A MÖBIUS CHARACTERIZATION OF SUBMANIFOLDS

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Abstract. In this paper, we study Möbius characterizations of submanifolds without umbilical points in a unit sphere $S^{n+p}(1)$. First of all, we proved that, for an n-dimensional ($n \geq 2$) submanifold $\mathbf{x} : M \mapsto S^{n+p}(1)$ without umbilical points and with vanishing Möbius form Φ, if $(n-2)\|\tilde{\mathbf{A}}\| \leq \sqrt{\frac{n-1}{n}}\big\{nR - \frac{1}{n}[(n-1)(2-\frac{1}{p})-1]\big\}$ is satisfied, then, $\mathbf{x}$ is Möbius equivalent to an open part of either the Riemannian product $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$ in $S^{n+1}(1)$, or the image of the conformal diffeomorphism σ of the standard cylinder $S^{n-1}(1) \times \mathbf{R}$ in $\mathbf{R}^{n+1}$, or the image of the conformal diffeomorphism τ of the Riemannian product $S^{n-1}(r) \times \mathbf{H}^1(\sqrt{1+r^2})$ in $\mathbf{H}^{n+1}$, or $\mathbf{x}$ is locally Möbius equivalent to the Veronese surface in $S^4(1)$. When $p = 1$, our pinching condition is the same as in Main Theorem of Hu and Li [6], in which they assumed that M is compact and the Möbius scalar curvature $n(n-1)R$ is constant. Secondly, we consider the Möbius sectional curvature of the immersion $\mathbf{x}$. We obtained that, for an n-dimensional compact submanifold $\mathbf{x} : M \mapsto S^{n+p}(1)$ without umbilical points and with vanishing form Φ, if the Möbius scalar curvature $n(n-1)R$ of the immersion $\mathbf{x}$ is constant and the Möbius sectional curvature K of the immersion $\mathbf{x}$ satisfies $K \geq 0$ when $p = 1$ and $K > 0$ when $p > 1$. then, $\mathbf{x}$ is Möbius equivalent to either the Riemannian product $S^k(r) \times S^{n-k}(\sqrt{1-r^2})$, for $k = 1, 2, \cdots, n-1$, in $S^{n+1}(1)$; or $\mathbf{x}$ is Möbius equivalent to a compact minimal submanifold with constant scalar curvature in $S^{n+p}(1)$.

1. Introduction

Let $\mathbf{x} : M \mapsto S^{n+p}(1)$ be an n-dimensional immersed submanifold in an $(n+p)$-dimensional unit sphere $S^{n+p}(1)$. In [11], Wang introduced a Möbius metric, Möbius form and the Möbius second fundamental form of the immersion $\mathbf{x}$. By making use of these Möbius invariants, he founded the fundamental formulas on Möbius geometry of submanifolds in $S^{n+p}(1)$. By following these results of Wang, the Möbius geometry on submanifolds in $S^{n+p}(1)$ was researched by many mathematicians (see. [6], [7], [8] and [9]). In particular, Li, Wang and Wu [8] studied the Möbius characterization of

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Veronese surface. They proved that if $\mathbf{x} : S^2(1) \mapsto S^m(1)$ is an immersion without umbilical points of the 2-sphere with vanishing Möbius form, then there exists a Möbius transformation $\tau : S^m(1) \mapsto S^m(1)$ such that $\tau \circ \mathbf{x} : S^2(1) \mapsto S^{2k}(1)$ is the Veronese surface, where $S^{2k}(1) \subset S^m(1)$ with $2 \le k \le [m/2]$. Furthermore, a kind of pinching problems on Möbius geometry of submanifolds in $S^{n+p}(1)$ was studied by Akivis and Goldberg [2], Hu and Li [6] and so on.

Let $\mathbf{x} : M \mapsto S^{n+p}(1)$ be an n-dimensional immersed submanifold in $S^{n+p}(1)$. We choose a local orthonormal basis $\{e_i\}$ for the induced metric $I = d\mathbf{x} \cdot d\mathbf{x}$ with dual basis $\{\theta_i\}$. Let $II = \sum_{i,j,\alpha} h^\alpha_{ij}\theta_i\theta_j e_\alpha$ be the second fundamental form of the immersion $\mathbf{x}$ and $\vec{H} = \sum_\alpha H^\alpha e_\alpha$ the mean curvature vector of the immersion $\mathbf{x}$, where $\{e_\alpha\}$ is a local orthonormal basis for the normal bundle of $\mathbf{x}$. By putting $\rho^2 = \dfrac{n}{n-1}\{\sum_{\alpha,i,j}(h^\alpha_{ij})^2 - n\|\vec{H}\|^2\}$, the Möbius metric of the immersion $\mathbf{x}$ is defined by $g = \rho^2 d\mathbf{x}\cdot d\mathbf{x}$, which is a Möbius invariant. $\Phi = \Sigma_{i,\alpha} C^\alpha_i \theta_i e_\alpha$ and $\mathbf{A} = \rho^2 \sum_{i,j} A_{ij}\theta_i\theta_j$ are Möbius form and Blaschke tensor of the immersion $\mathbf{x}$, respectively, where C^α_i and A_{ij} are defined by formulas (2.13) and (2.14) in section 2. It was proved that Φ and $\mathbf{A}$ are Möbius invariants (cf. [11]).

In particular, Akivis and Goldberg [1], [2] and Wang [11] proved that two hypersurfaces $\mathbf{x} : M \mapsto S^{n+1}(1)$ and $\tilde{\mathbf{x}} : \tilde{M} \mapsto S^{n+1}(1)$ are Möbius equivalent if and only if there exists a diffeomorphism $\sigma' : M \mapsto \tilde{M}$ which preserves the Möbius metric and the Möbius shape operator such that $\mathbf{x} = \sigma' \circ \tilde{\mathbf{x}}$.

Let $\mathbf{H}^{n+p}$ be an $(n+p)$-dimensional hyperbolic space defined by

$$\mathbf{H}^{n+p} = \{(y_0, y_1) \in \mathbf{R}^+ \times \mathbf{R}^{n+p} | -y_0^2 + y_1 \cdot y_1 = -1\}.$$

We denote the open hemisphere in $S^{n+p}(1)$ whose first coordinate is positive by $S^{n+p}_+(1)$. We consider conformal diffeomorphisms $\sigma_p : \mathbf{R}^{n+p} \mapsto S^{n+p}(1)\backslash\{(-1, 0)\}$ and $\tau_p : \mathbf{H}^{n+p} \mapsto S^{n+p}_+(1)$ defined by :

$$\sigma_p(u) = (\frac{1-|u|^2}{1+|u|^2}, \frac{2u}{1+|u|^2}), \quad u \in \mathbf{R}^{n+p}, \tag{1.1}$$

$$\tau_p(y_0, y_1) = (\frac{1}{y_0}, \frac{y_1}{y_0}), \quad (y_0, y_1) \in \mathbf{H}^{n+p}, \tag{1.2}$$

respectively. The conformal diffeomorphisms σ_p and τ_p assign any submanifold in $\mathbf{R}^{n+p}$ or $\mathbf{H}^{n+p}$ to a submanifold in $S^{n+p}(1)$. If $p = 1$, we denote σ_1 and τ_1 by σ and τ. In [7], Li, Liu, Wang and Zhao classified Möbius isoparametric hypersurfaces with two distinct principal curvatures. They obtained the following:

Theorem 1.1. *Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be a Möbius isoparametric hypersurface with two distinct principal curvatures. Then $\mathbf{x}$ is Möbius equivalent to an open part of one of the following Möbius isoparametric hypersurfaces in $S^{n+1}(1)$:*

(1) *the Riemannian product $S^k(r) \times S^{n-k}(\sqrt{1-r^2})$ in $S^{n+1}(1)$,*
(2) *the image of σ of the standard cylinder $S^k(1) \times \mathbf{R}^{n-k}$ in $\mathbf{R}^{n+1}$,*
(3) *the image of τ of the Riemannian product $S^k(r) \times \mathbf{H}^{n-k}(\sqrt{1+r^2})$ in $\mathbf{H}^{n+1}$.*

A submanifold $\mathbf{x} : M \mapsto S^{n+p}(1)$ is called Möbius isotropic if $\Phi \equiv 0$ and $\mathbf{A} = \lambda d\mathbf{x} \cdot d\mathbf{x}$ for some function λ. In [9], Liu, Wang and Zhao proved the following:

Theorem 1.2. *Any Möbius isotropic submanifolds in $S^{n+p}(1)$ is Möbius equivalent to an open part of one of the following Möbius isotropic submanifolds:*

(1) *a minimal submanifold with constant scalar curvature in $S^{n+p}(1)$,*
(2) *the image of σ_p of a minimal submanifold with constant scalar curvature in $\mathbf{R}^{n+p}$,*
(3) *the image of τ_p of a minimal submanifolds with constant scalar curvature in $\mathbf{H}^{n+p}$.*

On the other hand, Hu and Li [6] studied a pinching problem on the squared norm of the Blaschke tensor of the immersion $\mathbf{x}$ and obtained the following:

Theorem 1.3. *Let $\mathbf{x} : M \to S^{n+p}(1)$ be an n-dimensional $(n \geq 3)$ compact submanifold without umbilical points and with vanishing Möbius form Φ in $S^{n+p}(1)$. If the Möbius scalar curvature $n(n-1)R \geq \frac{(n-1)(n-2)}{n}$ is constant and if*

$$\|\tilde{\mathbf{A}}\| \leq \sqrt{\frac{n-1}{n}}\left(\frac{n}{n-2}R - \frac{1}{n}\right),$$

then, either $\mathbf{x}$ is Möbius equivalent to a minimal submanifold with constant scalar curvature in $S^{n+p}(1)$ or $\mathbf{x}$ is Möbius equivalent to $S^1(r) \times S^{n-1}(\sqrt{\frac{1}{1+c^2} - r^2})$ in $S^{n+1}(1/\sqrt{1+c^2})$ for some constant $c \geq 0, r = \sqrt{\frac{nR}{(n-2)(1+c^2)}}$, where $\tilde{\mathbf{A}} = \rho^2 \sum_{ij} \tilde{A}_{ij}\theta_i\theta_j$ with $\tilde{A}_{ij} = A_{ij} - \frac{1}{n}\sum\limits_k A_{kk}\delta_{ij}$.

