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A GAP THEOREM OF SELF-SHRINKERS

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ABSTRACT. In this paper, we study complete self-shrinkers in Euclidean space and prove that an n-dimensional complete self-shrinker with polynomial volume growth in Euclidean space $\mathbb{R}^{n+1}$ is isometric to either $\mathbb{R}^n$, $S^n(\sqrt{n})$, or $\mathbb{R}^{n-m} \times S^m(\sqrt{m})$, $1 \leq m \leq n-1$, if the squared norm S of the second fundamental form is constant and satisfies $S < \frac{10}{7}$.

1. Introduction

Let $X : M \to \mathbb{R}^{n+1}$ be a smooth n-dimensional immersed hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. The immersed hypersurface M is called a self-shrinker if it satisfies the quasilinear elliptic system:

$$\mathbf{H} = -X^N,$$

where $\mathbf{H}$ denotes the mean curvature vector of M, and X^N denotes the orthogonal projection of X onto the normal bundle of M.

It it known that self-shrinkers play an important role in the study of the mean curvature flow because they describe all possible blow up at a given singularity of a mean curvature flow.

For $n = 1$, Abresch and Langer [1] classified all smooth closed self-shrinker curves in $\mathbb{R}^2$ and showed that the round circle is the only embedded self-shrinkers. For $n \geq 2$, Huisken [9] studied compact self-shrinkers. He proved that if M is an n-dimensional compact self-shrinker with non-negative mean curvature H in $\mathbb{R}^{n+1}$, then $X(M) = S^n(\sqrt{n})$. We should notice that the condition of non-negative mean curvature is essential. In fact, let Δ and ∇ denote the Laplacian and the gradient operator on the self-shrinker, respectively and $\langle \cdot, \cdot \rangle$ denotes the standard inner product of $\mathbb{R}^{n+1}$. Because

$$\Delta H - \langle X, \nabla H \rangle + SH - H = 0,$$

we obtain $H > 0$ from the maximum principle if the mean curvature is non-negative. Furthermore, Angenent [2] has constructed compact embedded self-shrinker torus $S^1 \times S^{n-1}$ in $\mathbb{R}^{n+1}$.

Huisken [10] and Colding and Minicozzi [5] have studied complete and non-compact self-shrinkers in $\mathbb{R}^{n+1}$. They have proved that if M is an n-dimensional complete embedded self-shrinker in $\mathbb{R}^{n+1}$ with $H \geq 0$ and with polynomial volume

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growth, then M is isometric to either the hyperplane $\mathbb{R}^n$, the round sphere $S^n(\sqrt{n})$, or a cylinder $S^m(\sqrt{m}) \times \mathbb{R}^{n-m}$, $1 \le m \le n-1$.

Without the condition $H \ge 0$, Le and Sesum [11] proved that if M is an n-dimensional complete embedded self-shrinker with polynomial volume growth and $S < 1$ in Euclidean space $\mathbb{R}^{n+1}$, then $S = 0$ and M is isometric to the hyperplane $\mathbb{R}^n$, where S denotes the squared norm of the second fundamental form. Furthermore, Cao and Li [3] have studied the general case. They have proved that if M is an n-dimensional complete self-shrinker with polynomial volume growth and $S \le 1$ in Euclidean space $\mathbb{R}^{n+1}$, then M is isometric to either the hyperplane $\mathbb{R}^n$, the round sphere $S^n(\sqrt{n})$, or a cylinder $S^m(\sqrt{m}) \times \mathbb{R}^{n-m}$, $1 \le m \le n-1$.

Recently, Ding and Xin [6] have studied the second gap on the squared norm of the second fundamental form and they have proved that if M is an n-dimensional complete self-shrinker with polynomial volume growth in Euclidean space $\mathbb{R}^{n+1}$, there exists a positive number $\delta = 0.022$ such that if $1 \le S \le 1+0.022$, then $S = 1$.

Motivated by the above results of Le and Sesum, Cao and Li, Ding and Xin, we consider the second gap for the squared norm of the second fundamental form and prove the following classification theorem for self-shrinkers:

Theorem 1.1. *Let M be an n-dimensional complete self-shrinker with polynomial volume growth in $\mathbb{R}^{n+1}$. If the squared norm S of the second fundamental form is constant and satisfies*

$$S \le 1 + \frac{3}{7},$$

then M is isometric to one of the following:

(1) *the hyperplane $\mathbb{R}^n$,*
(2) *a cylinder $\mathbb{R}^{n-m} \times S^m(\sqrt{m})$, for $1 \le m \le n-1$,*
(3) *the round sphere $S^n(\sqrt{n})$.*

2. Preliminaries

In this section, we give some notation and formulas. Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional self-shrinker in $\mathbb{R}^{n+1}$. Let $\{e_1, \cdots, e_n, e_{n+1}\}$ be a local orthonormal basis along M with dual coframe $\{\omega_1, \cdots, \omega_n, \omega_{n+1}\}$, such that $\{e_1, \cdots, e_n\}$ is a local orthonormal basis of M and e_{n+1} is normal to M. Then we have

$$\omega_{n+1} = 0, \quad \omega_{n+1i} = -\sum_{j=1}^{n} h_{ij}\omega_j, \quad h_{ij} = h_{ji},$$

where h_{ij} denotes the component of the second fundamental form of M. $\mathbf{H} = \sum_{j=1}^{n} h_{jj} e_{n+1}$ is the mean curvature vector field, $H = |\mathbf{H}| = \sum_{j=1}^{n} h_{jj}$ is the mean curvature and $II = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j e_{n+1}$ is the second fundamental form of M. The Gauss equations and Codazzi equations are given by

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}, \tag{2.1}$$

$$h_{ijk} = h_{ikj}, \tag{2.2}$$

where R_{ijkl} is the component of curvature tensor, and the covariant derivative of h_{ij} is defined by

$$\sum_{k=1}^{n} h_{ijk}\omega_k = dh_{ij} + \sum_{k=1}^{n} h_{kj}\omega_{ki} + \sum_{k=1}^{n} h_{ik}\omega_{kj}.$$

Let

$$F_i = \nabla_i F,\ F_{ij} = \nabla_j \nabla_i F,\ h_{ijk} = \nabla_k h_{ij},\ \text{and } h_{ijkl} = \nabla_l \nabla_k h_{ij},$$

where ∇_j is the covariant differentiation operator. We have

$$h_{ijkl} - h_{ijlk} = \sum_{m=1}^{n} h_{im} R_{mjkl} + \sum_{m=1}^{n} h_{mj} R_{mikl}. \tag{2.3}$$

The following elliptic operator $\mathcal{L}$ was introduced by Colding and Minicozzi in [5]:

$$\mathcal{L} f = \Delta f - \langle X, \nabla f \rangle, \tag{2.4}$$

where Δ and ∇ denote the Laplacian and the gradient operator on the self-shrinker, respectively and $\langle \cdot, \cdot \rangle$ denotes the standard inner product of $\mathbb{R}^{n+1}$. By a direct calculation, we have

$$\mathcal{L} h_{ij} = (1-S) h_{ij}, \quad \mathcal{L} H = H(1-S), \quad \mathcal{L} X_i = -X_i, \quad \mathcal{L}|X|^2 = 2(n - |X|^2), \tag{2.5}$$

$$\frac{1}{2}\mathcal{L} S = \sum_{i,j,k} h_{ijk}^2 + S(1-S). \tag{2.6}$$

If S is constant, then we obtain from (2.6)

$$\sum_{i,j,k} h_{ijk}^2 = S(S-1). \tag{2.7}$$

Hence one has either

$$S = 0, \quad \text{or} \quad S = 1, \quad \text{or} \quad S > 1. \tag{2.8}$$

We can choose a local field of orthonormal frames on M^n such that, at the point that we consider,

$$h_{ij} = \begin{cases} \lambda_i, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j. \end{cases}$$

Then

$$S = \sum_{i,j} h_{ij}^2 = \sum_i \lambda_i^2,$$

where λ_i is called the principal curvature of M. From (2.1) and (2.3), we get

$$h_{ijij} - h_{jiji} = (\lambda_i - \lambda_j)\lambda_i \lambda_j. \tag{2.9}$$

By a direct calculation, we obtain

$$\sum_{i,j,k,l} h_{ijkl}^2 = S(S-1)(S-2) + 3(A - 2B), \tag{2.10}$$

where $A = \sum_{i,j,k} \lambda_i^2 h_{ijk}^2$, $B = \sum_{i,j,k} \lambda_i \lambda_j h_{ijk}^2$.

