}^N$ given by $\varphi(p) - \varphi(p_0) = \widetilde{y}(p) B$ such that $\frac{\partial}{\partial \widetilde{y}_1}|_{p_0}, \ldots, \frac{\partial}{\partial \widetilde{y}_n}|_{p_0}$ span $T_{p_0} M$ and at p_0, $g(\frac{\partial}{\partial \widetilde{y}_i}, \frac{\partial}{\partial \widetilde{y}_j}) = \delta_{ij}$, where $B = (b_{\alpha\beta}) \in O(N)$ is an $N \times N$ orthogonal matrix.

$$\begin{aligned}|\nabla z_\alpha|^2(p_0) &= g(\nabla z_\alpha, \nabla z_\alpha) \\ &= \sum_{\beta,\gamma=1}^{N} q_{\alpha\gamma} q_{\alpha\beta} g(\nabla y_\gamma, \nabla y_\beta)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\beta,\gamma=1}^{N} q_{\alpha\gamma} q_{\alpha\beta} g\left(\sum_{\mu=1}^{N} b_{\gamma\mu}\nabla\widetilde{y}_\mu, \sum_{\nu=1}^{N} b_{\beta\nu}\nabla\widetilde{y}_\nu\right) \\
&= \sum_{\beta,\gamma,\mu,\nu=1}^{N} q_{\alpha\gamma} b_{\gamma\mu} q_{\alpha\beta} b_{\beta\nu} g(\nabla\widetilde{y}_\mu, \nabla\widetilde{y}_\nu) \\
&= \sum_{j=1}^{n}\left(\sum_{\beta=1}^{N} q_{\alpha\beta} b_{\beta j}\right)^2 \le 1,
\end{aligned} \tag{2.12}
$$

since QB is an orthogonal matrix when B and Q are orthogonal matrices. Therefore, (2.11) holds because p_0 is an arbitrary point. Since Lemma 2.4 also holds for z_α from the definition of z_α, for any positive constant $t > \frac{1}{2}$, we have, from Lemma 2.4 and (2.11),

$$
\begin{aligned}
&\sum_{\alpha=1}^{N}(\lambda_{\alpha+1}-\lambda_1)\int_\Omega |\nabla z_\alpha|^2 u_1^{t+1} \\
&\quad \ge \sum_{j=1}^{n}(\lambda_{j+1}-\lambda_1)\int_\Omega |\nabla z_j|^2 u_1^{t+1} + (\lambda_{n+1}-\lambda_1)\sum_{A=n+1}^{N}\int_\Omega |\nabla z_A|^2 u_1^{t+1} \\
&\quad = \sum_{j=1}^{n}(\lambda_{j+1}-\lambda_1)\int_\Omega |\nabla z_j|^2 u_1^{t+1} + (\lambda_{n+1}-\lambda_1)\int_\Omega\left(n-\sum_{j=1}^{n}|\nabla z_j|^2\right)u_1^{t+1} \\
&\quad = \sum_{j=1}^{n}(\lambda_{j+1}-\lambda_1)\int_\Omega |\nabla z_j|^2 u_1^{t+1} + (\lambda_{n+1}-\lambda_1)\int_\Omega\sum_{j=1}^{n}(1-|\nabla z_j|^2)u_1^{t+1} \\
&\quad \ge \sum_{j=1}^{n}(\lambda_{j+1}-\lambda_1)\int_\Omega |\nabla z_j|^2 u_1^{t+1} + \int_\Omega\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_1)(1-|\nabla z_j|^2)u_1^{t+1} \\
&\quad = \sum_{j=1}^{n}(\lambda_{j+1}-\lambda_1)\int_\Omega u_1^{t+1}.
\end{aligned} \tag{2.13}
$$

On the other hand, from the Stokes theorem and the Cauchy–Schwarz inequality, we obtain

$$
\begin{aligned}
\int_\Omega |\nabla z_\alpha|^2 u_1^{t+1} &= -\int_\Omega z_\alpha u_1(u_1^t\Delta z_\alpha + (1+t)u_1^{t-1}\nabla z_\alpha\cdot\nabla u_1) \\
&\le \|z_\alpha u_1\|\cdot\|u_1^t\Delta z_\alpha + (1+t)u_1^{t-1}\nabla z_\alpha\cdot\nabla u_1\|,
\end{aligned} \tag{2.14}
$$

and

$$
\begin{aligned}
\int_\Omega |\nabla z_\alpha|^2 u_1^2 &= -\int_\Omega z_\alpha u_1(u_1\Delta z_\alpha + 2\nabla z_\alpha\cdot\nabla u_1) \\
&\le \|z_\alpha u_1\|\cdot\|u_1\Delta z_\alpha + 2\nabla z_\alpha\cdot\nabla u_1\|.
\end{aligned} \tag{2.15}
$$