Remark 1.4. In the original statement of the theorem 1.3 of Hu and Li [6], they did not write out the condition that M has no umbilical points. But this condition is necessary for their proof. Further, We should note that these assumptions that M is compact and the Möbius scalar curvature $n(n-1)R$ is constant play an important role in the proof of Theorem 1.3 of Hu and Li [6].

In this paper, first of all, we prove the following:

Main Theorem 1. *Let $\mathbf{x} : M \to S^{n+p}(1)$ be an n-dimensional $(n \geq 2)$ submanifold without umbilical points and with vanishing Möbius form Φ, if*

$$(n-2)\|\tilde{\mathbf{A}}\| \leq \sqrt{\frac{n-1}{n}}\left\{nR - \frac{1}{n}\left[(n-1)(2-\frac{1}{p}) - 1\right]\right\}, \tag{1.3}$$

then $\mathbf{x}$ *is locally Möbius equivalent to either the Veronese surface in* $S^4(1)$*, or* $\mathbf{x}$ *is Möbius equivalent to an open part of one of the following Möbius isoparametric hypersurfaces in* $S^{n+1}(1)$*:*

(1) *the Riemannian product* $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$ *in* $S^{n+1}(1)$*,*
(2) *the image of* σ *of the standard cylinder* $S^{n-1}(1) \times \mathbf{R}$ *in* $\mathbf{R}^{n+1}$*,*
(3) *the image of* τ *of the Riemannian product* $S^{n-1}(r) \times \mathbf{H}^1(\sqrt{1+r^2})$ *in* $\mathbf{H}^{n+1}$*,*

where $n(n-1)R$ *denotes the Möbius scalar curvature of the immersion* $\mathbf{x}$ *and* $\tilde{\mathbf{A}} = \rho^2 \sum_{ij} \tilde{A}_{ij}\theta_i\theta_j$ *with* $\tilde{A}_{ij} = A_{ij} - \frac{1}{n}\sum\limits_k A_{kk}\delta_{ij}$*.*

Remark 1.5. In our Main Theorem 1, we do not assume the global condition that M is compact and we do not need to assume that the Möbius scalar curvature is constant. Further, when $p = 1$ and ($n \geq 3$) our pinching condition is the same as in Hu and Li [6]. Since Hu and Li [6] assumed that M is compact, the cases of 2 and 3 above in Main Theorem 1 do not appear in their theorem. If $n = 2$, since the Möbius metric g is flat, we know that $R \equiv 0$. Main Theorem 1 reduces to the Theorem 5.1 in [11].

Since Riemannian product $S^k(r) \times S^{n-k}(\sqrt{1-r^2})$, for $k = 1, 2, \cdots, n-1$, have nonnegative Möbius sectional curvature and they do not satisfy the inequality in Theorem 1.3 of Hu and Li [6] except $k = 1$ or $k = n-1$ (see Proposition 3.2 and Remark 3.3 in section 3), we will consider the immersion $\mathbf{x}$ with nonnegative Möbius sectional curvature and prove the following:

Main Theorem 2. *Let* $\mathbf{x} : M \mapsto S^{n+p}(1)$ *be an* n*-dimensional compact submanifold without umbilical points and with vanishing Möbius form* Φ *and constant Möbius scalar curvature* $n(n-1)R$ *in* $S^{n+p}(1)$*. If the Möbius sectional curvature* K *of* M *satisfies*

$$\begin{cases} K \geq 0, & \text{if } p = 1 \\ K > 0, & \text{if } p > 1, \end{cases}$$

then, $\mathbf{x}$ *is Möbius equivalent to the Riemannian product* $S^k(r) \times S^{n-k}(\sqrt{1-r^2})$*, for* $k = 1, 2, \cdots, n-1$*, in* $S^{n+1}(1)$*; or* $\mathbf{x}$ *is Möbius equivalent to an* n*-dimensional compact minimal submanifold with constant scalar curvature in* $S^{n+p}(1)$*.*

2. Preliminaries and fundamental formulas on Möbius geometry

In this section, we review the definitions of Möbius invariants and give the fundamental formulas on Möbius geometry of submanifolds in $S^{n+p}(1)$, which can be found in [11].

Let $\mathbf{R}_1^{n+p+2}$ be the Lorentzian space with inner product

$$< x, w >= -x_0w_0 + x_1w_1 + \cdots + x_{n+p+1}w_{n+p+1}, \tag{2.1}$$

where $x = (x_0, x_1, \cdots, x_{n+p+1})$ and $w = (w_0, w_1, \cdots, w_{n+p+1})$. Let $\mathbf{x} : M \mapsto S^{n+p}(1)$ be an n-dimensional submanifold of $S^{n+p}(1)$ without umbilical points. Putting

$$Y = \rho(1, \mathbf{x}), \quad \rho^2 = \frac{n}{n-1}(\|II\|^2 - n\|\vec{H}\|^2) > 0, \tag{2.2}$$

then, $Y : M \mapsto \mathbf{R}_1^{n+p+2}$ is called *Möbius position vector* of $\mathbf{x}$. It is easy to prove that

$$g = < dY, dY >= \rho^2 d\mathbf{x} \cdot d\mathbf{x}$$

is a Möbius invariant which is recalled *Möbius metric* of the immersion $\mathbf{x}$. Let Δ denote the Laplacian on M with respect to the Möbius metric g. Defining

$$N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2}(1 + n^2 R)Y, \tag{2.3}$$

we can infer

$$< \Delta Y, Y >= -n, \quad < \Delta Y, dY >= 0, \quad < \Delta Y, \Delta Y >= 1 + n^2 R, \tag{2.4}$$

$$< Y, Y >= 0, \quad < N, Y >= 1, \quad < N, N >= 0, \tag{2.5}$$

where $n(n-1)R$ denotes the Möbius scalar curvature of the immersion $\mathbf{x}$. Let $\{E_1, \cdots, E_n\}$ denote a local orthonormal frame on (M, g) with dual frame $\{\omega_1, \cdots, \omega_n\}$. Putting $Y_i = E_i(Y)$, then we have, from (2.2), (2.4) and (2.5),

$$< Y_i, Y >=< Y_i, N >= 0, \quad < Y_i, Y_j >= \delta_{ij}, \quad 1 \le i, j \le n. \tag{2.6}$$

Let V be the orthogonal complement to the subspace $\mathrm{Span}\{Y, N, Y_1, \cdots, Y_n\}$ in $\mathbf{R}_1^{n+p+2}$. Along M, we have the following orthogonal decomposition:

$$\mathbf{R}_1^{n+p+2} = \mathrm{Span}\{Y, N\} \oplus \mathrm{Span}\{Y_1, \cdots, Y_n\} \oplus V, \tag{2.7}$$

where V is called *Möbius normal bundle* of the immersion $\mathbf{x}$. It is not difficult to prove that

$$E_\alpha = (H^\alpha, H^\alpha \mathbf{x} + e_\alpha), \quad n+1 \le \alpha \le \ n+p, \tag{2.8}$$

is a local orthonormal frame of V. Then $\{Y, N, Y_1, \cdots, Y_n, E_{n+1}, \cdots, E_{n+p}\}$ forms a moving frame in $\mathbf{R}_1^{n+p+2}$ along M. We use the following range of indices throughout this paper:

$$1 \le i, j, k, l, m \le n, \quad n+1 \le \alpha, \beta \le n+p.$$

The structure equations on M with respect to the Möbius metric g can be written as follows:

$$dY = \sum_i Y_i \omega_i, \tag{2.9}$$

$$dN = \sum_{i,j} A_{ij}\omega_j Y_i + \sum_{i,\alpha} C_i^\alpha \omega_i E_\alpha, \tag{2.10}$$

$$dY_i = -\sum_j A_{ij}\omega_j Y - \omega_i N + \sum_j \omega_{ij} Y_j + \sum_{j,\alpha} B_{ij}^\alpha \omega_j E_\alpha, \tag{2.11}$$

$$dE_\alpha = -\sum_i C_i^\alpha \omega_i Y - \sum_{i,j} B_{ij}^\alpha \omega_j Y_i + \sum_\beta \omega_{\alpha\beta} E_\beta, \tag{2.12}$$

where ω_{ij} is the connection form with respect to the Möbius metric g, $\omega_{\alpha\beta}$ is the normal connection form of $\mathbf{x} : M \to S^{n+p}(1)$, which is a Möbius invariant. $\mathbf{A} = \sum_{i,j} A_{ij}\omega_i \otimes \omega_j$ and $\Phi = \sum_{i,\alpha} C_i^\alpha \omega_i(\rho^{-1}e_\alpha)$ are called *Blaschke tensor* and *Möbius form* of the immersion $\mathbf{x}$, respectively, where

$$C_i^\alpha = -\rho^{-2}\{H_{,i}^\alpha + \sum_j (h_{ij}^\alpha - H^\alpha \delta_{ij}) e_j(\log\rho)\}, \tag{2.13}$$

$$\begin{aligned} A_{ij} = &-\rho^{-2}\{\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - \sum_\alpha H^\alpha h_{ij}^\alpha\} \\ &- \frac{1}{2}\rho^{-2}\big(\|\nabla(\log\rho)\|^2 - 1 + \|\vec{H}\|^2\big)\delta_{ij}. \end{aligned} \tag{2.14}$$

