Calc. Var.
DOI 10.1007/s00526-011-0399-6

Calculus of Variations

A lower bound for eigenvalues of a clamped plate problem

Qing-Ming Cheng · Guoxin Wei

Received: 12 May 2010 / Accepted: 18 December 2010

Abstract In this paper, we study eigenvalues of a clamped plate problem. We obtain a lower bound for eigenvalues, which gives an important improvement of results due to Levine and Protter.

Mathematics Subject Classification (2000) 35P15

1 Introduction

Let Ω be a bounded domain with piecewise smooth boundary $\partial\Omega$ in an n-dimensional Euclidean space $\mathbf{R}^n$. The following is called *Dirichlet eigenvalue problem of Laplacian*:

$$\begin{cases} \Delta u = -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

It is well known that the spectrum of this eigenvalue problem (1.1) is real and discrete (cf. [5,8,2,13]).

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty,$$

where each λ_i has finite multiplicity which is repeated according to its multiplicity.

Let $V(\Omega)$ denote the volume of Ω and let B_n denote the volume of the unit ball in $\mathbf{R}^n$. Then the following Weyl's asymptotic formula holds

Communicated by J. Jost.

Q.-M. Cheng
Department of Mathematics, Faculty of Science and Engineering, Saga University, Saga 840-8502, Japan
e-mail: cheng@ms.saga-u.ac.jp

G. Wei (✉)
School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China
e-mail: weigx03@mails.tsinghua.edu.cn

Published online: 15 January 2011

$$\lambda_k \sim \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to \infty. \tag{1.2}$$

From this asymptotic formula, one can infer

$$\frac{1}{k}\sum_{i=1}^{k} \lambda_i \sim \frac{n}{n+2}\frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to \infty. \tag{1.3}$$

Furthermore, Pólya [15] proved that

$$\lambda_k \geq \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \ldots, \tag{1.4}$$

if Ω is a tiling domain in $\mathbf{R}^n$. Moreover, he proposed the following:

Conjecture of Pólya *If Ω is a bounded domain in $\mathbf{R}^n$, then eigenvalue λ_k of the eigenvalue problem* (1.1) *satisfies*

$$\lambda_k \geq \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \ldots \tag{1.5}$$

Li and Yau [10] (cf. [4, 11]) proved the following

$$\frac{1}{k}\sum_{i=1}^{k} \lambda_i \geq \frac{n}{n+2}\frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \ldots \tag{1.6}$$

The formula (1.3) shows that the result of Li and Yau is the best possible in the sense of the average. From this formula, one can derive

$$\lambda_k \geq \frac{n}{n+2}\frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \ldots, \tag{1.7}$$

which gives a partial solution for the conjecture of Pólya with a factor $\frac{n}{n+2}$.

Furthermore, Melas [12] obtained the following estimate which is an improvement of (1.6).

$$\frac{1}{k}\sum_{i=1}^{k} \lambda_i \geq \frac{n}{n+2}\frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}} + c_n \frac{V(\Omega)}{I(\Omega)}, \quad \text{for } k = 1, 2, \ldots, \tag{1.8}$$

where c_n is a constant depending only on the dimension n and

$$I(\Omega) = \min_{a \in \mathbf{R}^n} \int_{\Omega} |x - a|^2 dx$$

is called *the moment of inertia* of Ω.

For a bounded domain in an n-dimensional complete Riemannian manifold, Cheng and Yang [7] have also given a lower bound for eigenvalues, recently.

Our purpose in this paper is to study eigenvalues of the following clamped plate problem. Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M^n. The following is called *a clamped plate problem*, which describes characteristic vibrations of a clamped plate:

$$\begin{cases} \Delta^2 u = \Gamma u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where Δ is the Laplacian on M^n and ν denotes the outward unit normal to the boundary $\partial\Omega$. It is well known that this problem has a real and discrete spectrum (cf. [6,17])

$$0 < \Gamma_1 \le \Gamma_2 \le \cdots \le \Gamma_k \le \cdots \to +\infty,$$

where each Γ_i has finite multiplicity which is repeated according to its multiplicity.