From (2.10), (2.13), (2.14) and (2.15), we derive

$$\begin{aligned}
&\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_1)\int_\Omega u_1^{t+1}\\
&\quad\le \sum_{\alpha=1}^{N}(\lambda_{\alpha+1}-\lambda_1)\int_\Omega |\nabla z_\alpha|^2 u_1^{t+1}\\
&\quad\le \sum_{\alpha=1}^{N}\frac{\int_\Omega |\nabla z_\alpha|^2 u_1^2}{\|z_\alpha u_1\|^2}\int_\Omega |\nabla z_\alpha|^2 u_1^{t+1}\\
&\quad\le \sum_{\alpha=1}^{N}\|u_1\Delta z_\alpha + 2\nabla z_\alpha\cdot\nabla u_1\|\cdot\|u_1^t\Delta z_\alpha + (1+t)u_1^{t-1}\nabla z_\alpha\cdot\nabla u_1\|\\
&\quad\le \sqrt{\sum_{\alpha=1}^{N}\|u_1\Delta z_\alpha + 2\nabla z_\alpha\cdot\nabla u_1\|^2\cdot\sum_{\alpha=1}^{N}\|u_1^t\Delta z_\alpha + (1+t)u_1^{t-1}\nabla z_\alpha\cdot\nabla u_1\|^2}.
\end{aligned} \tag{2.16}$$

Since Lemma 2.4 also holds for z_α from the definition of z_α, we have

$$\begin{aligned}
\sum_{\alpha=1}^{N}\|u_1\Delta z_\alpha + 2\nabla z_\alpha\cdot\nabla u_1\|^2 &= \int_\Omega (n^2|H|^2u_1^2 + 4|\nabla u_1|^2)\\
&\le n^2\sup_\Omega |H|^2 + 4\lambda_1
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
&\sum_{\alpha=1}^{N}\|u_1^t\Delta z_\alpha + (1+t)u_1^{t-1}\nabla z_\alpha\cdot\nabla u_1\|^2\\
&\quad= \int_\Omega\left(n^2|H|^2u_1^{2t} + \frac{(1+t)^2}{t^2}|\nabla u_1^t|^2\right)\\
&\quad\le \left(n^2\sup_\Omega |H|^2 + \frac{(1+t)^2}{2t-1}\lambda_1\right)\int_\Omega u_1^{2t}.
\end{aligned} \tag{2.18}$$

Putting (2.17) and (2.18) into (2.16), we obtain

$$\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_1)\le B(t)\sqrt{(n^2\sup_\Omega |H|^2 + 4\lambda_1)\left(n^2\sup_\Omega |H|^2 + \frac{(1+t)^2}{2t-1}\lambda_1\right)}, \tag{2.19}$$

where

$$B(t) = \frac{\sqrt{\int_\Omega u_1^{2t}}}{\int_\Omega u_1^{t+1}}.$$

Since the spectrum of the Dirichlet eigenvalue problem of the Laplacian is an invariant of isometries, we know that the above inequality holds for any isometric immersion from M into a Euclidean space. Now we define Φ by

$$\Phi = \{\varphi; \varphi \text{ is an isometric immersion from } M \text{ into a Euclidean space}\}.$$

Defining

$$H_0^2 = \inf_{\varphi\in\Phi} \sup_{\Omega} |H|^2,$$

we have

$$\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_1) \le B(t)\sqrt{(n^2H_0^2+4\lambda_1)\left(n^2H_0^2+\frac{(1+t)^2}{2t-1}\lambda_1\right)}. \tag{2.20}$$