We define two functions f_3 and f_4 as follows:

$$f_3 = \sum_{i,j,k} h_{ij} h_{jk} h_{ki} = \sum_{j=1}^{n} \lambda_j^3, \qquad f_4 = \sum_{i,j,k,l} h_{ij} h_{jk} h_{kl} h_{li} = \sum_{j=1}^{n} \lambda_j^4.$$

Then we have the following lemma.

Lemma 2.1. *Let M be an n-dimensional complete self-shrinker without boundary and with polynomial volume growth in $\mathbb{R}^{n+1}$. Then*

$$\mathcal{L}f_3 = 3(1-S)f_3 + 6\sum_{i,j,k}\lambda_i h_{ijk}^2, \tag{2.11}$$

$$\mathcal{L}f_4 = 4(1-S)f_4 + 4(2A+B). \tag{2.12}$$

Proof. By the definition of f_3, f_4, $\mathcal{L}f_3$ and $\mathcal{L}f_4$, we have the following calculations:

$$f_{3m} = 3\sum_{i,j,k} h_{ijm}h_{jk}h_{ki},$$

$$f_{3mm} = 3\sum_{i,j,k} h_{jk}h_{ki}h_{ijmm} + 3\sum_{i,j,k} h_{ijm}h_{jkm}h_{ki} + 3\sum_{i,j,k} h_{ijm}h_{jk}h_{kim},$$

$$\Delta f_3 = \sum_m f_{3mm} = 3\sum_{i,j,k} h_{jk}h_{ki}\Delta h_{ij} + 6\sum_{i,j,m}\lambda_i h_{ijm}^2,$$

$$\langle X, \nabla f_3\rangle = 3\sum_{i,j,k} h_{jk}h_{ki}\langle X, \nabla h_{ij}\rangle,$$

$$\begin{aligned}\mathcal{L}f_3 &= \Delta f_3 - \langle X, \nabla f_3\rangle \\ &= 3\sum_{i,j,k} h_{jk}h_{ki}\mathcal{L}h_{ij} + 6\sum_{i,j,m}\lambda_i h_{ijm}^2 \\ &= 3(1-S)f_3 + 6\sum_{i,j,k}\lambda_i h_{ijk}^2,\end{aligned}$$

and

$$f_{4m} = 4\sum_{i,j,k,l} h_{ijm}h_{jk}h_{kl}h_{li},$$

$$\begin{aligned}f_{4mm} = {}& 4\sum_{i,j,k,l} h_{ijmm}h_{jk}h_{kl}h_{li} + 4\sum_{i,j,k,l} h_{ijm}h_{jkm}h_{kl}h_{li} \\ &+ 4\sum_{i,j,k,l} h_{ijm}h_{jk}h_{klm}h_{li} + 4\sum_{i,j,k,l} h_{ijm}h_{jk}h_{kl}h_{lim},\end{aligned}$$

$$\Delta f_4 = \sum_m f_{4mm} = 4\sum_{i,j,k,l} h_{jk}h_{kl}h_{li}\Delta h_{ij} + 4\sum_{i,j,m}\lambda_i^2 h_{ijm}^2 + 4\sum_{i,j,k}\lambda_i\lambda_j h_{ijk}^2 + 4\sum_{i,j,m}\lambda_j^2 h_{ijm}^2,$$

$$\langle X, \nabla f_4\rangle = 4\sum_{i,j,k,l} h_{jk}h_{kl}h_{li}\langle X, \nabla h_{ij}\rangle,$$

$$\begin{aligned}\mathcal{L}f_4 &= \Delta f_4 - \langle X, \nabla f_4\rangle \\ &= 4\sum_{i,j,k,l} h_{jk}h_{kl}h_{li}\mathcal{L}h_{ij} + 8\sum_{i,j,m}\lambda_i^2 h_{ijm}^2 + 4\sum_{i,j,m}\lambda_i\lambda_j h_{ijm}^2 \\ &= 4(1-S)f_4 + 4(2A+B).\end{aligned}$$

□

3. Some estimates

In this section, we will give some estimates which are needed to prove our theorem. From now on, we denote

$$S-1=tS,$$

where t is a positive constant if we assume that S is constant and $S>1$. Then

$$(1-t)S=1, \qquad \sum_{i,j,k} h_{ijk}^2 = tS^2.$$

By a direct calculation, one obtains

$$\begin{aligned}\sum_{i,j,k,l} h_{ijkl}^2 \geq& \sum_i h_{iiii}^2 + \frac{3}{4}\sum_{i\neq j}(h_{ijij}+h_{jiji})^2 + \frac{3}{4}\sum_{i\neq j}(h_{ijij}-h_{jiji})^2\\ =& \sum_i h_{iiii}^2 + \frac{3}{4}\sum_{i\neq j}(h_{ijij}+h_{jiji})^2\\ &+\frac{3}{2}\left[S\sum_i \lambda_i^4 - (\sum_i \lambda_i^3)^2\right].\end{aligned} \tag{3.1}$$

We next have to estimate $S\sum_i \lambda_i^4 - (\sum_i \lambda_i^3)^2$ since we want to give the estimate of $\sum_{i,j,k,l} h_{ijkl}^2$. Define

$$f \equiv \sum_i \lambda_i^4 - \frac{1}{S}\left(\sum_i \lambda_i^3\right)^2 = f_4 - \frac{1}{S}(f_3)^2.$$

Firstly, we have

Lemma 3.1. *There is one point $x\in M$ such that the following identity holds at the point:*

$$\begin{aligned}&tS^2\left[\frac{c}{S}\left(\sum_i \lambda_i^3\right)^2 - \sum_i \lambda_i^4\right]\\ =&c\left(\sum_{i,j,k} 2\lambda_i h_{ijk}^2\right)\sum_i \lambda_i^3 - (2A+B)S + 3c\sum_j\left(\sum_i \lambda_i^2 h_{iij}\right)^2,\end{aligned} \tag{3.2}$$

where c is a real number.

Proof. Define a function

$$F=\frac{1}{4}S\sum_i \lambda_i^4 - \frac{1}{6}c\left(\sum_i \lambda_i^3\right)^2 = \frac{1}{4}Sf_4 - \frac{1}{6}c(f_3)^2.$$

We have from Lemma 2.1 that

$$\begin{aligned}\mathcal{L}F =& \mathcal{L}\big(\frac{1}{4}Sf_4 - \frac{1}{6}c(f_3)^2\big)\\ =& S(1-S)f_4 - c\Big((1-S)f_3^2 + 2\sum_{i,j,k}\lambda_i h_{ijk}^2 f_3\\ &+3\sum_j(\sum_i \lambda_i^2 h_{iij})^2\Big)+(2A+B)S.\end{aligned} \tag{3.3}$$

Since S is constant, we know, from (2.7) and (2.10), that $\sum_{i,j,k} h_{ijk}^2$ and $\sum_{i,j,k,l} h_{ijkl}^2$ are bounded. Thus, it is not difficult to check that F, $\mathcal{L}F$ and $|\nabla F|$ are bounded. Since M is an n-dimensional complete self-shrinker with polynomial volume growth, we have

$$\int_M \mathcal{L}F e^{-\frac{|X|^2}{2}} dv = 0$$

(see the corollary 3.10 in [5]). Hence there is a point $x \in M$ such that

$$S(1-S)f_4 - c\Big((1-S)f_3^2 + 2\sum_{i,j,k}\lambda_i h_{ijk}^2 f_3 + 3\sum_j(\sum_i \lambda_i^2 h_{iij})^2\Big) + (2A+B)S = 0$$

at the point because of the continuity of the function. $\square$

Secondly, we have

Lemma 3.2.

$$f = f_4 - \frac{f_3^2}{S} \geq \frac{(\lambda_1-\lambda_2)^2}{\lambda_1^2+\lambda_2^2}(\lambda_1\lambda_2)^2,$$

where $\lambda_1 = \max_i\{\lambda_i\}$, $\lambda_2 = \min_i\{\lambda_i\}$.