For the eigenvalues of the clamped plate problem, Agmon [1] and Pleijel [14] gave the following asymptotic formula,

$$\Gamma_k \sim \frac{16\pi^4}{\left(B_n V(\Omega)\right)^{\frac{4}{n}}} k^{\frac{4}{n}}, \quad k \to \infty. \tag{1.10}$$

This implies that

$$\frac{1}{k}\sum_{j=1}^{k} \Gamma_j \sim \frac{n}{n+4} \frac{16\pi^4}{\left(B_n V(\Omega)\right)^{\frac{4}{n}}} k^{\frac{4}{n}}, \quad k \to \infty. \tag{1.11}$$

Furthermore, Levine and Protter [9] proved that the eigenvalues of the clamped plate problem satisfy

$$\frac{1}{k}\sum_{j=1}^{k} \Gamma_j \ge \frac{n}{n+4} \frac{16\pi^4}{\left(B_n V(\Omega)\right)^{\frac{4}{n}}} k^{\frac{4}{n}}. \tag{1.12}$$

The formula (1.11) shows that the coefficient of $k^{\frac{2}{n}}$ is the best possible constant.

In this paper, we give an important improvement of the result due to Levine and Protter [9] by adding to its right hand side two terms of lower order in k. In fact, we prove the following:

Theorem *Let Ω be a bounded domain in an n-dimensional Euclidean space $\mathbf{R}^n$. The eigenvalues of the clamped plate problem satisfy*

$$\begin{aligned}\frac{1}{k}\sum_{j=1}^{k} \Gamma_j \ge\ & \frac{n}{n+4} \frac{16\pi^4}{\left(B_n V(\Omega)\right)^{\frac{4}{n}}} k^{\frac{4}{n}} \\ & + \left(\frac{n+2}{12n(n+4)} - \frac{1}{1152n^2(n+4)}\right) \frac{V(\Omega)}{I(\Omega)} \frac{n}{n+2} \frac{4\pi^2}{\left(B_n V(\Omega)\right)^{\frac{2}{n}}} k^{\frac{2}{n}} \\ & + \left(\frac{1}{576n(n+4)} - \frac{1}{27648n^2(n+2)(n+4)}\right) \left(\frac{V(\Omega)}{I(\Omega)}\right)^2,\end{aligned} \tag{1.13}$$

where $I(\Omega)$ is the moment of inertia of Ω.

2 Proof of Theorem

For a bounded domain Ω, *the moment of inertia* of Ω is defined by

$$I(\Omega) = \min_{a \in \mathbf{R}^n} \int_\Omega |x-a|^2 dx.$$

By a translation of the origin and a suitable rotation of axes, we can assume that the center of mass is the origin and

$$I(\Omega) = \int_{\Omega} |x|^2 dx.$$

For reader's convenience, we first review the definition and serval properties of the symmetric decreasing rearrangements. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain. Its *symmetric rearrangement* Ω^* is the open ball with the same volume as Ω,

$$\Omega^* = \left\{ x \in \mathbf{R}^n | \;\; |x| < \left(\frac{\mathrm{Vol}(\Omega)}{B_n} \right)^{\frac{1}{n}} \right\}.$$

By using a symmetric rearrangement of Ω, we have

$$I(\Omega) = \int_{\Omega} |x|^2 dx \geq \int_{\Omega^*} |x|^2 dx = \frac{n}{n+2} V(\Omega) \left(\frac{V(\Omega)}{B_n} \right)^{\frac{2}{n}}. \tag{2.1}$$

Let h be a nonnegative bounded continuous function on Ω. We consider its *distribution function* $\mu_h(t)$ defined by

$$\mu_h(t) = \mathrm{Vol}(\{x \in \Omega | \;\; h(x) > t\}).$$

The distribution function can be viewed as a function from $[0, \infty)$ to $[0, V(\Omega)]$. The *symmetric decreasing rearrangement* h^* *of* h is defined by

$$h^*(x) = \inf\{t \geq 0 | \mu_h(t) < B_n |x|^n\}$$

for $x \in \Omega^*$. By definition, we know that $\mathrm{Vol}(\{x \in \Omega | h(x) > t\}) = \mathrm{Vol}(\{x \in \Omega^* | h^*(x) > t\}), \;\forall t > 0$ and $h^*(x)$ is a radially symmetric function.

Putting $g(|x|) := h^*(x)$, one gets that $g : [0, +\infty) \to [0, \sup h]$ is a non-increasing function of $|x|$. Using the well known properties of the symmetric decreasing rearrangement, we obtain

$$\int_{\mathbf{R}^n} h(x) dx = \int_{\mathbf{R}^n} h^*(x) dx = n B_n \int_0^{\infty} s^{n-1} g(s) ds \tag{2.2}$$

and

$$\int_{\mathbf{R}^n} |x|^4 h(x) dx \geq \int_{\mathbf{R}^n} |x|^4 h^*(x) dx = n B_n \int_0^{\infty} s^{n+3} g(s) ds. \tag{2.3}$$

Good sources of further information on rearrangements are [3,16].