Here Hess_{ij} and ∇ are the Hessian matrix and the gradient with respect to the induced metric $d\mathbf{x}\cdot d\mathbf{x}$. It was proved that $\Phi = \Sigma_{i,\alpha} C_i^\alpha \theta_i e_\alpha$ and $\mathbf{A} = \rho^2 \sum_{i,j} A_{ij}\theta_i\theta_j$ are Möbius invariants. $\mathbf{B} = \sum_{i,j,\alpha} B_{ij}^\alpha \omega_i\omega_j(\rho^{-1}e_\alpha)$ is called *Möbius second fundamental form* of the immersion $\mathbf{x}$, where

$$B_{ij}^\alpha = \rho^{-1}(h_{ij}^\alpha - H^\alpha\delta_{ij}). \tag{2.15}$$

Hence, we have

$$\sum_i B_{ii}^\alpha = 0, \quad \sum_{i,j,\alpha} (B_{ij}^\alpha)^2 = \frac{n-1}{n}. \tag{2.16}$$

We define the covariant derivative of $C_i^\alpha, A_{ij}, B_{ij}^\alpha$ by

$$\sum_j C_{i,j}^\alpha \omega_j = dC_i^\alpha + \sum_j C_j^\alpha \omega_{ji} + \sum_\beta C_i^\beta \omega_{\beta\alpha},$$

$$\sum_k A_{ij,k}\omega_k = dA_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki}, \tag{2.17}$$

$$\sum_k B_{ij,k}^\alpha \omega_k = dB_{ij}^\alpha + \sum_k B_{ik}^\alpha \omega_{kj} + \sum_k B_{kj}^\alpha \omega_{ki} + \sum_\beta B_{ij}^\beta \omega_{\beta\alpha}. \tag{2.18}$$

From the structure equations (2.9), (2.10), (2.11) and (2.12), we can infer

$$A_{ij,k} - A_{ik,j} = \sum_\alpha (B_{ik}^\alpha C_j^\alpha - B_{ij}^\alpha C_k^\alpha), \tag{2.19}$$

$$C^\alpha_{i,j} - C^\alpha_{j,i} = \sum_k (B^\alpha_{ik}A_{kj} - B^\alpha_{kj}A_{ki}), \tag{2.20}$$

$$B^\alpha_{ij,k} - B^\alpha_{ik,j} = \delta_{ij}C^\alpha_k - \delta_{ik}C^\alpha_j, \tag{2.21}$$

$$R_{ijkl} = \sum_\alpha (B^\alpha_{ik}B^\alpha_{jl} - B^\alpha_{il}B^\alpha_{jk}) + (\delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il}), \tag{2.22}$$

$$R_{\alpha\beta ij} = \sum_k (B^\alpha_{ik}B^\beta_{kj} - B^\beta_{ik}B^\alpha_{kj}), \tag{2.23}$$

where R_{ijkl} and $R_{\alpha\beta ij}$ denote the curvature tensor with respect to the Möbius metric g on M and the normal curvature tensor of the normal connection. $n(n-1)R = \sum_{i,j} R_{ijij}$ is the Möbius scalar curvature of the immersion $\mathbf{x} : M \to S^{n+p}(1)$. From (2.3) and the structure equation (2.11), we have, (cf. [11]),

$$\mathrm{tr}\mathbf{A} = \frac{1}{2n}(1 + n^2R). \tag{2.24}$$

By taking exterior differentiation of (2.17) and (2.18), and defining

$$\sum_l A_{ij,kl}\omega_l = dA_{ij,k} + \sum_l A_{lj,k}\omega_{li} + \sum_l A_{il,k}\omega_{lj} + \sum_l A_{ij,l}\omega_{lk},$$

$$\sum_l B^\alpha_{ij,kl}\omega_l = dB^\alpha_{ij,k} + \sum_l B^\alpha_{lj,k}\omega_{li} + \sum_l B^\alpha_{il,k}\omega_{lj} + \sum_l B^\alpha_{ij,l}\omega_{lk} + \sum_\beta B^\beta_{ij,k}\omega_{\beta\alpha},$$

we have the following Ricci identities

$$A_{ij,kl} - A_{ij,lk} = \sum_m A_{mj}R_{mikl} + \sum_m A_{im}R_{mjkl}, \tag{2.25}$$

$$B^\alpha_{ij,kl} - B^\alpha_{ij,lk} = \sum_m B^\alpha_{mj}R_{mikl} + \sum_m B^\alpha_{im}R_{mjkl} + \sum_\beta B^\beta_{ij}R_{\beta\alpha kl}. \tag{2.26}$$

For a matrix $A = (a_{ij})$ we denote by $N(A)$ the square of the norm of A, i.e.,

$$N(A) = \mathrm{tr}(AA^t) = \sum_{i,j}(a_{ij})^2,$$

where A^t denotes the transposed matrix of A. It is obvious that $N(A) = N(T^tAT)$ holds for any orthogonal matrix T.

The following algebraic lemmas will be used in order to prove our Main Theorems.

Lemma 2.1. ([5]). *Let A and B be symmetric $(n \times n)$-matrices. Then*

$$N(AB - BA) \le 2N(A) \cdot N(B) \tag{2.27}$$

and the equality holds for nonzero matrices A and B if and only if A and B can be transformed simultaneously by an orthogonal matrix into multiples of $\tilde{A}$ and $\tilde{B}$, respectively, where

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad \tilde{B} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Moreover, if A_1, A_2 and A_3 are $(n \times n)$-symmetric matrices and satisfy

$$N(A_\alpha A_\beta - A_\beta A_\alpha) = 2N(A_\alpha) \cdot N(A_\beta), \qquad 1 \leq \alpha, \beta \leq 3,$$

then at least one of the matrices A_α must be zero.

Lemma 2.2. (Cheng [4] and Santos [10]). *Let A and B be $n \times n$-symmetric matrices satisfying $\mathrm{tr}A = 0, \mathrm{tr}B = 0$ and $AB - BA = 0$. Then,*

$$\mathrm{tr}(B^2 A) \geq -\frac{n-2}{\sqrt{n(n-1)}}(\mathrm{tr}B^2)(\mathrm{tr}A^2)^{1/2}, \tag{2.28}$$

and the equality holds if and only if $(n-1)$ of the eigenvalues x_i of B and the corresponding eigenvalues y_i of A satisfy $|x_i| = \dfrac{(\mathrm{tr}B^2)^{1/2}}{\sqrt{n(n-1)}}$, $x_i x_j \geq 0$, $y_i = \dfrac{(\mathrm{tr}A^2)^{1/2}}{\sqrt{n(n-1)}}$.

3. Möbius invariants on typical examples

In this section, we shall study Möbius invariants on typical examples. These results in this section will be used in the proof of Main Theorem 1 and the results in the following proposition 3.2 will support our assumption in Main Theorem 2. Throughout this section, we shall make the following convention on the ranges of indices:

$$1 \leq i, j \leq n, \quad 1 \leq a, b \leq k, \quad k+1 \leq s, t \leq n.$$

The following Lemma 3.1 due to Li, Liu, Wang and Zhao [7] will be used

Lemma 3.1. *Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an n-dimensional hypersurface with two distinct principal curvatures with multiplicities k and $n-k$, respectively. Then the principal curvatures of the Möbius second fundamental form $\mathbf{B}$ of $\mathbf{x}$ are constant, which are given by*

$$\mu_1 = \frac{1}{n}\sqrt{\frac{(n-1)(n-k)}{k}}, \quad \mu_2 = -\frac{1}{n}\sqrt{\frac{(n-1)k}{(n-k)}}.$$

Proposition 3.2. *Let* $\mathbf{x}_1 : S^k(1) \mapsto \mathbf{R}^{k+1}$ *and* $\mathbf{x}_2 : S^{n-k}(1) \mapsto \mathbf{R}^{n-k+1}$ *be the standard embeddings of the unit spheres. Then, for Riemannian product* $\mathbf{x} : S^k(r) \times S^{n-k}(\sqrt{1-r^2}) \mapsto S^{n+1}(1)$ *defined by* $\mathbf{x} = (r\mathbf{x}_1, \sqrt{1-r^2}\mathbf{x}_2)$*, for any* $1 \leq k \leq n-1$ *and any* $0 < r < 1$*, we have*

$$\Phi = 0, \tag{3.1}$$

$$R = \frac{k-1}{n(n-k)} + \frac{(n-1)(n-2k)}{nk(n-k)}r^2, \tag{3.2}$$

$$(n-2k)^2\|\tilde{\mathbf{A}}\|^2 = \frac{k(n-k)}{n}(nR - \frac{n-2}{n})^2, \tag{3.3}$$

$$R_{abab} = \frac{n-1}{k(n-k)}(1-r^2), \quad R_{asas} = 0, \quad R_{stst} = \frac{n-1}{k(n-k)}r^2, \tag{3.4}$$

where R_{ijij} *denotes the Möbius sectional curvature of the plane section spanned by* $\{E_i, E_j\}$*.*