Proof. Since

$$Sf_4 - f_3^2 = \frac{1}{S}\sum_i(\lambda_i^2 S - f_3\lambda_i)^2,$$

we have that

$$\begin{aligned} Sf_4 - f_3^2 &\geq \frac{1}{S}\left(\lambda_1^2 S - f_3\lambda_1\right)^2 + \frac{1}{S}\left(\lambda_2^2 S - f_3\lambda_2\right)^2 \\ &= S\lambda_1^4 + S\lambda_2^4 + \frac{f_3^2(\lambda_1^2+\lambda_2^2)}{S} - 2(\lambda_1^3+\lambda_2^3)f_3 \\ &\geq S(\lambda_1^4+\lambda_2^4) - \frac{\lambda_1^2+\lambda_2^2}{S}\frac{(\lambda_1^3+\lambda_2^3)^2 S^2}{(\lambda_1^2+\lambda_2^2)^2} \\ &= \frac{S}{\lambda_1^2+\lambda_2^2}\lambda_1^2\lambda_2^2(\lambda_1-\lambda_2)^2. \end{aligned}$$

$\square$

Thirdly, one has

Lemma 3.3.

$$A - B \leq \frac{1}{3}(\lambda_1-\lambda_2)^2 tS^2(1-\alpha), \tag{3.4}$$

where $\alpha = \frac{\sum_i h_{iii}^2}{\sum_{i,j,k} h_{ijk}^2} = \frac{\sum_i h_{iii}^2}{tS^2}$.

Proof. By means of symmetry, we have

$$\begin{aligned} A-B &= \sum_{i,j,k}(\lambda_i^2-\lambda_i\lambda_j)h_{ijk}^2 \\ &= \frac{1}{3}\sum_{i,j,k}(\lambda_i^2+\lambda_j^2+\lambda_k^2-\lambda_i\lambda_j-\lambda_j\lambda_k-\lambda_k\lambda_i)h_{ijk}^2 \\ &= \frac{1}{3}\sum_{i,j}3(\lambda_i-\lambda_j)^2h_{iij}^2 \\ &\quad+\frac{1}{3}\sum_{i\neq j\neq k\neq i}(\lambda_i^2+\lambda_j^2+\lambda_k^2-\lambda_i\lambda_j-\lambda_j\lambda_k-\lambda_k\lambda_i)h_{ijk}^2. \end{aligned}$$

Without loss of generality, we can assume that $\lambda_i \le \lambda_j \le \lambda_k$ and consider

$$z = \lambda_i^2+\lambda_j^2+\lambda_k^2-\lambda_i\lambda_j-\lambda_j\lambda_k-\lambda_k\lambda_i$$

as a function of λ_j, which takes its maximum at one of the boundary points λ_i or λ_k. On the other hand,

$$z_{\lambda_j=\lambda_i} = z_{\lambda_j=\lambda_k} = (\lambda_i-\lambda_k)^2 \le (\lambda_1-\lambda_2)^2.$$

Hence we get

$$\begin{aligned} A-B \le& \frac{1}{3}\left[\sum_{i,j}3(\lambda_i-\lambda_j)^2h_{iij}^2+\sum_{i\neq j\neq k\neq i}(\lambda_1-\lambda_2)^2h_{ijk}^2\right] \\ \le& \frac{1}{3}(\lambda_1-\lambda_2)^2\left(\sum_{i,j,k}h_{ijk}^2-\sum_i h_{iii}^2\right). \end{aligned}$$

Combining (2.7) and the definition of α, we get $0 \le \alpha < 1$ and (3.4). □

From Lemma 3.1, one knows that the estimates of $\sum_k(\sum_i \lambda_i^2 h_{iik})^2$ and $(\sum_{i,j,k} h_{ijk}^2\lambda_i)^2$ are needed.

Lemma 3.4.

$$\sum_k\left(\sum_i \lambda_i^2 h_{iik}\right)^2 \le \frac{1+2\alpha}{3}tS^2 f, \tag{3.5}$$

where $\alpha = \dfrac{\sum_i h_{iii}^2}{tS^2}$.

Proof. Since $S=\sum_{ij}h_{ij}^2$ is constant, we have $\sum_i \lambda_i h_{iik}=0$. Then

$$\begin{aligned} \sum_k\left(\sum_i \lambda_i^2 h_{iik}\right)^2 &= \sum_k\left[\sum_i(\lambda_i^2-a\lambda_i)h_{iik}\right]^2 \\ &\le \sum_i(\lambda_i^2-a\lambda_i)^2\sum_{i,k}h_{iik}^2, \end{aligned}$$

for any constant a. Let $a=\frac{1}{S}f_3=\frac{1}{S}\sum_i\lambda_i^3$. We have

$$\sum_k\left(\sum_i \lambda_i^2 h_{iik}\right)^2 \le \left[\sum_i\lambda_i^4-\frac{1}{S}\left(\sum_i\lambda_i^3\right)^2\right]\sum_{i,k}h_{iik}^2 = f\sum_{i,k}h_{iik}^2. \tag{3.6}$$

Since

$$\sum_{i,j,k} h_{ijk}^2 = \sum_i h_{iii}^2 + 3\sum_{i\neq j} h_{iij}^2 + \sum_{i\neq j\neq k\neq i} h_{ijk}^2,$$

we have that

$$\sum_{i,k} h_{iik}^2 \leq \frac{1}{3}\Big(\sum_{i,j,k} h_{ijk}^2 + 2\sum_i h_{iii}^2\Big) = \frac{1}{3}(1+2\alpha)\sum_{i,j,k} h_{ijk}^2 = \frac{1}{3}(1+2\alpha)tS^2. \tag{3.7}$$

Combining (3.6) and (3.7), we get (3.5). □

Lemma 3.5.

$$\left(\sum_{i,j,k} \lambda_i h_{ijk}^2\right)^2 \leq \left[\frac{1}{3}(A+2B) - \frac{4}{3}\sum_k \frac{1}{S+2\lambda_k^2}\left(\sum_i \lambda_i^2 h_{iik}\right)^2\right] tS^2. \tag{3.8}$$

Proof. A straightforward computation gives

$$\begin{aligned}
&\left(\sum_{i,j,k} \lambda_i h_{ijk}^2\right)^2\\
&= \left\{\frac{1}{3}\sum_{i,j,k}\left[(\lambda_i+\lambda_j+\lambda_k)h_{ijk} - (a_i h_{jk} + a_j h_{ki} + a_k h_{ij})\right] h_{ijk}\right\}^2\\
&\leq \frac{1}{9}\sum_{i,j,k}\left[(\lambda_i+\lambda_j+\lambda_k)h_{ijk} - (a_i h_{jk} + a_j h_{ki} + a_k h_{ij})\right]^2 \sum_{i,j,k} h_{ijk}^2\\
&= \frac{1}{9}\left[3(A+2B) - 12\sum_{i,k} a_k\lambda_i^2 h_{iik} + 3\sum_k (S+2\lambda_k^2)a_k^2\right] tS^2,
\end{aligned}$$

for any constant $a_k \in \mathbb{R}$. Let

$$a_k = 2\frac{\sum_i \lambda_i^2 h_{iik}}{S+2\lambda_k^2}.$$

Then (3.8) follows. □

4. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. The proof has three parts. In the first part of the proof, we will show that $S > 1 + \frac{1}{5} = 1.2$ if $S > 1$. In the second part, we will prove that $S > \frac{1}{0.802} > 1.24688$ if $S > \frac{6}{5} = 1.2$. In the third part, we will show that $S > 1 + \frac{3}{7}$ if $S > \frac{1}{0.802}$.