One gets from the coarea formula that

$$\mu_h(t) = \int_t^{\sup h} \int_{\{h=s\}} |\nabla h|^{-1} d\sigma_s ds.$$

Since h^* is radial, we have

$$\begin{aligned} \mu_h(g(s)) &= \mathrm{Vol}\{x \in \Omega | h(x) > g(s)\} = \mathrm{Vol}\{x \in \Omega^* | h^*(x) > g(s)\} \\ &= \mathrm{Vol}\{x \in \Omega^* | g(|x|) > g(s)\} = B_n s^n. \end{aligned}$$

It follows that

$$nB_n s^{n-1} = \mu_h'(g(s))g'(s)$$

for almost every s. Putting $\tau := \sup|\nabla h|$, we obtain from the above equations and the isoperimetric inequality that

$$\begin{aligned}-\mu_h'(g(s)) &= \int_{\{h=g(s)\}} |\nabla h|^{-1} d\sigma_{g(s)} \geq \tau^{-1}\mathrm{Vol}_{n-1}(\{h = g(s)\}) \\ &\geq \tau^{-1} n B_n s^{n-1}.\end{aligned}$$

Therefore, one obtains

$$-\tau \leq g'(s) \leq 0 \tag{2.4}$$

for almost every s.

The following lemma will be used to prove our theorem.

Lemma 2.1 *Let $b \geq 1, \eta > 0$ and $\psi : [0, +\infty) \to [0, +\infty)$ be a decreasing smooth function such that*

$$-\eta \leq \psi'(s) \leq 0$$

and, for a constant $d < 1$,

$$\frac{\psi(0)^{\frac{2b+2}{b}}}{6b\eta^2(bA)^{\frac{2}{b}}} < d$$

with

$$A := \int_0^\infty s^{b-1}\psi(s)ds > 0.$$

Then, we have

$$\begin{aligned}\int_0^\infty s^{b+3}\psi(s)ds \geq& \frac{1}{b+4}(bA)^{\frac{b+4}{b}}\psi(0)^{-\frac{4}{b}} \\ &+ \left(\frac{1}{3b(b+4)\eta^2} - \frac{d}{6(b+2)^2(b+4)\eta^2}\right)(bA)^{\frac{b+2}{b}}\psi(0)^{\frac{2b-2}{b}} \\ &+ \left(\frac{1}{36b(b+4)\eta^4} - \frac{d}{36(b+2)^2(b+4)\eta^4}\right)A\psi(0)^4.\end{aligned} \tag{2.5}$$

Proof Defining

$$D := \int_0^\infty s^{b+1}\psi(s)ds,$$

one can prove from the same assertions as in the Lemma 1 of [12],

$$D = \int_0^\infty s^{b+1}\psi(s)ds \geq \frac{1}{b+2}(bA)^{\frac{b+2}{b}}\psi(0)^{-\frac{2}{b}} + \frac{A\psi(0)^2}{6(b+2)\eta^2}. \tag{2.6}$$