Proof. Since Riemannian product $\mathbf{x} : S^k(r) \times S^{n-k}(\sqrt{1-r^2}) \mapsto S^{n+1}(1)$ is the standard embedding, we know that the second fundamental form of $\mathbf{x}$ has two distinct principal curvatures $\frac{\sqrt{1-r^2}}{r}$ and $-\frac{r}{\sqrt{1-r^2}}$ with multiplicities k and $n-k$, respectively. Putting $c = \frac{\sqrt{1-r^2}}{r}$, we have

$$h_{ab} = c\delta_{ab}, \quad h_{as} = 0, \quad h_{st} = -\frac{1}{c}\delta_{st}, \tag{3.5}$$

$$H = \frac{1}{n}\sum_{i=1}^{n} h_{ii} = \frac{1}{n}\{kc - (n-k)\frac{1}{c}\}, \tag{3.6}$$

$$\|II\|^2 = kc^2 + (n-k)\frac{1}{c^2}, \tag{3.7}$$

$$\rho^2 = \frac{n}{n-1}(\|II\|^2 - nH^2) = \frac{k(n-k)}{n-1}\frac{(c^2+1)^2}{c^2}. \tag{3.8}$$

Hence, the Möbius metric g of the $\mathbf{x}$ is given by

$$g = \rho^2 d\mathbf{x} \cdot d\mathbf{x}.$$

Since ρ^2 is constant, from (2.13) and (2.14) , we have $C_i = 0$ and $A_{ij} = -\frac{1}{2}\rho^{-2}\{(H^2 - 1)\delta_{ij} - 2Hh_{ij}\}$, where C_i and A_{ij} denote components of Möbius form Φ and components of the Blaschke tensor $\mathbf{A}$. Hence, we infer $\Phi = 0$ and

$$A_{ab} = \frac{n-1}{2k(n-k)n^2}\{k(2n-k) - n^2r^2\}\delta_{ab}, \tag{3.9}$$

$$A_{as} = 0, \tag{3.10}$$

$$A_{st} = \frac{n-1}{2k(n-k)n^2}\{n^2r^2 - k^2\}\delta_{st}. \tag{3.11}$$

Thus, we have

$$\mathrm{tr}\mathbf{A} = \frac{n-1}{2k(n-k)n}\{k^2 + n(n-2k)r^2\}. \tag{3.12}$$

From (2.24), we obtain

$$R = \frac{k-1}{n(n-k)} + \frac{(n-1)(n-2k)}{k(n-k)n}r^2. \tag{3.13}$$

According to

$$\tilde{A}_{ij} = A_{ij} - \frac{1}{n}(\mathrm{tr}\mathbf{A})\delta_{ij},$$

we have

$$\tilde{A}_{ab} = \frac{n-1}{kn^2}\{k - nr^2\}\delta_{ab}, \tag{3.14}$$

$$\tilde{A}_{as} = 0, \tag{3.15}$$

$$\tilde{A}_{st} = \frac{n-1}{(n-k)n^2}\{nr^2 - k\}\delta_{st}. \tag{3.16}$$

Therefore, we infer

$$\|\tilde{\mathbf{A}}\|^2 = \frac{(n-1)^2}{k(n-k)n}(r^2 - \frac{k}{n})^2. \tag{3.17}$$

From (3.13) and (3.17), we obtain

$$(n-2k)^2\|\tilde{\mathbf{A}}\|^2 = \frac{k(n-k)}{n}(nR - \frac{n-2}{n})^2. \tag{3.18}$$

From Lemma 3.1, (2.22), (3.9), (3.10) and (3.11), we have

$$R_{abab} = B_{aa}B_{bb} + A_{aa} + A_{bb} = \frac{n-1}{k(n-k)}(1 - r^2), \tag{3.19}$$

$$R_{asas} = B_{aa}B_{ss} + A_{aa} + A_{ss} = 0, \tag{3.20}$$

$$R_{stst} = B_{ss}B_{tt} + A_{ss} + A_{tt} = \frac{n-1}{k(n-k)}r^2. \tag{3.21}$$

This completes the proof of Proposition 3.2. □

Remark 3.3. From (3.3), we know that $(n-2)\|\tilde{\mathbf{A}}\| = \sqrt{\frac{n-1}{n}}(nR - \frac{n-2}{n})$ if and only if $k = 1$ or $k = n-1$.

Proposition 3.4. *Let $\hat{\mathbf{x}} : S^k(1) \times \mathbf{R}^{n-k} \mapsto \mathbf{R}^{n+1}$ be the standard cylinder. Then, the hypersurface $\mathbf{x} = \sigma \circ \hat{\mathbf{x}} : S^k(1) \times \mathbf{R}^{n-k} \mapsto S^{n+1}(1)$ satisfies*

$$\Phi = 0, \tag{3.22}$$

$$R = \frac{k-1}{n(n-k)}, \tag{3.23}$$

$$\|\tilde{A}\|^2 = \frac{k(n-1)^2}{n^3(n-k)}, \tag{3.24}$$

$$R_{abab} = \frac{n-1}{k(n-k)}, \quad R_{asas} = 0, \quad R_{stst} = 0, \tag{3.25}$$

where σ is the conformal diffeomorphism defined by (1.1) with $p = 1$.

Proof. Since $\hat{\mathbf{x}} : S^k(1) \times \mathbf{R}^{n-k} \mapsto \mathbf{R}^{n+1}$ is the standard cylinder, we know that the second fundamental form of $\hat{\mathbf{x}}$ has two distinct principal curvatures 1 and 0 with multiplicities k and $n-k$, respectively. Let $\hat{h}_{ij}$ and $\hat{H}$ denote components of the second fundamental form $\hat{II}$ and the mean curvature of $\hat{\mathbf{x}}$, respectively. Then, we have

$$\hat{h}_{ab} = \delta_{ab}, \quad \hat{h}_{as} = 0, \quad \hat{h}_{st} = 0, \tag{3.26}$$

$$\hat{H} = \frac{k}{n}, \quad \|\hat{II}\|^2 = k. \tag{3.27}$$

By defining

$$\hat{\rho}^2 = \frac{n}{n-1}(\|\hat{II}\|^2 - n\hat{H}^2) = \frac{k(n-k)}{n-1},$$

then, the Möbius metric $\hat{g}$ of the $\hat{\mathbf{x}}$ is given by

$$\hat{g} = \hat{\rho}^2 d\hat{\mathbf{x}} \cdot d\hat{\mathbf{x}}.$$

Let $\{\hat{e}_i\}$ be an orthonormal basis for the first fundamental form $\hat{I} = d\hat{\mathbf{x}} \cdot d\hat{\mathbf{x}}$ with the dual basis $\{\hat{\theta}_i\}$. Define

$$\hat{C}_i = -\hat{\rho}^{-2}\big\{\hat{H}_{,i} + \sum_j (\hat{h}_{ij} - \hat{H}\delta_{ij})\hat{e}_j(\log\hat{\rho})\big\}, \tag{3.28}$$

$$\begin{aligned}\hat{A}_{ij} = &-\hat{\rho}^{-2}\big\{\mathrm{Hess}_{ij}(\log\hat{\rho}) - \hat{e}_i(\log\hat{\rho})\hat{e}_j(\log\hat{\rho}) - \hat{H}\hat{h}_{ij}\big\}\\ &- \frac{1}{2}\hat{\rho}^{-2}\big(\|\nabla(\log\hat{\rho})\|^2 + \hat{H}^2\big)\delta_{ij},\end{aligned} \tag{3.29}$$

$$\hat{B}_{ij} = \hat{\rho}^{-1}(\hat{h}_{ij} - \hat{H}\delta_{ij}). \tag{3.30}$$

Here Hess_{ij} and ∇ are the Hessian matrix and the gradient with respect to the induced metric $\hat{I} = d\hat{\mathbf{x}} \cdot d\hat{\mathbf{x}}$. $\hat{\Phi} = \sum_i \hat{C}_i\hat{\theta}_i\hat{e}_{n+1}$, $\hat{\mathbf{A}} = \hat{\rho}^2 \sum_{i,j} \hat{A}_{ij}\hat{\theta}_i\hat{\theta}_j$ and $\hat{\mathbf{B}} = \sum_{i,j} \hat{B}_{ij}\hat{\theta}_i\hat{\theta}_j(\hat{\rho}^{-1}\hat{e}_{n+1})$ is called *Möbius form*, *Blaschke tensor* and *Möbius second fundamental form* of the immersion $\hat{\mathbf{x}}$, respectively (cf. [9]).