Next, we need to estimate $B(t)$ as a function of t by making use of the same method as Brands [5]. Let $u = u_1^t - u_1\int_\Omega u_1^{t+1}$. We know that u is a trial function for λ_2. Hence, we have

$$\lambda_2 \le \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}.$$

According to a direct calculation, we obtain

$$\frac{\lambda_2}{\lambda_1} \le \frac{\frac{t^2}{2t-1}B(t)^2-1}{B(t)^2-1}$$

since $B(t)^2-1 = \frac{\int_\Omega u^2}{(\int_\Omega u_1^{t+1})^2} > 0$ for $t>1$. Let $a = \frac{\lambda_2}{\lambda_1} > 1$. We have

$$\left(a-\frac{t^2}{2t-1}\right)B(t)^2 \le a-1.$$

When $1<t<a+\sqrt{a^2-a}$, we can infer

$$B(t) \le \sqrt{\frac{(a-1)(2t-1)}{a(2t-1)-t^2}}.$$

Therefore, we obtain

$$\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_1) \le \sqrt{(n^2H_0^2+4\lambda_1)\left(n^2H_0^2+\frac{(1+t)^2}{2t-1}\lambda_1\right)\frac{(a-1)(2t-1)}{a(2t-1)-t^2}}. \tag{2.21}$$

Letting $b = \frac{n^2H_0^2}{\lambda_1}$ and defining a function

$$f(t) = \frac{b(2t-1)+(1+t)^2}{a(2t-1)-t^2}, \tag{2.22}$$

we have

$$\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_1) \le \sqrt{\lambda_1(a-1)(n^2H_0^2+4\lambda_1)f(t)}. \tag{2.23}$$

If we take $t = 1$, $f(1) = \frac{b+4}{a-1}$. Thus, we obtain the result of Chen and Cheng [6]. Furthermore, we try to get the minimum of $f(t)$ under $1 \le t \le a + \sqrt{a^2 - a}$. It is not difficult to prove that the minimum of $f(t)$ is attained at

$$t_0 = \frac{a + b - 1 + \sqrt{(a + b - 1)^2 + 8a(a + b + 1)}}{2(a + b + 1)}.$$

Since $g(s) = t_0$ as a function of $s = a+b$, is a decreasing function of s in the interval $[a, \infty)$, we have

$$1 = g(\infty) < t_0 \le g(a) < a + \sqrt{a^2 - a}.$$

By a direct computation, we have

$$\begin{aligned} &a(2t_0 - 1) - t_0^2 \\ &\quad = \frac{2\sqrt{(a + b - 1)^2 + 8a(a + b + 1)}}{4(a + b + 1)^2} \\ &\qquad \times \left(2a(a + b + 1) - (a + b - 1) - \sqrt{(a + b - 1)^2 + 8a(a + b + 1)}\right). \end{aligned}$$

From $\{3(a + b) + 1\}^2 - 8b(a + b + 1) = (a + b - 1)^2 + 8a(a + b + 1)$, we get

$$\begin{aligned} &b(2t_0 - 1) + (1 + t_0)^2 \\ &\quad = \frac{2\sqrt{(a + b - 1)^2 + 8a(a + b + 1)}}{4(a + b + 1)^2} \\ &\qquad \times \left(2b(a + b + 1) + 3(a + b) + 1 + \sqrt{(a + b - 1)^2 + 8a(a + b + 1)}\right). \end{aligned}$$

Thus, we have

$$\begin{aligned} f(t_0) &= \frac{2b(a + b + 1) + 3(a + b) + 1 + \sqrt{(a + b - 1)^2 + 8a(a + b + 1)}}{2a(a + b + 1) - (a + b - 1) - \sqrt{(a + b - 1)^2 + 8a(a + b + 1)}} \\ &= \frac{2(a + b + 1)^2\left\{(2a - 1)b + 3a + 1 + \sqrt{(a + b - 1)^2 + 8a(a + b + 1)}\right\}}{4a(a + b + 1)^2(a - 1)} \\ &= \frac{(2a - 1)b + 3a + 1 + \sqrt{(a + b - 1)^2 + 8a(a + b + 1)}}{2a(a - 1)}. \end{aligned} \tag{2.24}$$

From (2.23) and (2.24), we obtain

$$\begin{aligned} &\sum_{j=1}^{n}(\lambda_{j+1} - \lambda_1) \\ &\qquad \le \sqrt{\lambda_1(n^2H_0^2 + 4\lambda_1)\frac{(2a - 1)b + 3a + 1 + \sqrt{(a + b - 1)^2 + 8a(a + b + 1)}}{2a}} \end{aligned}$$

$$= \lambda_1 \sqrt{Q_0 \left(\frac{n^2 H_0^2}{\lambda_1} + 4 \right)}$$

because $a = \frac{\lambda_2}{\lambda_1}$ and $b = \frac{n^2 H_0^2}{\lambda_1}$. This finishes the proof of Theorem 2.1. □