Since the formula (2.6) holds for any constant $b \geq 1$, we have

$$
\begin{aligned}
&\int_0^\infty s^{b+3}\psi(s)ds \\
&\geq \frac{1}{b+4}((b+2)D)^{\frac{b+4}{b+2}}\psi(0)^{-\frac{2}{b+2}} + \frac{D\psi(0)^2}{6(b+4)\eta^2} \\
&\geq \frac{1}{b+4}\left[(bA)^{\frac{b+2}{b}}\psi(0)^{-\frac{2}{b}} + \frac{A\psi(0)^2}{6\eta^2}\right]^{\frac{b+4}{b+2}}\psi(0)^{-\frac{2}{b+2}} \\
&\quad + \frac{\psi(0)^2}{6(b+4)\eta^2}\left[\frac{1}{b+2}(bA)^{\frac{b+2}{b}}\psi(0)^{-\frac{2}{b}} + \frac{A\psi(0)^2}{6(b+2)\eta^2}\right] \\
&= \frac{1}{b+4}\left[(bA)^{\frac{b+2}{b}}\psi(0)^{-\frac{2}{b}} + \frac{A\psi(0)^2}{6\eta^2}\right]\left[(bA)^{\frac{b+2}{b}}\psi(0)^{-\frac{2}{b}}\right]^{\frac{2}{b+2}} \\
&\quad \times \left(1 + \frac{A\psi(0)^{\frac{2b+2}{b}}}{6(bA)^{\frac{b+2}{b}}\eta^2}\right)^{\frac{2}{b+2}}\psi(0)^{-\frac{2}{b+2}} \\
&\quad + \frac{1}{6(b+2)(b+4)\eta^2}(bA)^{\frac{b+2}{b}}\psi(0)^{\frac{2b-2}{b}} + \frac{A\psi(0)^4}{36(b+2)(b+4)\eta^4} \\
&\geq \frac{1}{b+4}\left[(bA)^{\frac{b+2}{b}}\psi(0)^{-\frac{2}{b}} + \frac{A\psi(0)^2}{6\eta^2}\right]\left[(bA)^{\frac{b+2}{b}}\psi(0)^{-\frac{2}{b}}\right]^{\frac{2}{b+2}} \\
&\quad \times \left\{1 + \frac{1}{b+2}\frac{A\psi(0)^{\frac{2b+2}{b}}}{6(bA)^{\frac{b+2}{b}}\eta^2}\left(2 - \frac{b}{b+2}\frac{A\psi(0)^{\frac{2b+2}{b}}}{6(bA)^{\frac{b+2}{b}}\eta^2}\right)\right\}\psi(0)^{-\frac{2}{b+2}} \\
&\qquad\qquad \text{(from the Taylor formula)} \\
&\quad + \frac{1}{6(b+2)(b+4)\eta^2}(bA)^{\frac{b+2}{b}}\psi(0)^{\frac{2b-2}{b}} + \frac{A\psi(0)^4}{36(b+2)(b+4)\eta^4} \\
&\geq \frac{1}{b+4}\left[(bA)^{\frac{b+2}{b}}\psi(0)^{-\frac{2}{b}} + \frac{A\psi(0)^2}{6\eta^2}\right]\left[(bA)^{\frac{b+2}{b}}\psi(0)^{-\frac{2}{b}}\right]^{\frac{2}{b+2}} \\
&\quad \times \left\{1 + \frac{1}{b+2}\frac{A\psi(0)^{\frac{2b+2}{b}}}{6(bA)^{\frac{b+2}{b}}\eta^2}\left(2 - \frac{b}{b+2}d\right)\right\}\psi(0)^{-\frac{2}{b+2}} \\
&\quad + \frac{1}{6(b+2)(b+4)\eta^2}(bA)^{\frac{b+2}{b}}\psi(0)^{\frac{2b-2}{b}} + \frac{A\psi(0)^4}{36(b+2)(b+4)\eta^4} \\
&= \frac{1}{b+4}(bA)^{\frac{b+4}{b}}\psi(0)^{-\frac{4}{b}} \\
&\quad + \left(\frac{1}{3b(b+4)\eta^2} - \frac{d}{6(b+2)^2(b+4)\eta^2}\right)(bA)^{\frac{b+2}{b}}\psi(0)^{\frac{2b-2}{b}} \\
&\quad + \left(\frac{1}{36b(b+4)\eta^4} - \frac{d}{36(b+2)^2(b+4)\eta^4}\right)A\psi(0)^4.
\end{aligned}
$$

This completes the proof of the lemma. □

Proof of Theorem Let u_j be an orthonormal eigenfunction corresponding to the eigenvalue Γ_j, that is, u_j satisfies

$$\begin{cases} \Delta^2 u_j = \Gamma_j u_j, & \text{in } \Omega, \\ u_j = \frac{\partial u_j}{\partial \nu} = 0, & \text{on } \partial\Omega, \\ \int_\Omega u_i(x) u_j(x) dx = \delta_{ij}, & \text{for any } i, j. \end{cases} \tag{2.7}$$

Thus, $\{u_j\}_{j=1}^{\infty}$ forms an orthonormal basis of $L^2(\Omega)$. We define a function φ_j by

$$\varphi_j(x) = \begin{cases} u_j(x), & x \in \Omega, \\ 0, & x \in \mathbf{R}^n \backslash \Omega. \end{cases}$$

Denote by $\widehat{\varphi}_j(z)$ the Fourier transform of $\varphi_j(x)$. For any $z \in \mathbf{R}^n$, we have by definition that

$$\widehat{\varphi}_j(z) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} \varphi_j(x) e^{i<x,z>} dx = (2\pi)^{-n/2} \int_{\Omega} u_j(x) e^{i<x,z>} dx. \tag{2.8}$$