Since $\hat{\rho}^2$ is constant, from (3.28) and (3.29), we have $\hat{C}_i = 0$ and $\hat{A}_{ij} = -\frac{1}{2}\hat{\rho}^{-2}\{\hat{H}^2\delta_{ij} - 2\hat{H}\hat{h}_{ij}\}$. Hence, we infer $\hat{\Phi} = 0$ and

$$\begin{aligned}
\hat{A}_{ab} &= -\frac{(n-1)(k-2n)}{2(n-k)n^2}\delta_{ab},\\
\hat{A}_{as} &= 0,\\
\hat{A}_{st} &= -\frac{(n-1)k}{2(n-k)n^2}\delta_{st}.
\end{aligned}$$

Thus, from Theorem 4.1 of Liu, Wang and Zhao [9], we know $\Phi = \hat{\Phi} = 0$ and

$$A_{ab} = \hat{A}_{ab} = -\frac{(n-1)(k-2n)}{2(n-k)n^2}\delta_{ab}, \tag{3.31}$$

$$A_{as} = \hat{A}_{as} = 0, \tag{3.32}$$

$$A_{st} = \hat{A}_{st} = -\frac{(n-1)k}{2(n-k)n^2}\delta_{st}. \tag{3.33}$$

Thus, we infer

$$\mathrm{tr}\mathbf{A} = \frac{(n-1)k}{2(n-k)n}. \tag{3.34}$$

From (2.24), we obtain

$$R = \frac{k-1}{n(n-k)}. \tag{3.35}$$

From

$$\tilde{A}_{ij} = A_{ij} - \frac{1}{n}\mathrm{tr}\mathbf{A}\delta_{ij},$$

we have

$$\begin{aligned}
\tilde{A}_{ab} &= \frac{n-1}{n^2}\delta_{ab},\\
\tilde{A}_{as} &= 0,\\
\tilde{A}_{st} &= -\frac{(n-1)k}{(n-k)n^2}\delta_{st}.
\end{aligned}$$

Therefore, we infer

$$\|\tilde{\mathbf{A}}\|^2 = \frac{(n-1)^2k}{(n-k)n^3}. \tag{3.36}$$

From (3.35) and (3.36), we obtain

$$(n-2k)^2\|\tilde{\mathbf{A}}\|^2 = \frac{k(n-k)}{n}(nR - \frac{n-2}{n})^2. \tag{3.37}$$

From Lemma 3.1, (2.22), (3.31), (3.32) and (3.33), we have

$$
\begin{aligned}
R_{abab} &= B_{aa}B_{bb} + A_{aa} + A_{bb} = \frac{n-1}{k(n-k)},\\
R_{asas} &= B_{aa}B_{ss} + A_{aa} + A_{ss} = 0,\\
R_{stst} &= B_{ss}B_{tt} + A_{ss} + A_{tt} = 0.
\end{aligned}
$$

This completes the proof of Proposition 3.4. □

Remark 3.5. From (3.23) and (3.24), we know that $(n-2)\|\tilde{\mathbf{A}}\| = \sqrt{\frac{n-1}{n}}(nR - \frac{n-2}{n})$ if and only if $k = n-1$.

Proposition 3.6. *Let $\bar{\mathbf{x}} : S^k(r) \times \mathbf{H}^{n-k}(\sqrt{1+r^2}) \mapsto \mathbf{H}^{n+1}$ be the standard embedding. Then, the hypersurface $\mathbf{x} = \tau \circ \bar{\mathbf{x}} : S^k(r) \times \mathbf{H}^{n-k}(\sqrt{1+r^2}) \mapsto S^{n+1}(1)$ satisfies*

$$\Phi = 0, \tag{3.38}$$

$$R = \frac{k-1}{n(n-k)} - \frac{(n-1)(n-2k)}{nk(n-k)}r^2, \tag{3.39}$$

$$(2k-n)\|\tilde{\mathbf{A}}\| = \sqrt{\frac{k(n-k)}{n}}(nR - \frac{n-2}{n}), \tag{3.40}$$

$$R_{abab} = \frac{n-1}{k(n-k)}(1+r^2), \quad R_{asas} = 0, \quad R_{stst} = -\frac{n-1}{k(n-k)}r^2, \tag{3.41}$$

where τ is the conformal diffeomorphism defined by (1.2) with $p = 1$.

Proof. Since $\bar{\mathbf{x}} : S^k(1) \times \mathbf{H}^{n-k}(\sqrt{1+r^2}) \mapsto \mathbf{H}^{n+1}$ is the standard embedding, we know that the second fundamental form of $\bar{\mathbf{x}}$ has two distinct principal curvatures $\frac{\sqrt{1+r^2}}{r} = d$ and $\frac{r}{\sqrt{1+r^2}}$ with multiplicities k and $n-k$, respectively. Let $\bar{h}_{ij}$ and $\bar{H}$ denote the components of the second fundamental form $\bar{II}$ and the mean curvature of $\bar{\mathbf{x}}$, respectively. Then, we have

$$
\begin{aligned}
&\bar{h}_{ab} = d\delta_{ab}, \quad \bar{h}_{as} = 0, \quad \bar{h}_{st} = \frac{1}{d}\delta_{st},\\
&\bar{H} = \frac{1}{n}\{kd + (n-k)d\}, \quad \|\bar{II}\|^2 = kd^2 + (n-k)\frac{1}{d^2}.
\end{aligned}
$$

By defining

$$\bar{\rho}^2 = \frac{n}{n-1}(\|\bar{II}\|^2 - n\bar{H}^2) = \frac{k(n-k)}{n-1}\frac{(d^2-1)^2}{d^2},$$

then, the Möbius metric $\bar{g}$ of the $\bar{\mathbf{x}}$ is given by

$$\bar{g} = \bar{\rho}^2 d\bar{\mathbf{x}} \cdot d\bar{\mathbf{x}}.$$

Let $\{\bar{e}_i\}$ be an orthonormal basis for the first fundamental form $\bar{I} = d\bar{\mathbf{x}} \cdot d\bar{\mathbf{x}}$ with the dual basis $\{\bar{\theta}_i\}$. Define

$$\bar{C}_i = -(\bar{\rho})^{-2}\big\{\bar{H}_{,i} + \sum_j (\bar{h}_{ij} - \bar{H}\delta_{ij})\bar{e}_j(\log\bar{\rho})\big\}, \tag{3.42}$$

$$\begin{aligned}\bar{A}_{ij} &= -(\bar{\rho})^{-2}\big\{\mathrm{Hess}_{ij}(\log\bar{\rho}) - \bar{e}_i(\log\bar{\rho})\bar{e}_j(\log\bar{\rho}) - \bar{H}\bar{h}_{ij}\big\}\\ &\quad - \frac{1}{2}(\bar{\rho})^{-2}\big(\|\nabla(\log\bar{\rho})\|^2 + 1 + \bar{H}^2\big)\delta_{ij},\end{aligned} \tag{3.43}$$

$$\bar{B}_{ij} = (\bar{\rho})^{-1}(\bar{h}_{ij} - \bar{H}\delta_{ij}). \tag{3.44}$$

Here Hess_{ij} and ∇ are the Hessian matrix and the gradient with respect to the induced metric $\bar{I} = d\bar{\mathbf{x}} \cdot d\bar{\mathbf{x}}$. $\bar{\Phi} = \Sigma_i \bar{C}_i\bar{\theta}_i\bar{e}_{n+1}$, $\bar{\mathbf{A}} = \bar{\rho}^2\sum_{ij}\bar{A}_{ij}\bar{\theta}_i\bar{\theta}_j$ and $\bar{\mathbf{B}} = \sum_{i,j}\bar{B}_{ij}\bar{\theta}_i\bar{\theta}_j((\bar{\rho})^{-1}\bar{e}_{n+1})$ is called *Möbius form*, *Blaschke tensor* and *Möbius second fundamental form* of the immersion $\bar{\mathbf{x}}$, respectively (cf. [9]).

Since $\bar{\rho}^2$ is constant, from (3.42) and (3.43), we have $\bar{C}_i = 0$ and $\bar{A}_{ij} = -\frac{1}{2}(\bar{\rho})^{-2}\{(1+\bar{H}^2)\delta_{ij} - 2\bar{H}\bar{h}_{ij}\}$. Hence, we infer $\bar{\Phi} = 0$ and

$$\begin{aligned}&\bar{A}_{ab} = \frac{n-1}{2k(n-k)n^2}\{k(2n-k) + n^2r^2\}\delta_{ab},\\ &\bar{A}_{as} = 0,\\ &\bar{A}_{st} = -\frac{n-1}{2k(n-k)n^2}\{k^2 + n^2r^2\}\delta_{st}.\end{aligned}$$

Thus, from Theorem 4.4 of Liu, Wang and Zhao [9], we know $\Phi = \bar{\Phi} = 0$ and

$$A_{ab} = \bar{A}_{ab} = \frac{n-1}{2k(n-k)n^2}\{k(2n-k) + n^2r^2\}\delta_{ab}, \tag{3.45}$$

$$A_{as} = \bar{A}_{as} = 0, \tag{3.46}$$

$$A_{st} = \bar{A}_{st} = -\frac{n-1}{2k(n-k)n^2}\{k^2 + n^2r^2\}\delta_{st}. \tag{3.47}$$