For any positive integer k, we have from (2.9)

$$\frac{\lambda_{k+1} - \lambda_1}{\sum_{j=\alpha+1}^{k} (\lambda_{k+1} - \lambda_j) A_{\alpha j}^2 + \int_\Omega |\nabla z_\alpha|^2 u_1^2} \le \frac{1}{\|z_\alpha u_1\|^2}. \tag{2.25}$$

From (2.14) and (2.15), we obtain

$$\begin{aligned} &\frac{(\lambda_{k+1} - \lambda_1) \int_\Omega |\nabla z_\alpha|^2 u_1^{t+1} \int_\Omega |\nabla z_\alpha|^2 u_1^2}{\sum_{j=\alpha+1}^{k} (\lambda_{k+1} - \lambda_j) A_{\alpha j}^2 + \int_\Omega |\nabla z_\alpha|^2 u_1^2} \\ &\quad = \frac{(\lambda_{k+1} - \lambda_1) \int_\Omega |\nabla z_\alpha|^2 u_1^{t+1}}{1 + \sum_{j=\alpha+1}^{k} (\lambda_{k+1} - \lambda_j) \frac{A_{\alpha j}^2}{\int_\Omega |\nabla z_\alpha|^2 u_1^2}} \\ &\quad \le \|u_1^t \Delta z_\alpha + (1+t) u_1^{t-1} \nabla z_\alpha \cdot \nabla u_1\| \cdot \|u_1 \Delta z_\alpha + 2 \nabla z_\alpha \cdot \nabla u_1\|. \end{aligned} \tag{2.26}$$

For any positive integer k, we can find some α_0 such that

$$\sum_{j=\alpha_0+1}^{k} \frac{(\lambda_{k+1} - \lambda_j) A_{\alpha_0 j}^2}{\int_\Omega |\nabla z_{\alpha_0}|^2 u_1^2} = \max_{1 \le \alpha \le N} \sum_{j=\alpha+1}^{k} \frac{(\lambda_{k+1} - \lambda_j) A_{\alpha j}^2}{\int_\Omega |\nabla z_\alpha|^2 u_1^2}.$$

Hence, from Lemma 2.4, we obtain

$$\begin{aligned} &\frac{n(\lambda_{k+1} - \lambda_1) \int_\Omega u_1^{t+1}}{1 + \sum_{j=\alpha_0+1}^{k} (\lambda_{k+1} - \lambda_j) \frac{A_{\alpha_0 j}^2}{\int_\Omega |\nabla z_{\alpha_0}|^2 u_1^2}} \\ &\quad \le \sum_{\alpha=1}^{N} \|u_1^t \Delta z_\alpha + (1+t) u_1^{t-1} \nabla z_\alpha \cdot \nabla u_1\| \cdot \|u_1 \Delta z_\alpha + 2 \nabla z_\alpha \cdot \nabla u_1\| \\ &\quad \le \sqrt{(n^2 \sup_\Omega |H|^2 + 4\lambda_1) \left(n^2 \sup_\Omega |H|^2 + \frac{(1+t)^2}{2t-1} \lambda_1 \right) \int_\Omega u_1^{2t}}, \end{aligned}$$

that is, we have

$$\begin{aligned} &\frac{n(\lambda_{k+1} - \lambda_1)}{1 + \sum_{j=\alpha_0+1}^{k} (\lambda_{k+1} - \lambda_j) \frac{A_{\alpha_0 j}^2}{\int_\Omega |\nabla z_{\alpha_0}|^2 u_1^2}} \\ &\quad \le B(t) \sqrt{(n^2 \sup_\Omega |H|^2 + 4\lambda_1) \left(n^2 \sup_\Omega |H|^2 + \frac{(1+t)^2}{2t-1} \lambda_1 \right)}. \end{aligned} \tag{2.27}$$

On the other hand, we have

$$\int_\Omega |\nabla u_1^{t-1}|^2 u_1^2 = \frac{(t-1)^2}{2t-1} \int_\Omega \nabla u_1 \cdot \nabla u_1^{2t-1} = \frac{(t-1)^2}{2t-1} \lambda_1 \int_\Omega u_1^{2t}. \tag{2.28}$$

Letting

$$D_j = \int_\Omega u_1^t u_j,$$

we know

$$u_1^t = \sum_{j=1}^\infty D_j u_j, \quad \int_\Omega u_1^{2t} = \sum_{j=1}^\infty D_j^2. \tag{2.29}$$