From the Plancherel formula, we have

$$\int_{\mathbf{R}^n} \widehat{\varphi}_i(z) \widehat{\varphi}_j(z) dz = \delta_{ij}$$

for any i, j. Since $\{u_j\}_{j=1}^{\infty}$ is an orthonormal basis in $L^2(\Omega)$, the Bessel inequality implies that

$$\sum_{j=1}^{k} |\widehat{\varphi}_j(z)|^2 \le (2\pi)^{-n} \int_{\Omega} |e^{i<x,z>}|^2 dx = (2\pi)^{-n} V(\Omega). \tag{2.9}$$

For each $q = 1, \dots, n$, $j = 1, \dots, k$, we deduce from the divergence theorem and $u_j|_{\partial\Omega} = \frac{\partial u_j}{\partial \nu}|_{\partial\Omega} = 0$ that

$$\begin{aligned} z_q^2 \widehat{\varphi}_j(z) &= (2\pi)^{-n/2} \int_{\mathbf{R}^n} \varphi_j(x) (-i)^2 \frac{\partial^2 e^{i\langle x,z\rangle}}{\partial x_q^2} dx \\ &= -(2\pi)^{-n/2} \int_{\mathbf{R}^n} \frac{\partial^2 \varphi_j(x)}{\partial x_q^2} e^{i\langle x,z\rangle} dx \\ &= -\widehat{\frac{\partial^2 \varphi_j}{\partial x_q^2}}(z). \end{aligned} \tag{2.10}$$

It follows from the Parseval's identity that

$$\begin{aligned}
\int_{\mathbf{R}^n} |z|^4 |\widehat{\varphi}_j(z)|^2 dz &= \int_{\mathbf{R}^n} \left| |z|^2 \widehat{\varphi}_j(z) \right|^2 dz \\
&= \int_{\mathbf{R}^n} \left| \sum_{q=1}^{n} \widehat{\frac{\partial^2 \varphi_j}{\partial x_q^2}}(z) \right|^2 dz \\
&= \int_{\Omega} \left(\sum_{q=1}^{n} \frac{\partial^2 u_j}{\partial x_q^2} \right)^2 dx \\
&= \int_{\Omega} |\Delta u_j(x)|^2 dx \\
&= \int_{\Omega} u_j(x) \Delta^2 u_j(x) dx \\
&= \int_{\Omega} \Gamma_j u_j^2(x) dx \\
&= \Gamma_j. \qquad (2.11)
\end{aligned}$$

Since

$$\nabla \widehat{\varphi}_j(z) = (2\pi)^{-n/2} \int_{\Omega} i x u_j(x) e^{i\langle x,z\rangle} dx, \qquad (2.12)$$

we obtain

$$\sum_{j=1}^{k} |\nabla \widehat{\varphi}_j(z)|^2 \leq (2\pi)^{-n} \int_{\Omega} |i x e^{i\langle x,z\rangle}|^2 dx = (2\pi)^{-n} I(\Omega). \qquad (2.13)$$

Putting

$$h(z) := \sum_{j=1}^{k} |\widehat{\varphi}_j(z)|^2,$$

one derives from (2.9) that $0 \leq h(z) \leq (2\pi)^{-n} V(\Omega)$, it follows from (2.13) and the Cauchy–Schwarz inequality that

$$\begin{aligned}
|\nabla h(z)| &\leq 2 \left(\sum_{j=1}^{k} |\widehat{\varphi}_j(z)|^2 \right)^{1/2} \left(\sum_{j=1}^{k} |\nabla \widehat{\varphi}_j(z)|^2 \right)^{1/2} \\
&\leq 2(2\pi)^{-n} \sqrt{V(\Omega) I(\Omega)} \qquad (2.14)
\end{aligned}$$

for every $z \in \mathbf{R}^n$. From the Parseval's identity, we derive

$$\int_{\mathbf{R}^n} h(z) dz = \sum_{j=1}^{k} \int_{\Omega} |u_j(x)|^2 dx = k. \qquad (2.15)$$

Applying the symmetric decreasing rearrangement to h and noting that $\tau = \sup|\nabla h| \leq 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)} := \eta$, we obtain, from (2.4),

$$-\eta \leq -\tau \leq g'(s) \leq 0 \tag{2.16}$$

for almost every s. According to (2.2) and (2.15), we infer

$$k = \int_{\mathbf{R}^n} h(z)dz = \int_{\mathbf{R}^n} h^*(z)dz = nB_n \int_0^\infty s^{n-1}g(s)ds. \tag{2.17}$$

From (2.3) and (2.11), we obtain

$$\begin{aligned}\sum_{j=1}^{k}\Gamma_j &= \int_{\mathbf{R}^n} |z|^4 h(z)dz \\ &\geq \int_{\mathbf{R}^n} |z|^4 h^*(z)dz \\ &= nB_n \int_0^\infty s^{n+3}g(s)ds.\end{aligned} \tag{2.18}$$