Thus, we infer

$$\mathrm{tr}\mathbf{A} = \frac{n-1}{2k(n-k)n}\{k^2 - n(n-2k)r^2\}. \tag{3.48}$$

From (2.24), we obtain

$$R = \frac{k-1}{n(n-k)} - \frac{(n-1)(n-2k)}{nk(n-k)}r^2. \tag{3.49}$$

From

$$\tilde{A}_{ij} = A_{ij} - \frac{1}{n}\mathrm{tr}\mathbf{A}\delta_{ij},$$

we have

$$\begin{aligned}
\tilde{A}_{ab} &= \frac{n-1}{kn^2}(k+nr^2)\delta_{ab},\\
\tilde{A}_{as} &= 0,\\
\tilde{A}_{st} &= -\frac{n-1}{(n-k)n^2}(k+nr^2)\delta_{st}.
\end{aligned}$$

Therefore, we infer

$$\|\tilde{\mathbf{A}}\|^2 = \frac{(n-1)^2}{k(n-k)n}(r^2+\frac{k}{n})^2.$$

From (3.49) and (3.50), we obtain

$$(2k-n)\|\tilde{\mathbf{A}}\| = \sqrt{\frac{k(n-k)}{n}}(nR-\frac{n-2}{n}).$$

From Lemma 3.1, (2.22), (3.45), (3.46) and (3.47), we have

$$\begin{aligned}
R_{abab} &= B_{aa}B_{bb}+A_{aa}+A_{bb} = \frac{n-1}{k(n-k)}(1+r^2),\\
R_{asas} &= B_{aa}B_{ss}+A_{aa}+A_{ss} = 0,\\
R_{stst} &= B_{ss}B_{tt}+A_{ss}+A_{tt} = -\frac{n-1}{k(n-k)}r^2.
\end{aligned}$$

This completes the proof of Proposition 3.6. □

Remark 3.7. From (3.40), we know that $(n-2)\|\tilde{\mathbf{A}}\| = \sqrt{\frac{n-1}{n}}(nR-\frac{n-2}{n})$ if and only if $k=n-1$.

4. Proofs of Main Theorems

In this section, we will prove our Main Theorems.

Proof of Main Theorem 1. Since the Möbius form $\Phi = \sum_{i,\alpha} C_i^\alpha e_\alpha \equiv 0$, we have, by (2.19), (2.20) and (2.21), that

$$A_{ij,k} = A_{ik,j}, \quad B^\alpha_{ij,k} = B^\alpha_{ik,j}, \quad \sum_k B^\alpha_{ik}A_{kj} = \sum_k B^\alpha_{kj}A_{ki}, \text{ for any } \alpha. \tag{4.1}$$

From the definition $\Delta B^\alpha_{ij} = \sum_k B^\alpha_{ij,kk}$ of the Laplacian of the Möbius second fundamental form of the immersion $\mathbf{x}$, we have

$$\frac{1}{2}\Delta\sum_{i,j,\alpha}(B^\alpha_{ij})^2 = \sum_{i,j,k,\alpha}(B^\alpha_{ij,k})^2 + \sum_{i,j,\alpha}B^\alpha_{ij}\Delta B^\alpha_{ij}. \tag{4.2}$$

From (2.16), we have

$$\sum_{i,j,k,\alpha}(B^\alpha_{ij,k})^2+\sum_{i,j,\alpha}B^\alpha_{ij}\Delta B^\alpha_{ij}=0. \tag{4.3}$$

From (2.16), (2.22), (2.23), (2.26) and (4.1), we have, by a direct calculation, that

$$\begin{aligned}\sum_{i,j,\alpha}B^\alpha_{ij}\Delta B^\alpha_{ij}=&-2\sum_{\alpha,\beta}\big[\mathrm{tr}(B_\alpha^2B_\beta^2)-\mathrm{tr}\{(B_\alpha B_\beta)^2\}\big]\\&-\sum_{\alpha,\beta}\big\{\mathrm{tr}(B_\alpha B_\beta)\big\}^2+n\sum_\alpha\mathrm{tr}(AB_\alpha^2)+\frac{n-1}{n}\mathrm{tr}A,\end{aligned} \tag{4.4}$$

where B_α and A denote the $n\times n$-symmetric matrices (B^α_{ij}) and (A_{ij}) respectively. Putting $\tilde A=(\tilde A_{ij})$ with

$$\tilde A_{ij}=A_{ij}-\frac{1}{n}(\mathrm{tr}A)\delta_{ij}, \tag{4.5}$$

we have

$$\|A\|^2=\sum_{i,j}(A_{ij})^2=\sum_{i,j}(\tilde A_{ij})^2+\frac{1}{n}(\mathrm{tr}A)^2=\|\tilde A\|^2+\frac{1}{n}(\mathrm{tr}A)^2, \tag{4.6}$$

From (4.5), we have

$$\mathrm{tr}\tilde A=0,\quad \mathrm{tr}(\tilde AB_\alpha^2)=\mathrm{tr}(AB_\alpha^2)-\frac{1}{n}(\mathrm{tr}A)(\mathrm{tr}B_\alpha^2). \tag{4.7}$$

From (4.1), we know that $B_\alpha A=AB_\alpha$. Therefore $B_\alpha\tilde A=\tilde AB_\alpha$ holds. From Lemma 2.2, we have

$$\mathrm{tr}(AB_\alpha^2)\geq-\frac{n-2}{\sqrt{n(n-1)}}\mathrm{tr}B_\alpha^2\|\tilde A\|+\frac{1}{n}(\mathrm{tr}A)(\mathrm{tr}B_\alpha^2). \tag{4.8}$$

Case (i) where $p=1$. Put $B^{n+1}_{ij}=B_{ij}$ and $B_{n+1}=B$. Since $BA=AB$ holds from (4.1), we can choose a local orthonormal basis $\{E_1,E_2,\cdots,E_n\}$ such that $B_{ij}=\mu_i\delta_{ij}$ and $A_{ij}=\lambda_i\delta_{ij}$. Thus, we have from (4.4), (2.16) and (4.8)

$$\begin{aligned}\textstyle\sum_{i,j}B_{ij}\Delta B_{ij}\ &=\ -(\mathrm{tr}B^2)^2+n\mathrm{tr}(AB^2)+\tfrac{n-1}{n}\mathrm{tr}A\\&\geq\ -(\tfrac{n-1}{n})^2-\sqrt{\tfrac{n-1}{n}}(n-2)\|\tilde A\|+2\tfrac{n-1}{n}\mathrm{tr}A\\&=\ \sqrt{\tfrac{n-1}{n}}\big[\sqrt{\tfrac{n-1}{n}}(nR-\tfrac{n-2}{n})-(n-2)\|\tilde{\mathbf{A}}\|\big],\end{aligned} \tag{4.9}$$

where $\|\tilde{\mathbf{A}}\|^2=\|\tilde A\|^2$ and $\mathrm{tr}\mathbf{A}=\mathrm{tr}A$ are used. From the assumption (1.3) in Main Theorem 1, we know that the right hand side of formula (4.9) is nonnegative. Therefore, from (4.3) and (4.9), we obtain

$$B_{ij,k}=0,\quad\text{for all}\ \ i,j,k\ \ \text{and}\ \sum_{i,j}B_{ij}\Delta B_{ij}=0. \tag{4.10}$$

Hence the equality in (4.9) holds. We have

$$\sqrt{\frac{n-1}{n}}(nR - \frac{n-2}{n}) - (n-2)\|\tilde{\mathbf{A}}\| = 0. \tag{4.11}$$

Further, the inequality (4.8) becomes equality. From Lemma 2.2, we know that $(n-1)$ of the eigenvalues μ_i of B satisfy $|\mu_i| = \dfrac{(\mathrm{tr}B^2)^{1/2}}{\sqrt{n(n-1)}} = \dfrac{1}{n}$ and $\mu_i\mu_j \geq 0$, which yields that the $(n-1)$ of μ_i's are equal and constant. Since $\mathrm{tr}B = 0$ and $\sum_{i,j} B_{ij}^2 = \dfrac{n-1}{n}$ hold, we know that B has two distinct principal curvatures, which are all constant. Therefore, we obtain $\mathbf{x} : M \mapsto S^{n+1}(1)$ is a Möbius isoparametric hypersurface with two distinct principal curvatures. By the result of Li, Liu, Wang and Zhao [7], we have that $\mathbf{x}$ is Möbius equivalent to an open part of the Riemannian product $S^k(r) \times S^{n-k}(\sqrt{1-r^2})$ in $S^{n+1}(1)$, or an open part of the image of σ of the standard cylinder $S^k(1) \times \mathbf{R}^{n-k}$ in $\mathbf{R}^{n+1}$ or an open part of the image of τ of $S^k(r) \times \mathbf{H}^{n-k}(\sqrt{1+r^2})$ in $\mathbf{H}^{n+1}$, for $k = 1, 2, \cdots, n-1$. From Remark 3.3, Remark 3.5 and Remark 3.7, we know that formula (4.11) holds if and only if $k = n-1$. Hence, Main Theorem 1 is true in this case.