Taking $f = u_1^{t-1}$ in (2.6), we get

$$\begin{aligned}
\int_\Omega |\nabla u_1^{t-1}|^2 u_1^2 &= -2\int_\Omega u_1^t \nabla u_1^{t-1} \cdot \nabla u_1 - \int_\Omega u_1^{t+1} \Delta u_1^{t-1} \\
&= -\sum_{j=1}^\infty D_j \left(2\int_\Omega u_j \nabla u_1^{t-1} \cdot \nabla u_1 + \int_\Omega u_j u_1 \Delta u_1^{t-1} \right) \\
&= -\sum_{j=1}^\infty D_j \left(\int_\Omega u_j \Delta u_1^t - \int_\Omega u_j u_1^{t-1} \Delta u_1 \right) \\
&= -\sum_{j=1}^\infty D_j \left(\int_\Omega u_1^t \Delta u_j - \int_\Omega u_j u_1^{t-1} \Delta u_1 \right) \\
&= \sum_{j=1}^\infty D_j \left(\lambda_j \int_\Omega u_1^t u_j - \lambda_1 \int_\Omega u_j u_1^t \right) \\
&= \sum_{j=2}^\infty (\lambda_j - \lambda_1) D_j^2.
\end{aligned}$$

Thus, we infer

$$\sum_{j=2}^\infty (\lambda_j - \lambda_1) D_j^2 = \frac{(t-1)^2}{2t-1} \lambda_1 \int_\Omega u_1^{2t}. \tag{2.30}$$

Defining

$$\beta_j = \frac{D_j}{\sqrt{\frac{(t-1)^2}{2t-1} \lambda_1 \int_\Omega u_1^{2t}}}, \tag{2.31}$$

we have

$$\sum_{j=2}^{\infty}(\lambda_j - \lambda_1)\beta_j^2 = 1. \tag{2.32}$$

For any positive integer l,

$$\begin{aligned}
1 &= \sum_{j=2}^{\infty}(\lambda_j - \lambda_1)\beta_j^2 \\
&= \sum_{j=2}^{l}(\lambda_j - \lambda_1)\beta_j^2 + \sum_{j=l+1}^{\infty}(\lambda_j - \lambda_1)\beta_j^2 \\
&\geq \sum_{j=2}^{l}(\lambda_j - \lambda_1)\beta_j^2 + (\lambda_{l+1} - \lambda_1)\sum_{j=l+1}^{\infty}\beta_j^2 \\
&= \sum_{j=2}^{l}(\lambda_j - \lambda_1)\beta_j^2 + (\lambda_{l+1} - \lambda_1)\sum_{j=2}^{\infty}\beta_j^2 - (\lambda_{l+1} - \lambda_1)\sum_{j=2}^{l}\beta_j^2 \\
&= \sum_{j=2}^{l}(\lambda_j - \lambda_{l+1})\beta_j^2 + (\lambda_{l+1} - \lambda_1)\sum_{j=2}^{\infty}\beta_j^2,
\end{aligned}$$

namely,

$$(\lambda_{l+1} - \lambda_1)\sum_{j=2}^{\infty}\beta_j^2 \leq 1 + \sum_{j=2}^{l}(\lambda_{l+1} - \lambda_j)\beta_j^2. \tag{2.33}$$

From (2.29) and (2.30), we infer

$$\frac{(t-1)^2}{2t-1}\lambda_1\sum_{j=1}^{\infty}\beta_j^2 = 1. \tag{2.34}$$

Since

$$\left(\int_\Omega u_1^{t+1}\right)^2 = D_1^2 = \frac{(t-1)^2}{2t-1}\lambda_1\beta_1^2\int_\Omega u_1^{2t},$$

according to the definition of $B(t)$, we have

$$B(t)^2 = \frac{\int_\Omega u_1^{2t}}{(\int_\Omega u_1^{t+1})^2} = \frac{1}{\frac{(t-1)^2}{2t-1}\lambda_1\beta_1^2} = \frac{1}{1 - \frac{(t-1)^2}{2t-1}\lambda_1\sum_{j=2}^{\infty}\beta_j^2}. \tag{2.35}$$