In order to apply Lemma 2.1, from (2.17) and the definition of A, we take

$$\psi(s) = g(s), \quad A = \frac{k}{nB_n}, \quad \eta = 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)}, \tag{2.19}$$

from (2.1), we deduce that

$$\eta \geq 2(2\pi)^{-n}\left(\frac{n}{n+2}\right)^{\frac{1}{2}} B_n^{-\frac{1}{n}} V(\Omega)^{\frac{n+1}{n}}. \tag{2.20}$$

On the other hand, $0 < g(0) \leq \sup h^*(z) = \sup h(z) \leq (2\pi)^{-n}V(\Omega)$, we have from (2.1), (2.19) and (2.20) that

$$\begin{aligned}\frac{g(0)^{\frac{2n+2}{n}}}{6n\eta^2(nA)^{\frac{2}{n}}} &\leq \frac{((2\pi)^{-n}V(\Omega))^{\frac{2n+2}{n}}}{6n\left(2(2\pi)^{-n}\left(\frac{n}{n+2}\right)^{\frac{1}{2}} B_n^{-\frac{1}{n}} V(\Omega)^{\frac{n+1}{n}}\right)^2 \left(\frac{k}{B_n}\right)^{\frac{2}{n}}} \\ &= \frac{n+2}{24n^2}\frac{B_n^{\frac{4}{n}}}{(2\pi)^2 k^{\frac{2}{n}}} \leq \frac{n+2}{24n^2}\frac{B_n^{\frac{4}{n}}}{(2\pi)^2}.\end{aligned}$$

By a direct calculation, one sees from $B_n = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}$ that

$$\frac{B_n^{\frac{4}{n}}}{(2\pi)^2} < \frac{1}{2}, \tag{2.21}$$

where $\Gamma\left(\frac{n}{2}\right)$ is the Gamma function. From the above arguments, one has

$$\frac{g(0)^{\frac{2n+2}{n}}}{6n\eta^2(nA)^{\frac{2}{n}}} \leq \frac{n+2}{48n^2} < 1. \tag{2.22}$$

Hence we know that the function $\psi(s) = g(s)$ satisfies the conditions in Lemma 2.1 with $b = n$ and

$$\eta = 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)}, \quad d = \frac{n+2}{48n^2}.$$

From Lemma 2.1 and (2.18), we conclude

$$\begin{aligned}
\sum_{j=1}^{k}\Gamma_j &\geq nB_n\int_0^\infty s^{n+3}g(s)ds \\
&\geq \frac{n}{n+4}(B_n)^{-\frac{4}{n}}k^{\frac{n+4}{n}}g(0)^{-\frac{4}{n}} \\
&\quad+\left(\frac{1}{3(n+4)\eta^2}-\frac{1}{288n(n+2)(n+4)\eta^2}\right)k^{\frac{n+2}{n}}(B_n)^{-\frac{2}{n}}g(0)^{\frac{2n-2}{n}} \\
&\quad+\left(\frac{1}{36n(n+4)\eta^4}-\frac{1}{1728n^2(n+2)(n+4)\eta^4}\right)kg(0)^4.
\end{aligned} \tag{2.23}$$

Defining a function F by

$$\begin{aligned}
F(t) &= \frac{n}{n+4}(B_n)^{-\frac{4}{n}}k^{\frac{n+4}{n}}t^{-\frac{4}{n}} \\
&\quad+\left(\frac{1}{3(n+4)\eta^2}-\frac{1}{288n(n+2)(n+4)\eta^2}\right)k^{\frac{n+2}{n}}(B_n)^{-\frac{2}{n}}t^{\frac{2n-2}{n}} \\
&\quad+\left(\frac{1}{36n(n+4)\eta^4}-\frac{1}{1728n^2(n+2)(n+4)\eta^4}\right)kt^4.
\end{aligned} \tag{2.24}$$

