Math. Ann. (2007) 337:159–175
DOI 10.1007/s00208-006-0030-x

Mathematische Annalen

Bounds on eigenvalues of Dirichlet Laplacian

Qing-Ming Cheng · Hongcang Yang

Received: 25 February 2005 / Revised: 15 June 2006 /
Published online: 3 August 2006

Abstract In this paper, we investigate an eigenvalue problem of Dirichlet Laplacian on a bounded domain Ω in an n-dimensional Euclidean space $\mathbf{R}^n$. If λ_{k+1} is the $(k+1)$th eigenvalue of Dirichlet Laplacian on Ω, then, we prove that, for $n \geq 41$ and $k \geq 41$, $\lambda_{k+1} \leq k^{\frac{2}{n}}\lambda_1$ and, for any n and k, $\lambda_{k+1} \leq C_0(n,k)k^{\frac{2}{n}}\lambda_1$ with $C_0(n,k) \leq j^2_{n/2,1}/j^2_{n/2-1,1}$, where $j_{p,k}$ denotes the k-th positive zero of the standard Bessel function $J_p(x)$ of the first kind of order p. From the asymptotic formula of Weyl and the partial solution of the conjecture of Pólya, we know that our estimates are optimal in the sense of order of k.

Mathematics Subject Classification (2001) 35P15 · 58G25

Q.-M. Cheng was partially Supported by a Grant-in-Aid for Scientific Research from the Japan Society for the Promotion of Science.

H. Yang was partially Supported by Chinese NSF, SF of CAS and NSF of USA.

Q.-M. Cheng (✉)
Department of Mathematics, Faculty of Science and Engineering, Saga University,
Saga 840-8502, Japan
e-mail: cheng@ms.saga-u.ac.jp

H. Yang
International Centre for Theoretical Physics, P. O. Box 586, 34100 Trieste, Italy

H. Yang
Department of Mathematics, University of California, Berkeley, CA 94720, USA

H. Yang
Academy of Mathematics and Systematical Sciences, CAS, Beijing 100080, China
e-mail: yanghc2@netease.com

1 Introduction

An eigenvalue problem of Dirichlet Laplacian on a bounded domain Ω with smooth boundary $\partial\Omega$ in an n-dimensional Euclidean space $\mathbf{R}^n$ is

$$\begin{cases} \triangle u = -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

which is also called a fixed membrane problem, where Δ is the Laplacian in $\mathbf{R}^n$. This problem has a real and purely discrete spectrum

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty.$$

Here each eigenvalue is repeated from its multiplicity.

It is well known that Kac [13] (cf. [18]) posed a question "Can one hear the shape of a drum?" which is the title of a his famous article in 1966. A mathematical interpretation of the question is that if two domains are isospectral, is it necessarily true that they are isometric? Hence, it is very important to study the properties of the spectrum of the eigenvalue problem of Dirichlet Laplacian on a bounded domain Ω in $\mathbf{R}^n$.

In the early part of twentieth century, Hilbert conjectured that the research of the asymptotic behavior of the eigenvalue λ_k of the eigenvalue problem (1.1) would yield results of the utmost importance. In 1911, Weyl proved that

$$\lambda_k \sim \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to \infty, \tag{1.2}$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$. Further, Pólya conjectured the eigenvalue λ_k should satisfy

$$\lambda_k \geq \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \tag{1.3}$$

for $k = 1, 2, \ldots$ (see [5]). On the conjecture of Pólya, Li and Yau [15] attacked it and obtained

$$\lambda_k \geq \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for} \quad k = 1, 2, \ldots. \tag{1.4}$$

On the other hand, Stewartson and Waechter [19] proposed to study an inverse problem: let ϕ be the set of all increasing sequences of positive numbers which tend to infinity, can one identify those sequences in ϕ which correspond to spectra of the eigenvalue problem (1.1) for some domain? The study of the universal inequalities plays an important role to restrict those sequences which are spectra. Although the universal inequlities for the eigenvalue λ_k of the eigenvalue problem (1.1) have been studied by many mathematicians, the main contributions have been obtained by Payne et al. [16, 17] (cf. Thompson [21]), Hile and Protter [12] and Yang [22]. Namely, Payne et al. [17] (cf. Thompson [21]) and Hile and Protter [12] proved, respectively,

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{nk} \sum_{i=1}^{k} \lambda_i \tag{1.5}$$

and

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{kn}{4}. \tag{1.6}$$

Further, in 1991, Yang [22] has proved a very sharp universal inequality:

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} \lambda_i (\lambda_{k+1} - \lambda_i). \tag{1.7}$$

From this inequality, one has

$$\begin{aligned} \lambda_{k+1} \leq & \left(1 + \frac{2}{n}\right) \frac{1}{k} \sum_{i=1}^{k} \lambda_i \\ & + \left[\left(\frac{2}{n} \frac{1}{k} \sum_{i=1}^{k} \lambda_k \right)^2 - \left(1 + \frac{4}{n}\right) \frac{1}{k} \sum_{j=1}^{k} \left(\lambda_j - \frac{1}{k} \sum_{i=1}^{k} \lambda_i \right)^2 \right]^{1/2}, \end{aligned} \tag{1.8}$$

which has been called Yang's inequality by Ashbaugh [1, 2].