From (2.27), (2.33) and (2.35), we have

Proposition 2.5. *Let M be an n-dimensional complete submanifold in $\mathbf{R}^N$, $\Omega \subset M$ be a bounded domain with a piecewise smooth boundary $\partial\Omega$. Then, for any positive integer k, there exists an integer α_0 with $1 \le \alpha_0 \le N$ such that eigenvalues of the Dirichlet eigenvalue problem of the Laplacian satisfy, for any positive integer l and $t > \frac{1}{2}$,*

$$\frac{n(\lambda_{k+1} - \lambda_1)}{1 + \sum_{j=\alpha_0+1}^{k} (\lambda_{k+1} - \lambda_j) \frac{A_{\alpha_0 j}^2}{\int_\Omega |\nabla z_{\alpha_0}|^2 u_1^2}}$$
$$\le \sqrt{\frac{(n^2 \sup_\Omega |H|^2 + 4\lambda_1)(n^2 \sup_\Omega |H|^2 + \frac{(1+t)^2}{2t-1}\lambda_1)}{1 - \frac{(t-1)^2}{2t-1} \frac{\lambda_1}{\lambda_{l+1} - \lambda_1} (1 + \sum_{j=2}^{l} (\lambda_{l+1} - \lambda_j) \beta_j^2)}}.$$

3. Estimates for Eigenvalues on Minimal Submanifolds

In this section, we will deal with eigenvalues of the Laplacian on bounded domains in complete minimal submanifolds of Euclidean spaces. Thus, let $\Omega \subset M$ be a bounded domain with a piecewise smooth boundary $\partial\Omega$ in an n-dimensional complete minimal submanifold M of the Euclidean space $\mathbf{R}^N$. We consider the following Dirichlet eigenvalue problem of the Laplacian:

$$\begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since M is an n-dimensional complete minimal submanifold in $\mathbf{R}^N$, we have from Lemma 2.4 and the definition of $C_{\alpha j}$,

$$C_{\alpha j} = 0$$

for any α and j. Hence, we have from (2.2),

$$2B_{\alpha j} = (\lambda_1 - \lambda_j) A_{\alpha j}. \tag{3.1}$$

For any α, we have

$$\begin{aligned} 0 &= -\frac{2}{t+1} \int_\Omega u_1^{t+1} \Delta z_\alpha \\ &= 2 \int_\Omega u_1^t \nabla z_\alpha \cdot \nabla u_1 \\ &= 2 \sum_{i=1}^{\infty} D_i B_{\alpha i} = \sum_{i=1}^{\infty} (\lambda_1 - \lambda_i) D_i A_{\alpha i}. \end{aligned}$$

Thus, from (2.31) we obtain

$$\sum_{i=2}^{\infty} (\lambda_i - \lambda_1) \beta_i A_{\alpha i} = 0. \tag{3.2}$$

For any positive integer $j \geq 2$, since

$$((\lambda_j - \lambda_1)\beta_j A_{\alpha j})^2 \leq \left(\sum_{i=2, i\neq j}^{\infty} (\lambda_i - \lambda_1)\beta_i^2\right)\left(\sum_{i=2, i\neq j}^{\infty} (\lambda_i - \lambda_1) A_{\alpha i}^2\right),$$

according to (2.8) and (2.32), we derive

$$(\lambda_j - \lambda_1)^2 \beta_j^2 A_{\alpha j}^2 \leq (1 - (\lambda_j - \lambda_1)\beta_j^2)\left(\int_\Omega |\nabla z_\alpha|^2 u_1^2 - (\lambda_j - \lambda_1) A_{\alpha j}^2\right).$$

Hence, we have

$$(\lambda_j - \lambda_1)\beta_j^2 + (\lambda_j - \lambda_1)\frac{A_{\alpha j}^2}{\int_\Omega |\nabla z_\alpha|^2 u_1^2} \leq 1. \tag{3.3}$$

From (2.9) and Lemma 2.4, we can get

$$(\lambda_{k+1} - \lambda_1)\sum_{\alpha=1}^{N} \|z_\alpha u_1\|^2 \leq n + \sum_{\alpha=1}^{N}\sum_{j=\alpha+1}^{k} (\lambda_{k+1} - \lambda_j) A_{\alpha j}^2. \tag{3.4}$$

On the other hand, from the Stokes theorem, we have

$$\int_\Omega u_1^{t+1} |\nabla z_\alpha|^2 = \frac{1}{2}\int_\Omega u_1^{t+1} \Delta z_\alpha^2 = -(t+1)\int_\Omega z_\alpha u_1^t \nabla z_\alpha \cdot \nabla u_1.$$