Proof of Theorem 1.1.

Part I (Claim: $S > 1 + \frac{1}{5} = \frac{6}{5}$ if $S > 1$)**.** Letting $c = 2$ and applying Lemma 3.1, we get

$$
\begin{aligned}
0 &= (S-1)[\frac{1}{2}Sf_4 - f_3^2] + (2\sum_{i,j,k}\lambda_i h_{ijk}^2)f_3 \\
&\quad - \frac{S}{2}(2A+B) + 3\sum_j(\sum_i \lambda_i^2 h_{iij})^2 \\
&\le (S-1)[\frac{1}{2}Sf_4 - f_3^2] + \frac{1}{2(S-1)}(2\sum_{i,j,k}\lambda_i h_{ijk}^2)^2 \\
&\quad + \frac{S-1}{2}f_3^2 - \frac{S}{2}(2A+B) + 3\sum_j(\sum_i \lambda_i^2 h_{iij})^2 \\
&\le \frac{S-1}{2}(Sf_4 - f_3^2) + S\left[\frac{2}{3}(A+2B) - \frac{8}{9S}\sum_k(\sum_i \lambda_i^2 h_{iik})^2\right] \\
&\quad - \frac{S}{2}(2A+B) + 3\sum_j(\sum_i \lambda_i^2 h_{iij})^2 \\
&\le \frac{S-1}{2}(Sf_4 - f_3^2) - \frac{S}{6}[2A-5B] + \frac{19}{27}(1+2\alpha)tS^2 f \\
&= -\frac{S}{6}(2A-5B) + [\frac{65}{54} + \frac{38}{2}\alpha]tS^2 f
\end{aligned} \tag{4.1}
$$

at the point x. Then it follows that

$$
-\frac{65}{9}tSf \le -\frac{65(2A-5B)}{65+76\alpha}. \tag{4.2}
$$

On the other hand, we have

$$
\frac{3}{2}Sf \le S(S-1)(S-2) + 3(A-2B). \tag{4.3}
$$

Combining (4.2) and (4.3), we obtain

$$
\begin{aligned}
&\frac{3}{2}[1 - \frac{130}{27}t]Sf \\
&\le S(S-1)(S-2) + 3(A-2B) - \frac{65(2A-5B)}{65+76\alpha} \\
&= S(S-1)(S-2) + 4(A-B) - \frac{65(3A-3B)}{65+76\alpha} - \frac{76\alpha(A+2B)}{65+76\alpha} \\
&\le S(S-1)(S-2) + [4 - \frac{195}{65+76\alpha}](A-2B) \qquad (\text{Since } A+2B \ge 0) \\
&\le S(S-1)(S-2) + [4 - \frac{195}{65+76\alpha}]\frac{1-\alpha}{3}(\lambda_1-\lambda_2)^2 tS^2.
\end{aligned}
$$

Letting $y = 65 + 76\alpha$, we get

$$\begin{aligned}(4-\frac{195}{65+76\alpha})(1-\alpha) &= \frac{1}{76}(4-\frac{195}{y})(141-y)\\ = \frac{1}{76}(564+195-\frac{195\times 141}{y}-4y) &\le \frac{1}{76}(759-2\sqrt{4\times 195\times 141}) \equiv 3\gamma_1,\end{aligned}$$

where $\gamma_1 = 0.4198\cdots < 0.42$.

Since we assume $t \le \frac{1}{6}$, that is, $1 \le S \le 1+\frac{1}{5} = \frac{6}{5}$, we have that

$$S(S-1)(S-2) + \frac{3\gamma_1}{3}(\lambda_1-\lambda_2)^2 tS^2 \ge \frac{3}{2}(1-\frac{130}{27}t)\frac{(\lambda_1-\lambda_2)^2}{\lambda_1^2+\lambda_2^2}(\lambda_1\lambda_2)^2 S. \tag{4.4}$$

We next consider two cases:

Case 1 $(\lambda_1(x)\lambda_2(x) \ge 0)$. We see from (4.4) that

$$S(S-1)(S-2) \ge -\gamma_1(\lambda_1-\lambda_2)^2 tS^2 \ge -S\gamma_1 tS^2,$$

that is,

$$S-2 \ge -\gamma_1 S.$$

Then

$$S \ge \frac{2}{1+\gamma_1} \ge \frac{2}{1+0.42} > 1.4 > 1.2 = \frac{6}{5}.$$

Case 2 $(\lambda_1(x)\lambda_2(x) < 0)$. From (4.4), we obtain

$$\begin{aligned}(S-1)(S-2)+\gamma_1 StS &\ge (S-1)(S-2)+\gamma_1(\lambda_1^2+\lambda_2^2)tS\\ &\ge 2\gamma_1\lambda_1\lambda_2 tS + \frac{3}{2}(1-\frac{130}{27}t)(\lambda_1\lambda_2)^2\\ &\ge 2\gamma_1\lambda_1\lambda_2 tS + \frac{3}{2}(1-\frac{130}{27}\frac{1}{6})(\lambda_1\lambda_2)^2\\ &\ge -\frac{4\gamma_1^2 t^2 S^2}{4[\frac{3}{2}\times(1-\frac{130}{27\times 6})]},\end{aligned}$$

that is,

$$S \ge \frac{16+27\gamma_1^2}{8+8\gamma_1+27\gamma_1^2} > 1.286 > 1.2 = \frac{6}{5}.$$

Hence we have proved

$$S > 1+\frac{1}{5} = \frac{6}{5}.$$

Part II (Claim: $S > \frac{1}{0.802} > 1.24688$ if $S > \frac{6}{5}$). Letting $c = \frac{9}{5}$ and applying Lemma 3.1, we have

$$
\begin{aligned}
0 &= (S-1)[\frac{5}{9}Sf_4 - f_3^2] + (2\sum_{i,j,k}\lambda_i h_{ijk}^2)f_3 \\
&\quad - \frac{5S}{9}(2A+B) + 3\sum_j(\sum_i \lambda_i^2 h_{iij})^2 \\
&\leq (S-1)[\frac{5}{9}Sf_4 - f_3^2] + \frac{9}{16(S-1)}(2\sum_{i,j,k}\lambda_i h_{ijk}^2)^2 \\
&\quad + \frac{4(S-1)}{9}f_3^2 - \frac{5S}{9}(2A+B) + 3\sum_j(\sum_i \lambda_i^2 h_{iij})^2 \\
&\leq \frac{5tS}{9}(Sf) + \frac{9}{16tS}(2\sum_{i,j,k}\lambda_i h_{ijk}^2)^2 - \frac{5}{9}S(2A+B) + 3\sum_j(\sum_i \lambda_i^2 h_{iij})^2 \\
&\leq \frac{5}{9}tS^2 f + \frac{1}{9\times 16tS}(2\sum_{i,j,k}\lambda_i h_{ijk}^2)^2 - \frac{5}{9}S(2A+B) \qquad (4.5)\\
&\quad + \frac{5}{9}\left[\frac{4}{3}(A+2B)S - \frac{16}{9}\sum_k(\sum_i \lambda_i^2 h_{iik})^2\right] + 3\sum_j(\sum_i \lambda_i^2 h_{iij})^2 \\
&\leq \frac{5}{9}tS^2 f + \frac{1}{144tS}(2\sum_{i,j,k}\lambda_i h_{ijk}^2)^2 + \frac{20}{27}(A+2B)S \\
&\quad - \frac{5}{9}(2A+B)S + \frac{163}{81\times 3}(1+2\alpha)tS^2 f \\
&= \frac{5}{9}tS^2 f + \frac{1}{144tS}(2\sum_{i,j,k}\lambda_i h_{ijk}^2)^2 - \frac{5(2A-5B)S}{27} \\
&\quad + \frac{163}{81\times 3}(1+2\alpha)tS^2 f,
\end{aligned}
$$

at the point x, that is,

$$
0 \leq 3tSf + \frac{3}{80tS^2}(2\sum_{i,j,k}\lambda_i h_{ijk}^2)^2 - (2A-5B) + \frac{163}{81\times 3} \times \frac{27}{5}(1+2\alpha)tSf. \tag{4.6}
$$