Case (ii) where $p \geq 2$. Define $\sigma_{\alpha\beta} = \sum\limits_{i,j} B_{ij}^{\alpha} B_{ij}^{\beta}$. Since the $(p \times p)$-matrix $(\sigma_{\alpha\beta})$ is symmetric, we can choose $E_{n+1}, \cdots, E_{n+p}$ such that $(\sigma_{\alpha\beta})$ is diagonal, that is,

$$\sigma_{\alpha\beta} = \sigma_\alpha \delta_{\alpha\beta}. \tag{4.12}$$

From Lemma 2.1, we have

$$\begin{aligned}
&-\sum_{\alpha,\beta} N(B_\alpha B_\beta - B_\beta B_\alpha) - \sum_{\alpha,\beta} \{\mathrm{tr}(B_\alpha B_\beta)\}^2 \\
&\geq -2\sum_{\alpha\neq\beta} \sigma_\alpha \sigma_\beta - \sum_\alpha \sigma_\alpha^2 \\
&= -2(\sum_\alpha \sigma_\alpha)^2 + \sum_\alpha \sigma_\alpha^2 \\
&\geq -2(\frac{n-1}{n})^2 + \frac{1}{p}(\sum_\alpha \sigma_\alpha)^2 \\
&= -(2 - \frac{1}{p})(\frac{n-1}{n})^2.
\end{aligned} \tag{4.13}$$

From (4.4),(4.8), (4.13), we have

$$
\begin{aligned}
(4.14)\qquad & \sum_{i,j,\alpha} B_{ij}^{\alpha}\Delta B_{ij}^{\alpha} \\
\geq & -(2-\frac{1}{p})(\frac{n-1}{n})^2 + n\sum_{\alpha}\mathrm{tr}(AB_{\alpha}^2) + \frac{n-1}{n}\mathrm{tr}A \\
= & -(2-\frac{1}{p})(\frac{n-1}{n})^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}\sum_{\alpha}\mathrm{tr}B_{\alpha}^2\|\tilde{A}\| \\
& + \mathrm{tr}A\sum_{\alpha}\mathrm{tr}B_{\alpha}^2 + \frac{n-1}{n}\mathrm{tr}A \\
= & -(2-\frac{1}{p})(\frac{n-1}{n})^2 - \sqrt{\frac{n-1}{n}}(n-2)\|\tilde{A}\| + 2\frac{n-1}{n}\mathrm{tr}A \\
= & \sqrt{\frac{n-1}{n}}\{\sqrt{\frac{n-1}{n}}(nR - \frac{1}{n}[(n-1)(2-\frac{1}{p})-1]) - (n-2)\|\tilde{\mathbf{A}}\|\},
\end{aligned}
$$

where $\|\tilde{\mathbf{A}}\|^2 = \|\tilde{A}\|^2$ and $\mathrm{tr}\mathbf{A} = \mathrm{tr}A$ are used. From the assumption (1.3) in Main Theorem 1, we know that the right hand side of (4.14) is nonnegative. Therefore, from (4.3) and (4.14), we obtain $B_{ij,k}^{\alpha} = 0$, for all i, j, k, α, and $\sum_{i,j,\alpha} B_{ij}^{\alpha}\Delta B_{ij}^{\alpha} = 0$. Hence, the above inequalities become equalities. Thus, we have

$$
(4.15)\qquad (n-2)\|\tilde{\mathbf{A}}\| = \sqrt{\frac{n-1}{n}}\big\{nR - \frac{1}{n}\big[(n-1)(2-\frac{1}{p})-1\big]\big\},
$$

and

$$
(4.16)\qquad \sigma_{n+1} = \sigma_{n+2} = \cdots = \sigma_{n+p}
$$

because of $\frac{1}{p}(\sum_{\alpha}\sigma_{\alpha})^2 = \sum_{\alpha}\sigma_{\alpha}^2$. From Lemma 2.1, we know that at most two of the matrices $B_{\alpha} = (B_{ij}^{\alpha})$ are nonzero. From (2.16), we have $\sum_{\alpha}\sigma_{\alpha} = \frac{n-1}{n}$. Hence, (4.16) yields $p = 2$ and we may assume that

$$
(4.17)\qquad B_{n+1} = \lambda\widetilde{A},\quad B_{n+2} = \mu\widetilde{B},\quad \lambda, \mu \neq 0,
$$

where $\widetilde{A}$ and $\widetilde{B}$ are defined in Lemma 2.1. Therefore, we have

$$
(4.18)\qquad B_{12}^{n+1} = B_{21}^{n+1} = \lambda,\quad B_{ij}^{n+1} = 0,\qquad (i,j) \notin \{(1,2),(2,1)\},
$$

$$
(4.19)\qquad B_{11}^{n+2} = \mu,\quad B_{22}^{n+2} = -\mu,\quad B_{ij}^{n+2} = 0,\quad (i,j) \notin \{(1,1),(2,2)\}.
$$

Since the inequality (4.8) becomes equality, from Lemma 2.2, we know that, for each α, $(n-1)$ of the eigenvalues μ_i^{α} of $B_{\alpha} = (B_{ij}^{\alpha})$ satisfy $|\mu_i^{\alpha}| = \dfrac{(\mathrm{tr}B_{\alpha}^2)^{1/2}}{\sqrt{n(n-1)}}$ and $\mu_i^{\alpha}\mu_j^{\alpha} \geq 0$, which infer that the $(n-1)$ of μ_i^{α} are equal. From (4.19), we have the eigenvalues of $B_{n+2} = (B_{ij}^{n+2})$ are $\mu, -\mu, 0, 0, \cdots 0$. Since $\mu \neq 0$, we infer $n = 2$.

From (4.18), we can infer, by an algebraic method, that the eigenvalues of B_{n+1} are $\lambda, -\lambda$. Since $n = p = 2$ holds, from (4.1), (4.18) and (4.19), we have

$$A_{11} = A_{22}, \quad A_{12} = A_{21} = 0. \tag{4.20}$$

Therefore, $\mathbf{x} : M^2 \mapsto S^4(1)$ is a Möbius isotropic submanifold in S^4. Thus, we have $\|\tilde{\mathbf{A}}\| = 0$. Since $n = 2$ and $p = 2$ hold, from (4.15), we have $R = \frac{1}{8}$. We obtain $\mathrm{tr}A = \frac{3}{8}$. Hence $A_{11} = A_{22} = \frac{3}{16}$. From Liu, Wang and Zhao [9], we obtain that $\mathbf{x} : M^2 \mapsto S^4(1)$ is Möbius equivalent to an open part of either a minimal surface $\tilde{\mathbf{x}} : M^2 \mapsto S^4(1)$ with constant scalar curvature in $S^4(1)$, or the image of σ_2 of a minimal surface with constant scalar curvature in $\mathbf{R}^4$ or the image of τ_2 of a minimal surface with constant scalar curvature in $\mathbf{H}^4$. For a surface, Gaussian curvature is constant if and only if the scalar curvature is constant. From the Proposition 4.1 and Theorem 4.2 of Bryant [3], we know that a minimal surface with constant scalar curvature in $\mathbf{R}^4$ is totally geodesic and a minimal surface with constant scalar curvature in $\mathbf{H}^4$ is also totally geodesic. Since $\mathbf{x} : M^2 \mapsto S^4(1)$ has no umbilical ponits, we infer that $\mathbf{x} : M^2 \mapsto S^4(1)$ is Möbius equivalent to an open part of a minimal surface $\tilde{\mathbf{x}} : M^2 \mapsto S^4(1)$ with constant scalar curvature in $S^4(1)$. From the Gauss equation of the minimal surface $\tilde{\mathbf{x}} : M^2 \mapsto S^4(1)$ with constant scalar curvature in $S^4(1)$, we know that the squared norm of the second fundamental form of this minimal surface is constant. According to the definition (2.2) of ρ, ρ^2 is constant. From (2.14), we have $\rho^2 = \frac{8}{3}$. Thus, the squared norm of the second fundamental form of $\tilde{\mathbf{x}}$ must be $\frac{4}{3}$, i.e. $\|II\|^2 = \frac{4}{3}$. Therefore, from the result of Chern, do Carmo and Kobayashi [5], we obtain that $\tilde{\mathbf{x}} : M^2 \mapsto S^4(1)$ is locally a Veronese surface in $S^4(1)$. This finishes the proof of Main Theorem 1.

Proof of Main Theorem 2. Since the Möbius form $\Phi = \sum_{i,\alpha} C_i^\alpha e_\alpha \equiv 0$ holds, we have

$$A_{ij,k} = A_{ik,j}, \quad B^\alpha_{ij,k} = B^\alpha_{ik,j}, \quad \sum_k B^\alpha_{ik} A_{kj} = \sum_k B^\alpha_{kj} A_{ki}. \tag{4.21}$$

Hence, for any α, $B_\alpha A = A B_\alpha$, where $A = (A_{ij})$ and $B_\alpha = (B^\alpha_{ik})$. For any fixed α, we can choose the basis $\{E_i\}$ such that $A = (A_{ij})$ and $B_\alpha = (B^\alpha_{ik})$ are diagonal, that is,

$$A_{ij} = \lambda_i \delta_{ij}, \quad B^\alpha_{ij} = \mu^\alpha_i \delta_{ij}. \tag{4.22}$$

Since $n(n-1)R$ is constant, from (2.24), we have that $\mathrm{tr}\mathbf{A} = \mathrm{tr}A = \sum_i A_{ii}$ is constant. From (2.25), (4.21), (4.22), we infer

$$\begin{aligned}
\frac{1}{2}\Delta\|\mathbf{A}\|^2 &= \sum_{i,j,k}(A_{ij,k})^2 + \sum_{i,j,k} A_{ij}A_{ij,kk} \\
&= \sum_{i,j,k}(A_{ij,k})^2 + \sum_{i,j,k} A_{ij}A_{kk,ij} + \sum_{i,j,k,l} A_{ij}A_{li}R_{lkjk} + \sum_{i,j,k,l} A_{ij}A_{kl}R_{lijk} \\
&= \sum_{i,j,k}(A_{ij,k})^2 + \frac{1}{2}\sum_{i,k} R_{ikik}(\lambda_i - \lambda_k)^2.
\end{aligned} \tag{4.23}$$

When $p > 1$, from the assumption $K > 0$ in Main Theorem 2, by integrating (4.23), we have

$$R_{ikik}(\lambda_i - \lambda_k)^2 = 0.$$

Therefore, we know that $\lambda_i = \lambda_k$, that is, $\mathbf{x} : M \mapsto S^{n+p}(1)$ is a Möbius isotropic submanifold in $S^{n+p}(1)$ with positive Möbius sectional curvature. From the result in [9], we know that $\mathbf{x}$ is Möbius equivalent to the compact minimal submanifolds with constant scalar curvature in $S^{n+p}(1)$.