From Lemma 2.4 and the Cauchy–Schwarz inequality, we get

$$\begin{aligned} n\int_\Omega u_1^{t+1} &= -(t+1)\sum_{\alpha=1}^{N}\int_\Omega z_\alpha u_1^t \nabla z_\alpha \cdot \nabla u_1 \\ &\leq (t+1)\left(\sum_{\alpha=1}^{N}\|z_\alpha u_1\|^2\right)^{\frac{1}{2}}\left(\sum_{\alpha=1}^{N}\int_\Omega u_1^{2t-2}(\nabla z_\alpha \cdot \nabla u_1)^2\right)^{\frac{1}{2}} \\ &= (t+1)\left(\sum_{\alpha=1}^{N}\|z_\alpha u_1\|^2\right)^{\frac{1}{2}}\left(\int_\Omega u_1^{2t-2}|\nabla u_1|^2\right)^{\frac{1}{2}} \\ &= (t+1)\left(\sum_{\alpha=1}^{N}\|z_\alpha u_1\|^2\right)^{\frac{1}{2}}\left(\frac{1}{2t-1}\int_\Omega \nabla u_1^{2t-1} \cdot \nabla u_1\right)^{\frac{1}{2}} \\ &= (t+1)\left(\sum_{\alpha=1}^{N}\|z_\alpha u_1\|^2\right)^{\frac{1}{2}}\left(\frac{\lambda_1}{2t-1}\int_\Omega u_1^{2t}\right)^{\frac{1}{2}}, \end{aligned}$$

namely,

$$\sum_{\alpha=1}^{N}\|z_\alpha u_1\|^2 \geq \frac{n^2(\int_\Omega u_1^{t+1})^2}{\frac{(t+1)^2}{2t-1}\lambda_1\int_\Omega u_1^{2t}} = \frac{n^2}{\frac{(t+1)^2}{2t-1}\lambda_1 B(t)^2}. \tag{3.5}$$

From (3.3)–(3.5), (2.35) and (2.33), we have

$$\begin{aligned}
&\frac{n^2(\lambda_{k+1}-\lambda_1)}{n+\sum_{\alpha=1}^N\sum_{j=\alpha+1}^k(\lambda_{k+1}-\lambda_j)A_{\alpha j}^2}\\
&\quad\le \frac{(t+1)^2}{2t-1}\lambda_1 B(t)^2\\
&\quad= \frac{(t+1)^2}{2t-1}\frac{\lambda_1}{1-\frac{(t-1)^2}{2t-1}\lambda_1\sum_{j=2}^\infty\beta_j^2}\\
&\quad\le \frac{(t+1)^2}{2t-1}\frac{\lambda_1}{1-\frac{(t-1)^2}{2t-1}\frac{\lambda_1}{\lambda_{l+1}-\lambda_1}(1+\sum_{j=2}^l(\lambda_{l+1}-\lambda_j)\beta_j^2)}\\
&\quad\le \frac{(t+1)^2}{2t-1}\frac{\lambda_1}{1-\frac{(t-1)^2}{2t-1}\frac{\lambda_1}{\lambda_{l+1}-\lambda_1}\left(1+\sum_{j=2}^l\frac{\lambda_{l+1}-\lambda_j}{\lambda_j-\lambda_1}[1-(\lambda_j-\lambda_1)\frac{A_{\alpha j}^2}{\int_\Omega|\nabla z_\alpha|^2u_1^2}]\right)}.
\end{aligned}$$

Defining

$$\sigma_{\alpha l}=\lambda_1+\frac{\lambda_{l+1}-\lambda_1}{1+\sum_{j=2}^l\frac{\lambda_{l+1}-\lambda_j}{\lambda_j-\lambda_1}\left[1-(\lambda_j-\lambda_1)\frac{A_{\alpha j}^2}{\int_\Omega|\nabla z_\alpha|^2u_1^2}\right]}$$

and taking

$$t=\frac{2\sigma_{\alpha l}}{\sigma_{\alpha l}+\lambda_1},$$

we obtain the following theorem.

Theorem 3.1. *Let M be an n-dimensional complete minimal submanifold in $\mathbf{R}^N$, $\Omega\subset M$ be a bounded domain with a piecewise smooth boundary $\partial\Omega$. Then, for positive integers k, l, eigenvalues of the Dirichlet eigenvalue problem of the Laplacian satisfy, for $1\le\alpha\le N$,*

$$\frac{n^2(\lambda_{k+1}-\lambda_1)}{n+\sum_{\alpha=1}^N\sum_{j=\alpha+1}^k(\lambda_{k+1}-\lambda_j)A_{\alpha j}^2}\le 3\lambda_1+\frac{\lambda_1^2}{\sigma_{\alpha l}}. \tag{3.6}$$