Then

$$
-\frac{298+326\alpha}{45}tSf \leq \frac{3}{80tS^2}(2\sum_{i,j,k}\lambda_i h_{ijk}^2)^2 - (2A-5B). \tag{4.7}
$$

Since

$$
\begin{aligned}
(2\sum_{i,j,k}\lambda_i h_{ijk}^2)^2 &\leq 4S^2[\sum_i h_{iiii}^2 + \frac{1}{4}\sum_{i\neq j}(h_{ijij}+h_{jiji})^2] \\
&\leq 4S^2[\sum_i h_{iiii}^2 + \frac{3}{4}\sum_{i\neq j}(h_{ijij}+h_{jiji})^2] \qquad (4.8)\\
&\leq 4S^2[\sum_{i,j,k,l} h_{ijkl}^2 - \frac{3}{2}Sf] \\
&= 4S^2[S(S-1)(S-2) + 3(A-2B) - \frac{3}{2}Sf],
\end{aligned}
$$

one obtains

$$\frac{3}{2}Sf + \frac{(2\sum_{i,j,k}\lambda_i h_{ijk}^2)^2}{4S^2} \leq S(S-1)(S-2) + 3(A-2B). \tag{4.9}$$

We now assume $\frac{1}{6} < t \leq 0.198$, that is, $S \leq \frac{1}{0.802}$. Then we will get a contradiction.

From (4.7), we have

$$-\frac{298}{225}Sf \leq \frac{9}{40S^2}(2\sum_{i,j,k}\lambda_i h_{ijk}^2)^2 - \frac{298}{298+326\alpha}(2A-5B). \tag{4.10}$$

Noting $A+2B \geq 0$, we see from (4.9) and (4.10) that

$$\begin{aligned}\frac{79}{450}Sf &\leq S(S-1)(S-2) + \left[4 - \frac{3\times 298}{298+326\alpha}\right](A-B)\\ &\leq S(S-1)(S-2) + \left[4 - \frac{3\times 298}{298+326\alpha}\right]\frac{(\lambda_1-\lambda_2)^2}{3}tS^2(1-\alpha).\end{aligned} \tag{4.11}$$

On the other hand,

$$\begin{aligned}&\frac{1}{3}\left(4 - \frac{3\times 298}{298+326\alpha}\right)(1-\alpha)\\ &= \frac{1}{3\times 326}\left(4 - \frac{3\times 298}{Z}\right)(624-Z)\\ &= \frac{1}{3\times 326}(2496+894-4Z-\frac{3\times 298\times 624}{Z})\\ &\leq \frac{1}{3\times 326}(2496+894-2\sqrt{4\times 3\times 298\times 624})\\ &\equiv \gamma_2 = 0.41146\cdots < 0.4115,\end{aligned} \tag{4.12}$$

where $Z = 298+326\alpha$.

From (4.11), we have

$$\begin{aligned}0 &\leq S(S-1)(S-2) + \gamma_2(\lambda_1-\lambda_2)^2S^2t - \frac{79}{450}Sf\\ &\leq S(S-1)(S-2) + \gamma_2(\lambda_1-\lambda_2)^2S^2t\\ &\quad - \frac{79}{450}S\frac{(\lambda_1-\lambda_2)^2}{\lambda_1^2+\lambda_2^2}(\lambda_1\lambda_2)^2.\end{aligned} \tag{4.13}$$

Then it follows that

$$S(S-1)(S-2) \geq (\lambda_1-\lambda_2)^2\left(-\gamma_2 tS^2 + \frac{79}{450}(\lambda_1\lambda_2)^2\right). \tag{4.14}$$

We next consider two cases:

Case 1 ($\lambda_1(x)\lambda_2(x) > 0$). From (4.14), we have

$$S(S-1)(S-2) \geq (\lambda_1-\lambda_2)^2(-\gamma_2 tS^2) \geq -S\gamma_2 tS^2,$$

that is,

$$S-2 \geq -\gamma_2 S.$$

Then

$$S \geq \frac{2}{1+\gamma_2} \geq \frac{2}{1+0.42} > \frac{1}{0.802}.$$

Case 2 ($\lambda_1(x)\lambda_2(x) \le 0$). From (4.13), we obtain

$$\begin{aligned} S(S-1)(S-2)+\gamma_2 StS^2 &\ge S(S-1)(S-2)+\gamma_2(\lambda_1^2+\lambda_2^2)S^2t \\ &\ge \frac{79}{450}S\frac{(\lambda_1-\lambda_2)^2}{\lambda_1^2+\lambda_2^2}(\lambda_1\lambda_2)^2+\gamma_2(2\lambda_1\lambda_2)tS^2 \\ &\ge \frac{79}{450}S(\lambda_1\lambda_2)^2+2\gamma_2\lambda_1\lambda_2 tS^2 \\ &\ge -\frac{(2\gamma_2 tS^2)^2}{4\times\frac{79}{450}S} = -\frac{450}{79}\gamma_2^2t^2S^3, \end{aligned}$$

that is,

$$S \ge \frac{2+\frac{450}{79}\gamma_2^2}{1+\gamma_2+\frac{450}{79}\gamma_2^2} > 1.247456 > 1.2469 > \frac{1}{0.802}.$$

It is a contradiction, hence we have proved

$$S > \frac{1}{0.802}.$$

Part III (Claim: $S > \frac{10}{7}$ if $S > \frac{1}{0.802}$)**.** Before we prove the above claim, we will prove the following lemma.

Lemma 4.1. *Let M be an n-dimensional complete self-shrinker without boundary and with polynomial volume growth in $\mathbb{R}^{n+1}$. If the squared norm S of the second fundamental form is constant, then for any constant $\delta > 0$, $c_0 \ge 0$ and c_1 satisfying*

$$(\beta+t)c_0\delta = (\delta-1+\delta c_0)^2, \tag{4.15}$$

and $\beta \ge 0$, there exists a point $p_0 \in M$ such that, at p_0,

$$\begin{aligned} &tS^2(S-2) \\ &\ge (2-\delta t+c_1\delta)Sf-(5-2\delta+c_1\delta+\frac{\beta}{3})A+(6+\delta+2c_1\delta-\frac{2}{3}\beta)B \\ &\quad+\left[4\sqrt{\frac{2\beta}{3t}}-\frac{2}{t}-3(1+c_0)\delta\right]\frac{1}{S}\sum_k(\sum_i\lambda_i^2h_{iik})^2. \end{aligned} \tag{4.16}$$

Proof. From [6], we have

$$\int_M (A-2B-Sf)e^{-\frac{|X|^2}{2}}dv = 0. \tag{4.17}$$

Then for any constant c_1, we have

$$\int_M c_1S(A-2B)e^{-\frac{|X|^2}{2}}dv = \int_M c_1S^2fe^{-\frac{|X|^2}{2}}dv \tag{4.18}$$

since S is constant. From (3.3), we have

$$\begin{aligned} &\int_M (1-S)(cf_3^2-Sf_4)e^{-\frac{|X|^2}{2}}dv \\ &= \int_M [(2A+B)S-2cf_3\sum_{i,j,k}\lambda_ih_{ijk}^2-3c\sum_j(\sum_i\lambda_i^2h_{iij})^2]e^{-\frac{|X|^2}{2}}dv. \end{aligned} \tag{4.19}$$