Next, we consider the case where $p = 1$. In this case, we know that the Möbius sectional curvature of the immersion $\mathbf{x}$ is nonnegative. By integrating (4.23), we infer

$$A_{ij,k} = 0, \text{ for any } i, j, k, \quad R_{ikik}(\lambda_i - \lambda_k)^2 = 0. \tag{4.24}$$

From (2.22) and (4.22), we have $R_{ikik} = \mu_i\mu_k + \lambda_i + \lambda_k$ for $i \neq k$. Hence, we infer

$$(\mu_i\mu_k + \lambda_i + \lambda_k)(\lambda_i - \lambda_k)^2 = 0. \tag{4.25}$$

Form (4.24) and (2.17), we have

$$0 = d\lambda_i\delta_{ij} + (\lambda_i - \lambda_j)\omega_{ij}, \quad 1 \leq i, j \leq n. \tag{4.26}$$

Setting $i = j$ in (4.26), we obtain $d\lambda_i = 0$, that is, eigenvalues of (A_{ij}) are all constant. From (4.26), we infer that for $\lambda_i \neq \lambda_j$,

$$\omega_{ij} = 0. \tag{4.27}$$

Let $\lambda_1, \lambda_2, \cdots, \lambda_l$ are these distinct eigenvalues of $A = (A_{ij})$. We can assume $\lambda_1 < \lambda_2 < \cdots < \lambda_l$. From (4.25), we have

$$\lambda_i = \lambda_k, \quad \text{or} \quad \mu_i\mu_k + \lambda_i + \lambda_k = 0. \tag{4.28}$$

In the second case, we will prove that $A = (A_{ij})$ has at most three distinct eigenvalues. In fact, if we assume $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \cdots < \lambda_l$ are these distinct eigenvalues of $A = (A_{ij})$. Let $\lambda_1, \lambda_2, \lambda_i$ are the three distinct eigenvalues of $A = (A_{ij})$, we have

$$\mu_i\mu_1 + \lambda_i + \lambda_1 = 0.$$

$$\mu_i\mu_2 + \lambda_i + \lambda_2 = 0.$$

Hence, we have

$$\mu_i = -\frac{\lambda_1 - \lambda_2}{\mu_1 - \mu_2}, \tag{4.29}$$

$$\lambda_i = -\lambda_1 + \mu_1\frac{\lambda_1 - \lambda_2}{\mu_1 - \mu_2}. \tag{4.30}$$

Hence, for $r = 3, 4, \cdots, l$, we have $\lambda_r = \lambda_i$. This is a contradiction. Therefore, $A = (A_{ij})$ has at most three distinct eigenvalues.

(1). In the first case, we consider the case that (A_{ij}) only has one distinct eigenvalues. Since the Möbius form $\Phi = \sum_{i,\alpha} C_i^\alpha e_\alpha \equiv 0$, we know $\mathbf{x} : M \mapsto S^{n+1}(1)$ is a Möbius isotropic hypersurface in $S^{n+1}(1)$ with nonnegative Möbius sectional curvature. By the result in [9], we know that $\mathbf{x}$ is Möbius equivalent to a minimal hypersurface with constant scalar curvature in $S^{n+1}(1)$.

(2). We consider the second case that (A_{ij}) has two or three distinct eigenvalues. From (4.29), we know that at most three of the principal curvatures of (B_{ij}) are distinct. Since $\mathbf{x}$ has no umbilical points, we know that the distinct principal curvatures of (B_{ij}) is two or three.

(i) If two of the principal curvatures of (B_{ij}) are distinct, without lost of generality, we may assume $\mu_1 < \mu_2$. From (2.16), we know that μ_1 and μ_2 are constant, that is, $\mathbf{x} : M \mapsto S^{n+1}(1)$ is a Möbius isoparametric hypersurface with two distinct principal curvatures in $S^{n+1}(1)$. Since $\mathbf{x}$ is compact, from Theorem 1.1 in the introduction, we infer that $\mathbf{x}$ is Möbius equivalent to the Riemannian product $S^k(r) \times S^{n-k}(\sqrt{1 - r^2})$, for $k = 1, 2, \cdots, n - 1$.

(ii) If three of the principal curvatures of (B_{ij}) are distinct, without lose of generality, we may assume $\mu_1 < \mu_2 < \mu_3$. From (2.16) and (4.29), we know that μ_1, μ_2, μ_3

are constant. From the proof of Main Theorem 1, we infer

$$\begin{aligned}\frac{1}{2}\Delta\sum_{i,j}B_{ij}^2 &= \sum_{i,j,k}B_{ij,k}^2+\sum_{i,j}B_{ij}\Delta B_{ij}\\ &=\sum_{i,j,k}B_{ij,k}^2-(\mathrm{tr}B^2)^2+n\mathrm{tr}(AB^2)+\frac{n-1}{n}\mathrm{tr}A\\ &=\sum_{i,j,k}B_{ij,k}^2+\frac{1}{2}\sum_{i,j}(\mu_i-\mu_j)^2R_{ijij}\geq 0.\end{aligned}$$

Since $\sum_{i,j}B_{ij}^2$ is constant, we obtain $B_{ij,k}=0$ for any i,j,k. From (2.18), we have, for each $\mu_i\neq\mu_j$,

$$\omega_{ij}=0. \tag{4.31}$$

Hence, we know that the distributions of the eigenspaces with respect to μ_i are integrable. Since the distinct principal curvatures of M is three, we can write $M=M_1\times M_2\times M_3$, where M_i $(1\leq i\leq 3)$ is the integrable manifold corresponding to the principal curvature μ_i. Since μ_i's are constant, we know that M_i, $i=1,2,3$, are closed. Thus, they are compact because M is compact. From (2.22), we have, for $j,k,l\in[i]$,

$$R_{ijkl}=(\mu_i^2+2\lambda_i)(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}), \tag{4.32}$$

that is, M_i are constant curvature space with respect to the Möbius metric g. Putting $k_i=\mu_i^2+2\lambda_i,\quad 1\leq i\leq 3$, then, we have

$$\begin{aligned}k_1&=(\mu_1-\mu_2)(\mu_1-\mu_3)>0,\\ k_2&=(\mu_2-\mu_1)(\mu_2-\mu_3)<0,\\ k_3&=(\mu_3-\mu_1)(\mu_3-\mu_2)>0.\end{aligned} \tag{4.33}$$

Therefore, we may infer dim $M_2=1$. In fact, if dim $M_2\geq 2$ holds, by the assumption that the Möbius sectional curvature of M is nonnegative, we have $k_2\geq 0$. This is a contradiction.

Let (u,v,w) be a coordinate system for M such that $u\in M_1$, $v\in M_2$, $w\in M_3$ and $E_l=\frac{\partial}{\partial v}$, where $l=\mathrm{dim}M_1+1$. Then, from structure equations (2.9), (2.10), (2.11) and (2.12) and (4.31), by a direct and simple calculation, we obtain

$$N_v=\lambda_2 Y_v, \tag{4.34}$$

$$Y_{vv}=-\lambda_2 Y-N+\mu_2 E,\quad Y_{vj}=0,\text{for } j\neq l, \tag{4.35}$$

$$E_v=-\mu_2 Y_v, \tag{4.36}$$

where we denote E_{n+1} by E. From (4.35), we can write $Y = f(v) + F(u, w)$. Then, by (4.34), (4.35) and (4.36), we have

$$f'''(v) + k_2 f'(v) = 0, \tag{4.37}$$

where $k_2 = \mu_2^2 + 2\lambda_2 < 0$. The solution of (4.37) can be easily written as

$$f(v) = C_1 \frac{1}{\sqrt{-k_2}} \cosh(\sqrt{-k_2} v) + C_2 \frac{1}{\sqrt{-k_2}} \sinh(\sqrt{-k_2} v), \tag{4.38}$$

where $C_1, C_2 \in \mathbf{R}_1^{n+3}$ are constant vectors. From (4.38), we know that M_2 must be a hyperbola. This is a contradiction because M_2 is compact. Hence, the case (ii) does not occur, that is, M is a Möbius isoparametric hypersurface with two distinct principal curvatures. This completes the proof of Main Theorem 2.

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