Corollary 3.2. *Let M be an n-dimensional complete minimal submanifold in $\mathbf{R}^N$, $\Omega\subset M$ be a bounded domain with a piecewise smooth boundary $\partial\Omega$. Then, for the Dirichlet eigenvalue problem of the Laplacian, we have*

$$\frac{\lambda_2}{\lambda_1}\le\frac{n+3+\sqrt{n^2+10n+9}}{2n}. \tag{3.7}$$

Proof. Taking $k=l=1$ in (3.6), we have

$$n(\lambda_2-\lambda_1)\le 3\lambda_1+\frac{\lambda_1^2}{\lambda_2}.$$

The above inequality can be written by the following quadratic inequality:

$$n\left(\frac{\lambda_2}{\lambda_1}\right)^2 - (n+3)\frac{\lambda_2}{\lambda_1} - 1 \le 0.$$

Therefore, we can obtain (3.7). □

Remark 3.3. When $n = 2$, the inequality (3.7) becomes the following form:

$$\frac{\lambda_2}{\lambda_1} \le \frac{5+\sqrt{33}}{4}.$$

Thus, the result of Brands [5] for a bounded domain in the Euclidean space is also included here.

For any positive integer k, we can find some α_0 such that

$$\sum_{j=\alpha_0+1}^{k} \frac{(\lambda_{k+1}-\lambda_j)A_{\alpha_0 j}^2}{\int_\Omega |\nabla z_{\alpha_0}|^2 u_1^2} = \max_{1\le\alpha\le N} \sum_{j=\alpha+1}^{k} \frac{(\lambda_{k+1}-\lambda_j)A_{\alpha j}^2}{\int_\Omega |\nabla z_{\alpha}|^2 u_1^2}.$$

Then, from Lemma 2.4, we get

$$\begin{aligned} n &+ \sum_{\alpha=1}^{N}\sum_{j=\alpha+1}^{k}(\lambda_{k+1}-\lambda_j)A_{\alpha j}^2 \\ &\le n + \sum_{j=\alpha_0+1}^{k}(\lambda_{k+1}-\lambda_j)\frac{A_{\alpha_0 j}^2}{\int_\Omega |\nabla z_{\alpha_0}|^2 u_1^2}\sum_{\alpha=1}^{N}\int_\Omega |\nabla z_\alpha|^2 u_1^2 \\ &= n\left(1 + \sum_{j=\alpha_0+1}^{k}(\lambda_{k+1}-\lambda_j)\frac{A_{\alpha_0 j}^2}{\int_\Omega |\nabla z_{\alpha_0}|^2 u_1^2}\right). \end{aligned}$$

Therefore, we have the following corollary.

Corollary 3.4. *Let M be an n-dimensional complete minimal submanifold in $\mathbf{R}^N$, $\Omega \subset M$ be a bounded domain with a piecewise smooth boundary $\partial\Omega$. Then, for any positive integer k, there exists an integer α_0 with $1 \le \alpha_0 \le N$ such that eigenvalues of the Dirichlet eigenvalue problem of the Laplacian satisfy, for any positive integer l,*

$$\frac{n(\lambda_{k+1}-\lambda_1)}{1+\sum_{j=\alpha_0+1}^{k}(\lambda_{k+1}-\lambda_j)\frac{A_{\alpha_0 j}^2}{\int_\Omega |\nabla z_{\alpha_0}|^2 u_1^2}} \le 3\lambda_1 + \frac{\lambda_1^2}{\sigma_{\alpha_0 l}}. \tag{3.8}$$

Since (3.3) holds for any j and any α, from Corollary 3.4, we have the following corollary.

Corollary 3.5. *Let M be an n-dimensional complete minimal submanifold in $\mathbf{R}^N$, $\Omega \subset M$ be a bounded domain with a piecewise smooth boundary $\partial\Omega$. Then,* *for positive integers k, l, eigenvalues of the Dirichlet eigenvalue problem of the Laplacian satisfy, for $1 \leq \alpha \leq N$,*

$$\frac{n(\lambda_{k+1}-\lambda_1)}{1+\sum_{j=2}^{k}\frac{\lambda_{k+1}-\lambda_j}{\lambda_j-\lambda_1}} \leq \frac{n(\lambda_{k+1}-\lambda_1)}{1+\sum_{j=2}^{k}(\lambda_{k+1}-\lambda_j)(\frac{1}{\lambda_j-\lambda_1}-\beta_j^2)} \leq 3\lambda_1 + \frac{\lambda_1^2}{\sigma_{\alpha l}}.$$

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