Then

$$\begin{aligned}
&\int_M (c_1S^2f - tS^2f_4 + ctSf_3^2)e^{-\frac{|X|^2}{2}}dv\\
(4.20)\qquad &= \int_M \Big\{2cf_3\sum_{i,j,k}\lambda_i h_{ijk}^2 - (2A+B)S + 3c\sum_j(\sum_i \lambda_i^2 h_{iij})^2\\
&+ c_1S(A-2B)\Big\}e^{-\frac{|X|^2}{2}}dv.
\end{aligned}$$

Thus we have that there exists a point $p_0 \in M$ such that, at p_0,

$$\begin{aligned}
&c_1S^2f - tS^2f_4 + ctSf_3^2\\
(4.21)\qquad &= 2cf_3\sum_{i,j,k}\lambda_i h_{ijk}^2 - (2A+B)S + 3c\sum_j(\sum_i \lambda_i^2 h_{iij})^2 + c_1S(A-2B).
\end{aligned}$$

Then

$$\begin{aligned}
&c_1S^2f - tS(Sf_4 - f_3^2)\\
(4.22)\qquad &= (1-c)tSf_3^2 + 2cf_3\sum_{i,j,k}\lambda_i h_{ijk}^2 - (2A+B)S\\
&+ 3c\sum_j(\sum_i \lambda_i^2 h_{iij})^2 + c_1S(A-2B).
\end{aligned}$$

Putting $c = 1 + c_0$ with $c_0 \geq 0$, we get

$$\begin{aligned}
(4.23)\qquad (c_1S^2 - tS^2)f &= -c_0tSf_3^2 + 2(c_0+1)f_3\sum_{i,j,k}\lambda_i h_{ijk}^2 - (2A+B)S\\
&+ 3(1+c_0)\sum_j(\sum_i \lambda_i^2 h_{iij})^2 + c_1S(A-2B).
\end{aligned}$$

For any positive constant $\delta > 0$, we have from (4.22),

$$\begin{aligned}
\delta tSf &= c_1\delta Sf + c_0\delta tf_3^2 - 2(1+c_0)\frac{f_3}{S}\delta\sum_{i,j,k}\lambda_i h_{ijk}^2\\
(4.24)\qquad &+ (2\delta - c_1\delta)A + (\delta + 2c_1\delta)B\\
&- 3(1+c_0)\delta\frac{1}{S}\sum_j(\sum_i \lambda_i^2 h_{iij})^2.
\end{aligned}$$

Putting

$$(4.25)\qquad u_{ijkl} = \frac{1}{4}(h_{ijkl} + h_{jkli} + h_{klij} + h_{lijk}),$$

by a direct computation, we have

$$\begin{aligned}
\sum_{i,j,k,l}h_{ijkl}^2 &\geq \sum_{i,j,k,l}u_{ijkl}^2 + \frac{3}{4}\sum_{i,j}(h_{iijj} - h_{jjii})^2\\
(4.26)\qquad &= \sum_{i,j,k,l}u_{ijkl}^2 + \frac{3}{4}\sum_{i,j}(\lambda_i - \lambda_j)^2\lambda_i^2\lambda_j^2\\
&= \sum_{i,j,k,l}u_{ijkl}^2 + \frac{3}{2}(Sf_4 - f_3^2).
\end{aligned}$$

From (2.10) and (4.26), we obtain

$$S(S-1)(S-2)+3(A-2B)\geq\sum_{i,j,k,l}u_{ijkl}^2+\frac{3}{2}Sf. \tag{4.27}$$

By a direct calculation, we can get

$$\begin{aligned}\sum_{i,j,k,l}u_{ijkl}^2\geq&\frac{1}{2}Sf+\frac{4}{tS^2}\sum_i\lambda_i^2(\sum_j\lambda_j^2h_{jji})^2-2A\\&+\frac{2f_3}{S}\sum_{i,j,k}\lambda_ih_{ijk}^2+\frac{1}{S^2}(\sum_{i,j,k}\lambda_ih_{ijk}^2)^2.\end{aligned} \tag{4.28}$$

Combining (4.27) and (4.28), we have

$$\begin{aligned}&S(S-1)(S-2)+3(A-2B)\\&\geq 2Sf-2A+\frac{4}{tS^2}\sum_i\lambda_i^2(\sum_j\lambda_j^2h_{jji})^2\\&+\frac{2f_3}{S}\sum_{i,j,k}\lambda_ih_{ijk}^2+\frac{1}{S^2}(\sum_{i,j,k}\lambda_ih_{ijk}^2)^2.\end{aligned} \tag{4.29}$$

From (4.24), one has

$$\begin{aligned}&\delta tSf+\frac{4}{tS^2}\sum_i\lambda_i^2(\sum_j\lambda_j^2h_{jji})^2+\frac{2f_3}{S}\sum_{i,j,k}\lambda_ih_{ijk}^2+\frac{1}{S^2}(\sum_{i,j,k}\lambda_ih_{ijk}^2)^2\\&=c_1\delta Sf+c_0\delta tf_3^2-\frac{2[(1+c_0)\delta-1]f_3}{S}\sum_{i,j,k}\lambda_ih_{ijk}^2\\&+\frac{1}{S^2}(\sum_{i,j,k}\lambda_ih_{ijk}^2)^2+(2\delta-c_1\delta)A+(\delta+2c_1\delta)B\\&+\sum_j(\frac{4\lambda_j^2}{tS}-3(1+c_0)\delta)\frac{1}{S}(\sum_i\lambda_i^2h_{iij})^2\\&\geq c_1\delta Sf+\left[t-\frac{[(1+c_0)\delta-1]^2}{c_0\delta}\right]\frac{1}{tS^2}(\sum_{i,j,k}\lambda_ih_{ijk}^2)^2\\&+(2\delta-c_1\delta)A+(\delta+2c_1\delta)B\\&+\frac{1}{S}\sum_j(\frac{4\lambda_j^2}{tS}-3(1+c_0)\delta)(\sum_i\lambda_i^2h_{iij})^2.\end{aligned} \tag{4.30}$$

Taking δ and c, such that,

$$(\beta+t)c_0\delta=[(c_0+1)\delta-1]^2, \tag{4.31}$$

with $\beta \geq 0$, we have from Lemma 3.5

$$
\begin{aligned}
&\delta tSf + \frac{4}{tS^2}\sum_i \lambda_i^2(\sum_j \lambda_j^2 h_{jji})^2 + \frac{2f_3}{S}\sum_{i,j,k}\lambda_i h_{ijk}^2 + \frac{1}{S^2}(\sum_{i,j,k}\lambda_i h_{ijk}^2)^2 \\
&\geq c_1\delta Sf - \beta\Big[\frac{1}{3}(A+2B) - \frac{4}{3}\sum_k \frac{1}{S+2\lambda_k^2}(\sum_i \lambda_i^2 h_{iik})^2\Big] \\
&\quad + (2\delta - c_1\delta)A + (\delta + 2c_1\delta)B \\
&\quad + \frac{1}{S}\sum_j\Big(\frac{4\lambda_j^2}{tS} - 3(1+c_0)\delta\Big)(\sum_i \lambda_i^2 h_{iij})^2 \\
&= c_1\delta Sf + (2\delta - c_1\delta - \frac{\beta}{3})A + (\delta + 2c_1\delta - \frac{2\beta}{3})B \\
&\quad + \sum_k\Big[\frac{4}{3}\beta\frac{1}{S+2\lambda_k^2} + \frac{4\lambda_k^2}{tS^2} - \frac{3(1+c_0)\delta}{S}\Big](\sum_i \lambda_i^2 h_{iik})^2 \\
&\geq c_1\delta Sf + (2\delta - c_1\delta - \frac{\beta}{3})A + (\delta + 2c_1\delta - \frac{2\beta}{3})B \\
&\quad + \Big[4\sqrt{\frac{2\beta}{3t}} - \frac{2}{t} - 3(1+c_0)\delta\Big]\frac{1}{S}\sum_k(\sum_i \lambda_i^2 h_{iik})^2.
\end{aligned} \tag{4.32}
$$