It is not hard to prove from (2.20) that $\eta \geq (2\pi)^{-n}B_n^{-\frac{1}{n}}V(\Omega)^{\frac{n+1}{n}}$. Furthermore, it follows from (2.24) that

$$\begin{aligned}
&F'(t) \\
&\leq -\frac{4}{n+4}(B_n)^{-\frac{4}{n}}k^{\frac{n+4}{n}}t^{-1-\frac{4}{n}} \\
&\quad+\left(\frac{2(n-1)}{3n(n+4)}-\frac{(n-1)}{144n^2(n+2)(n+4)}\right)k^{\frac{n+2}{n}}(2\pi)^{2n}V(\Omega)^{-\frac{2(n+1)}{n}}t^{\frac{n-2}{n}} \\
&\quad+\left(\frac{1}{9n(n+4)}-\frac{1}{432n^2(n+2)(n+4)}\right)kt^3(2\pi)^{4n}(B_n)^{\frac{4}{n}}V(\Omega)^{-\frac{4(n+1)}{n}} \\
&= \frac{k}{n+4}t^{-\frac{n+4}{n}}\times\left\{\left(\frac{2(n-1)}{3n}-\frac{(n-1)}{144n^2(n+2)}\right)(2\pi)^{2n}k^{\frac{2}{n}}V(\Omega)^{-\frac{2(n+1)}{n}}t^{\frac{2n+2}{n}}\right. \\
&\quad\left.-4(B_n)^{-\frac{4}{n}}k^{\frac{4}{n}}+\left(\frac{1}{9n}-\frac{1}{432n^2(n+2)}\right)(2\pi)^{4n}(B_n)^{\frac{4}{n}}V(\Omega)^{-\frac{4(n+1)}{n}}t^{\frac{4n+4}{n}}\right\}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\frac{n+4}{k}t^{\frac{n+4}{n}}F'(t) \\
&\quad\leq\left(\frac{2(n-1)}{3n}-\frac{(n-1)}{144n^2(n+2)}\right)(2\pi)^{2n}k^{\frac{2}{n}}V(\Omega)^{-\frac{2(n+1)}{n}}t^{\frac{2n+2}{n}} \\
&\quad-4(B_n)^{-\frac{4}{n}}k^{\frac{4}{n}}+\left(\frac{1}{9n}-\frac{1}{432n^2(n+2)}\right)(2\pi)^{4n}(B_n)^{\frac{4}{n}}V(\Omega)^{-\frac{4(n+1)}{n}}t^{\frac{4n+4}{n}}.
\end{aligned} \tag{2.25}$$

Since the right hand side of (2.25) is an increasing function of t, if it is not larger than 0 at $t = (2\pi)^{-n}V(\Omega)$, that is,

$$\begin{aligned}&\left(\frac{2(n-1)}{3n} - \frac{(n-1)}{144n^2(n+2)}\right)(2\pi)^{2n}k^{\frac{2}{n}}V(\Omega)^{-\frac{2(n+1)}{n}}((2\pi)^{-n}V(\Omega))^{\frac{2n+2}{n}}\\&+\left(\frac{1}{9n} - \frac{1}{432n^2(n+2)}\right)(2\pi)^{4n}(B_n)^{\frac{4}{n}}V(\Omega)^{-\frac{4(n+1)}{n}}((2\pi)^{-n}V(\Omega))^{\frac{4n+4}{n}}\\&-4(B_n)^{-\frac{4}{n}}k^{\frac{4}{n}} \le 0,\end{aligned} \tag{2.26}$$

then one has from (2.25) that $F'(t) \le 0$ on $(0, (2\pi)^{-n}V(\Omega)]$. Hence, $F(t)$ is decreasing on $(0, (2\pi)^{-n}V(\Omega)]$. Indeed, by a direct calculation, we have that (2.26) is equivalent to

$$\begin{aligned}&\left(\frac{(n-1)}{6n} - \frac{(n-1)}{576n^2(n+2)}\right)(2\pi)^{-2}k^{\frac{2}{n}}\\&+\left(\frac{1}{36n} - \frac{1}{1728n^2(n+2)}\right)(2\pi)^{-4}(B_n)^{\frac{4}{n}}\\&\le (B_n)^{-\frac{4}{n}}k^{\frac{4}{n}}.\end{aligned} \tag{2.27}$$

From (2.21), we can prove that $(2\pi)^{-2}(B_n)^{\frac{4}{n}} < 1$ and

$$\begin{aligned}&\left(\frac{(n-1)}{6n} - \frac{(n-1)}{576n^2(n+2)}\right)(2\pi)^{-2}k^{\frac{2}{n}}\\&+\left(\frac{1}{36n} - \frac{1}{1728n^2(n+2)}\right)(2\pi)^{-4}(B_n)^{\frac{4}{n}}\\&< \frac{1}{6}(2\pi)^{-2}k^{\frac{2}{n}} + \frac{1}{36n}(2\pi)^{-2}\\&< (2\pi)^{-2}\left\{\frac{1}{6}k^{\frac{4}{n}} + \frac{1}{36n}\right\}\\&< (2\pi)^{-2}k^{\frac{4}{n}} < (B_n)^{-\frac{4}{n}}k^{\frac{4}{n}},\end{aligned} \tag{2.28}$$

that is, $F(t)$ is a decreasing function on $(0, (2\pi)^{-n}V(\Omega)]$.