From (4.29), we have

$$
\begin{aligned}
&tS^2(S-2) + 3(A-2B) \\
&\geq 2Sf - 2A - \delta tSf + c_1\delta Sf + (2\delta - c_1\delta - \frac{\beta}{3})A + (\delta + 2c_1\delta - \frac{2\beta}{3})B \\
&\quad + \Big[4\sqrt{\frac{2\beta}{3t}} - \frac{2}{t} - 3(1+c_0)\delta\Big]\frac{1}{S}\sum_k(\sum_i \lambda_i^2 h_{iik})^2,
\end{aligned} \tag{4.33}
$$

that is,

$$
\begin{aligned}
&tS^2(S-2) \\
&\geq (2 - \delta t + c_1\delta)Sf - (5 - 2\delta + c_1\delta + \frac{\beta}{3})A \\
&\quad + (6 + \delta + 2c_1\delta - \frac{2\beta}{3})B \\
&\quad + \Big[4\sqrt{\frac{2\beta}{3t}} - \frac{2}{t} - 3(1+c_0)\delta\Big]\frac{1}{S}\sum_k(\sum_i \lambda_i^2 h_{iik})^2.
\end{aligned} \tag{4.34}
$$

□

Taking $6 + \delta + 2c_1\delta - \frac{2\beta}{3} = 5 - 2\delta + c_1\delta + \frac{\beta}{3}$, we have from (4.15) that $\beta = c_1\delta + 3\delta + 1$, $(\beta + t)c_0\delta = ((c_0+1)\delta - 1)^2$. Taking $\delta = \frac{17}{5}$, $c_0 = \frac{6}{17}$ and applying Lemma 4.1, we obtain $\beta = \frac{54}{5} - t$, $c_1 = -\frac{2}{17} - \frac{5}{17}t$,

$$
\begin{aligned}
&tS^2(S-2) \\
&\geq (\frac{8}{5} - \frac{22t}{5})Sf - (\frac{7}{5} - \frac{4t}{3})(A-B) \\
&\quad + \frac{1}{S}\Big[4\sqrt{\frac{2}{3}(\frac{54}{5t} - 1)} - \frac{2}{t} - \frac{69}{5}\Big]\sum_k(\sum_i \lambda_i^2 h_{iik})^2.
\end{aligned} \tag{4.35}
$$

Putting $g_1(t) = 4\sqrt{\frac{2}{3}(\frac{54}{5t}-1)} - \frac{2}{t} - \frac{69}{5}$, we can obtain that

$$g_1(t) < 0,$$

when $t > 0.1978$. Since

$$(g_1(t))^{'} = \frac{2}{t^2} - \frac{36\sqrt{6}}{5\sqrt{-1+\frac{54}{5t}}t^2} < 0 \tag{4.36}$$

when $1 > t > 0.14$, we have $g_1(t) \leq g_1(0.1978) < 0$ when $0.1978 < t \leq \frac{3}{10}$.

From Lemma 3.3 and Lemma 3.4, we have

$$\begin{aligned}
& tS^2(S-2) \\
\geq\; & (\frac{8}{5} - \frac{22t}{5})Sf - (\frac{7}{5} - \frac{4t}{3})(A-B) \\
& + \frac{1}{S}\left[4\sqrt{\frac{2}{3}(\frac{54}{5t}-1)} - \frac{2}{t} - \frac{69}{5}\right]\sum_k(\sum_i \lambda_i^2 h_{iik})^2 \\
\geq\; & (\frac{8}{5} - \frac{22t}{5})Sf - (\frac{7}{5} - \frac{4t}{3})\frac{1-\alpha}{3}(\lambda_1-\lambda_2)^2 tS^2 \\
& + \left[4\sqrt{\frac{2}{3}(\frac{54}{5t}-1)} - \frac{2}{t} - \frac{69}{5}\right]\frac{1+2\alpha}{3}tSf \\
=\; & -(\frac{7}{15} - \frac{4t}{9})(1-\alpha)(\lambda_1-\lambda_2)^2 tS^2 \\
& + \left\{\left[\frac{8}{5t} + \frac{4}{3}\sqrt{\frac{2}{3}(\frac{54}{5t}-1)} - \frac{2}{3t} - 9\right]\right. \\
& \left. + \left[\frac{8}{3}\sqrt{\frac{2}{3}(\frac{54}{5t}-1)} - \frac{4}{3t} - \frac{46}{5}\right]\alpha\right\} tSf.
\end{aligned} \tag{4.37}$$

If $t > \frac{3}{10}$, the result is obviously true. If $t \leq \frac{3}{10}$, we will obtain a contradiction. In this case, we have $0.198 \leq t \leq \frac{3}{10}$. Putting

$$a(t) = \frac{8}{5t} + \frac{4}{3}\sqrt{\frac{2}{3}(\frac{54}{5t}-1)} - \frac{2}{3t} - 9, \tag{4.38}$$

$$b(t) = -\frac{8}{3}\sqrt{\frac{2}{3}(\frac{54}{5t}-1)} + \frac{4}{3t} + \frac{46}{5}, \tag{4.39}$$

we have

$$\begin{aligned}
& tS^2(S-2) \\
\geq\; & -(\frac{7}{15} - \frac{4t}{9})(1-\alpha)(\lambda_1-\lambda_2)^2 tS^2 + [a(t) - b(t)\alpha]tSf.
\end{aligned} \tag{4.40}$$

Since $a^{'}(t) = -\frac{14}{15t^2} - \frac{12\sqrt{6}}{5\sqrt{-1+\frac{54}{5t}}t^2} < 0$, we have

$$a(t) \geq a(\frac{3}{10}) = -\frac{52}{9} + \frac{4}{3}\sqrt{\frac{70}{3}} \approx 0.662834 > 0. \tag{4.41}$$

Since $b'(t) = -\frac{4}{3t^2} + \frac{24\sqrt{6}}{5\sqrt{-1+\frac{54}{5t}t^2}} > 0$ if $t > 0.14$, we have that $b(t)$ is an increasing function of $t \in [0.198, \frac{3}{10}]$. Then

$$b(t) \le b(\frac{3}{10}) = \frac{614}{45} - \frac{8}{3}\sqrt{\frac{70}{3}} \approx 0.763221 > 0. \tag{4.42}$$

Therefore we get

$$\begin{aligned} & tS^2(S-2) \\ & \ge -(\frac{7}{15} - \frac{4t}{9})(1-\alpha)(\lambda_1 - \lambda_2)^2 tS^2 + [a(\frac{3}{10}) - b(\frac{3}{10})\alpha]tSf. \end{aligned} \tag{4.43}$$

We next consider two cases:

Case 1 ($a(\frac{3}{10}) - b(\frac{3}{10})\alpha \le 0$). In this case $\frac{a(\frac{3}{10})}{b(\frac{3}{10})} \le \alpha \le 1$. Since λ_1, λ_2 are the maximum and minimum of the principal curvatures at any point of M, we obtain, for any j,

$$\lambda_j + \lambda_1 \ge \lambda_2 + \lambda_1,$$
$$(\lambda_1 - \lambda_j)(\lambda_1 + \lambda_j) \ge (\lambda_1 - \lambda_j)(\lambda_1 + \lambda_2).$$