On the other hand, since $0 < g(0) \le (2\pi)^{-n}V(\Omega)$ and the right hand side of the formula (2.23) is $F(g(0))$, which is a decreasing function of $g(0)$ on $(0, (2\pi)^{-n}V(\Omega)]$, then we can replace $g(0)$ by $(2\pi)^{-n}V(\Omega)$ in (2.23) which gives inequality

$$\begin{aligned}\frac{1}{k}\sum_{j=1}^{k}\Gamma_j \ge{}& \frac{n}{n+4}\frac{16\pi^4}{(B_nV(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}}\\&+\left(\frac{n+2}{12n(n+4)} - \frac{1}{1152n^2(n+4)}\right)\frac{V(\Omega)}{I(\Omega)}\frac{n}{n+2}\frac{4\pi^2}{(B_nV(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}}\\&+\left(\frac{1}{576n(n+4)} - \frac{1}{27648n^2(n+2)(n+4)}\right)\left(\frac{V(\Omega)}{I(\Omega)}\right)^2.\end{aligned}$$

This completes the proof of Theorem. □

Acknowledgments This work was done, in part, while the second author was visiting to Saga University as a JSPS Fellow and he was supported in part by JSPS, he would like to express his thanks to Prof. Q.-M. Cheng for his encouragements and help. The first author was partly supported by a Grant-in-Aid for Scientific Research from JSPS. The second author was partly supported by grant No. 11001087 of NSFC.

References

1. Agmon, S.: On kernels, eigenvalues and eigenfunctions of operators related to elliptic problems. Commun. Pure Appl. Math. **18**, 627–663 (1965)
2. Ashbaugh, M.S., Benguria, R.D.: Universal bounds for the low eigenvalues of Neumann Laplacian in N dimensions. SIAM J. Math. Anal. **24**, 557–570 (1993)
3. Bandle, C.: Isoperimetric Inequalities and Applications. Pitman Monographs and Studies in Mathematics, vol. 7. Pitman, Boston (1980)
4. Berezin, F.A.: Covariant and contravariant symbols of operators. Izv. Akad. Nauk SSSR Ser. Mat. **36**, 1134–1167 (1972)
5. Chavel, I.: Eigenvalues in Riemannian Geometry. Academic Press, New York (1984)
6. Cheng, Q.-M., Yang, H.C.: Inequalities for eigenvalues of a clamped plate problem. Trans. Am. Math. Soc. **358**, 2625–2635 (2006)
7. Cheng, Q.-M., Yang, H.C.: Estimates for eigenvalues on Riemannian manifolds. J. Differ. Equ. **247**, 2270–2281 (2009)
8. Laptev, A.: Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces. J. Funct. Anal. **151**, 531–545 (1997)
9. Levine, H.A., Protter, M.H.: Unrestricted lower bounds for eigenvalues for classes of elliptic equations and systems of equations with applications to problems in elasticity. Math. Methods Appl. Sci. **7**(2), 210–222 (1985)
10. Li, P., Yau, S.T.: On the Schrödinger equations and the eigenvalue problem. Commun. Math. Phys. **88**, 309–318 (1983)
11. Lieb, E.: The number of bound states of one-body Schrödinger operators and the Weyl problem. Proc. Symp. Pure Math. **36**, 241–252 (1980)
12. Melas, A.D.: A lower bound for sums of eigenvalues of the Laplacian. Proc. Am. Math. Soc. **131**, 631–636 (2003)
13. Payne, L.E., Pólya, G., Weinberger, H.F.: On the ratio of consecutive eigenvalues. J. Math. Phys. **35**, 289–298 (1956)
14. Pleijel, A.: On the eigenvalues and eigenfunctions of elastic plates. Commun. Pure Appl. Math. **3**, 1–10 (1950)
15. Pólya, G.: On the eigenvalues of vibrating membranes. Proc. Lond. Math. Soc. **11**, 419–433 (1961)
16. Pólya, G., Szegö, G.: Isoperimetric Inequalities in Mathematical Physics. Annals of Mathematics Studies, Number 27. Princeton university press, Princeton (1951)
17. Wang, Q.L., Xia, C.Y.: Universal bounds for eigenvalues of the biharmonic operator on Riemannian manifolds. J. Funct. Anal. **245**, 334–352 (2007)