So we get

$$\lambda_j^2 - (\lambda_1 + \lambda_2)\lambda_j \le -\lambda_1\lambda_2,$$

and

$$f_4 - (\lambda_1 + \lambda_2)f_3 \le -\lambda_1\lambda_2 S.$$

Then

$$Sf = Sf_4 - f_3^2 \le -f_3^2 + (\lambda_1 + \lambda_2)Sf_3 - \lambda_1\lambda_2 S^2, \tag{4.44}$$

$$Sf \le \frac{(\lambda_1 - \lambda_2)^2}{4} S^2. \tag{4.45}$$

From (4.43) and (4.45), we have

$$\begin{aligned} & tS^2(S-2) \\ & \ge -(\frac{7}{15} - \frac{4t}{9})(1-\alpha)(\lambda_1 - \lambda_2)^2 tS^2 + [a(\frac{3}{10}) - b(\frac{3}{10})\alpha]\frac{(\lambda_1 - \lambda_2)^2}{4} tS^2. \end{aligned} \tag{4.46}$$

Since $a(\frac{3}{10}) - b(\frac{3}{10})\alpha \le 0$, using $-2\lambda_1\lambda_2 \le \lambda_1^2 + \lambda_2^2 \le S$, we see from (4.46)

$$\begin{aligned} S - 2 \ge \Big\{ & -2(\frac{7}{15} - \frac{4t}{9}) + \frac{1}{2}a(\frac{3}{10}) \\ & + \Big[2(\frac{7}{15} - \frac{4t}{9}) - \frac{1}{2}b(\frac{3}{10})\Big]\alpha \Big\} S. \end{aligned} \tag{4.47}$$

Since

$$\begin{aligned} & 2(\frac{7}{15} - \frac{4t}{9}) - \frac{1}{2}b(\frac{3}{10}) \\ & \ge 2(\frac{7}{15} - \frac{4}{9} \times \frac{3}{10}) - \frac{1}{2} \times 0.77 = \frac{2}{3} - \frac{1}{2} \times 0.77 > 0, \end{aligned} \tag{4.48}$$

we have from (4.47) that

$$S - 2 \ge -2(\frac{7}{15} - \frac{4t}{9})\Big[1 - \frac{a(\frac{3}{10})}{b(\frac{3}{10})}\Big] S. \tag{4.49}$$

On the other hand,

$$\frac{a(\frac{3}{10})}{b(\frac{3}{10})} \approx 0.86847 > 0.86.$$

Then from (4.49), we see

$$\begin{aligned} S-2 &\geq -2(\frac{7}{15}-\frac{4t}{9})(1-0.86)S \\ &\geq -2(\frac{7}{15}-\frac{4}{9}\times 0.198)\times 0.14S \\ &> 0.1061S, \end{aligned} \tag{4.50}$$

hence

$$S > \frac{2}{1+0.1061} > 1.8 > \frac{10}{7}. \tag{4.51}$$

This is impossible.

Case 2 ($a(\frac{3}{10})-b(\frac{3}{10})\alpha > 0$). In this case $\frac{a(\frac{3}{10})}{b(\frac{3}{10})} > \alpha \geq 0$. From Lemma 3.2 and (4.43), we obtain

$$\begin{aligned} &tS^2(S-2) \\ &\geq -(\frac{7}{15}-\frac{4t}{9})(1-\alpha)(\lambda_1-\lambda_2)^2 tS^2 + \left[a(\frac{3}{10})-b(\frac{3}{10})\alpha\right] tSf \\ &\geq -(\frac{7}{15}-\frac{4t}{9})(1-\alpha)(\lambda_1-\lambda_2)^2 tS^2 \\ &\quad + \left[a(\frac{3}{10})-b(\frac{3}{10})\alpha\right]\frac{(\lambda_1-\lambda_2)^2}{\lambda_1^2+\lambda_2^2}(\lambda_1\lambda_2)^2 tS. \end{aligned} \tag{4.52}$$

Putting $y=-\frac{\lambda_1\lambda_2}{S}$, we have $-\frac{1}{2} \leq y = -\frac{\lambda_1\lambda_2}{S} \leq \frac{1}{2}\frac{\lambda_1^2+\lambda_2^2}{S} \leq \frac{1}{2}$. Then we infer from (4.52) that

$$\begin{aligned} &S-2 \\ &\geq -(\frac{7}{15}-\frac{4t}{9})(1-\alpha)(1+2y)S + \left[a(\frac{3}{10})-b(\frac{3}{10})\alpha\right](1+2y)(-y)^2 S \\ &= \Big\{\left[-(\frac{7}{15}-\frac{4t}{9})(1+2y) + a(\frac{3}{10})(1+2y)y^2\right] \\ &\quad + \left[(\frac{7}{15}-\frac{4t}{9})(1+2y) - b(\frac{3}{10})(1+2y)y^2\right]\alpha\Big\} S. \end{aligned} \tag{4.53}$$

Defining two functions $\rho(y)$ and $\varrho(y)$ by

$$\rho(y) = -(\frac{7}{15}-\frac{4t}{9})(1+2y) + a(\frac{3}{10})(1+2y)y^2, \tag{4.54}$$

$$\varrho(y) = (\frac{7}{15}-\frac{4t}{9})(1+2y) - b(\frac{3}{10})(1+2y)y^2. \tag{4.55}$$

Since $n > 2$, we have $1 + 2y > 0$. Then

$$\begin{aligned} \varrho(y) &= (\frac{7}{15} - \frac{4t}{9})(1+2y) - b(\frac{3}{10})(1+2y)y^2 \\ &= (1+2y)[\frac{7}{15} - \frac{4t}{9} - b(\frac{3}{10})y^2] \\ &> (1+2y)[\frac{7}{15} - \frac{4}{9} \times \frac{3}{10} - 0.7633 \times \frac{1}{4}] \\ &= (1+2y) \times 0.142508 > 0, \end{aligned} \tag{4.56}$$

$$\begin{aligned} \rho(y) &= -(\frac{7}{15} - \frac{4t}{9})(1+2y) + a(\frac{3}{10})(1+2y)y^2 \\ &= (1+2y)[-\frac{7}{15} + \frac{4t}{9} + a(\frac{3}{10})y^2] \\ &< (1+2y)[-\frac{7}{15} + \frac{4}{9} \times \frac{3}{10} + 0.663 \times \frac{1}{4}] \\ &= (1+2y) \times (-0.1676) < 0. \end{aligned} \tag{4.57}$$

By a direct calculation, we obtain

$$\begin{aligned} \rho^{'}(y) &= -2(\frac{7}{15} - \frac{4t}{9}) + a(\frac{3}{10})[2y + 6y^2] \\ &= -2[\frac{7}{15} - \frac{4t}{9} + a(\frac{3}{10})y + 3a(\frac{3}{10})y^2] \\ &< -2[\frac{7}{15} - \frac{4}{9} \times \frac{3}{10} - \frac{1}{12}a(\frac{3}{10})] \\ &< -2[\frac{1}{3} - \frac{1}{12} \times 0.663] < 0. \end{aligned} \tag{4.58}$$

It follows that

$$\rho(y) \geq \rho(\frac{1}{2}) = -2(\frac{7}{15} - \frac{4t}{9}) + \frac{1}{2}a(\frac{3}{10}). \tag{4.59}$$

From the above arguments, we have

$$\begin{aligned} S - 2 &\geq (\rho(y) + \varrho(y)\alpha)S \\ &\geq (-2(\frac{7}{15} - \frac{4t}{9}) + \frac{1}{2}a(\frac{3}{10}))S \\ &> (-\frac{14}{15}S + \frac{8}{9}(S-1) + \frac{1}{2} \times 0.66 \times S) \\ &= -\frac{2}{45}S + 0.33S - \frac{8}{9} > 0.28S - \frac{8}{9}. \end{aligned} \tag{4.60}$$

Then

$$S > \frac{\frac{10}{9}}{1 - 0.28} > 1.54 > \frac{10}{7}. \tag{4.61}$$

It is a contradiction.

Hence, we have $t > \frac{3}{10}$, that is, $S > \frac{10}{7}$ if $S > 1$. This completes the proof of Theorem 1.1. $\square$

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