Remark 1.1 Since the importance of Yang's inequalities (1.7) and (1.8) has been emphasized by Ashbaugh in [1, 2], it is certainly important for readers to know the original proof of Yang's inequalities. Although, Ashbaugh [1, 2] has published a proof of Yang's inequalities in his survey papers, Yang has never published his original proof. Hence, we shall give a proof of Yang's inequality (1.7) by following his original method in Appendix.

Recently, for eigenvalue problems of Dirichlet Laplacian on either a bounded domain in an n-dimensional unit sphere, or an n-dimensional compact minimal submanifold in a unit sphere, or a bounded domain in an n-dimensional complex projective space, or an n-dimensional compact homogeneous Riemannian manifold, or a compact complex submanifold in an m-dimensional complex projective space, we also obtained the universal inequalities on higher eigenvalues in [6] and [8], which are sharper than the old results in corresponding cases (cf. 6–12, 14, 17, 20, 23).

In this paper, we want to study the bound of λ_{k+1}/λ_1 for a bounded domain Ω in $\mathbf{R}^n$. We shall prove that, for $n \geq 41$ and $k \geq 41$,

$$\lambda_{k+1} \leq k^{2/n}\lambda_1$$

holds. For any n and k, we obtain

$$\lambda_{k+1} \leq C_0(n,k)k^{2/n}\lambda_1,$$

where

$$C_0(n,k) = \begin{cases} \dfrac{j^2_{n/2,1}}{j^2_{n/2-1,1}}, & \text{for } k = 1 \\ 1 + \dfrac{a(\min\{n, k-1\})}{n}, & \text{for } k \geq 2 \end{cases}$$

and $a(1) \leq 2.64, a(2) \leq 2.27$ and $a(p) \leq 2.2 - 4\log(1 + \frac{p-3}{50})$ for $p \geq 3$ is a constant depending only on p, and $j_{p,k}$ denotes the k-th positive zero of the standard Bessel function $J_p(x)$ of the first kind of order p.

Remark 1.2 From Weyl's asymptotic formula (1.2) and the partial solution (1.4) of the conjecture of Pólya, we know that our estimates of λ_{k+1}/λ_1 are best possible in the sense of order of k.

This paper is organized as follows. In Sect. 2, we give a general recursion formula for any positive real numbers $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{k+1}$ satisfying a condition. In Sect. 3, we study an eigenvalue problem of Dirichlet Laplacian on a bounded domain in $\mathbf{R}^n$. We shall prove our main estimates of the λ_{k+1}. In Appendix, we shall give an outline of proof of Yang's inequality (1.7). The idea of the proof comes from the original preprint of Yang [22].

2 A general recursion formula

In this section, we shall prove a general recursion formula on any positive real numbers satisfying some conditions, which plays an important role in proofs of our results.

Theorem 2.1 *Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{k+1}$ be any positive real numbers satisfying*

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}\lambda_i(\lambda_{k+1} - \lambda_i). \tag{2.1}$$

Define

$$\Lambda_k = \frac{1}{k}\sum_{i=1}^{k}\lambda_i, \quad T_k = \frac{1}{k}\sum_{i=1}^{k}\lambda_i^2, \quad F_k = \left(1 + \frac{2}{n}\right)\Lambda_k^2 - T_k. \tag{2.2}$$

Then, we have

$$F_{k+1} \leq C(n,k)\left(\frac{k+1}{k}\right)^{\frac{4}{n}} F_k, \tag{2.3}$$

where

$$C(n,k) = 1 - \frac{1}{3n}\left(\frac{k}{k+1}\right)^{\frac{4}{n}} \frac{\left(1+\frac{2}{n}\right)\left(1+\frac{4}{n}\right)}{(k+1)^3} < 1.$$

Proof Putting

$$p_{k+1} = \Lambda_{k+1} - \left(1 + \frac{2}{n}\frac{1}{1+k}\right)\Lambda_k,$$

since

$$\lambda_{k+1} = (k+1)\Lambda_{k+1} - k\Lambda_k = (k+1)\left[p_{k+1} + \left(1+\frac{2}{n}\right)\frac{1}{k+1}\Lambda_k\right], \tag{2.4}$$

we have

$$\begin{aligned} F_{k+1} &= \left(1+\frac{2}{n}\right)\Lambda_{k+1}^2 - \frac{k}{k+1}\left(1+\frac{2}{n}\right)\Lambda_k^2 - \frac{1}{k+1}\lambda_{k+1}^2 + \frac{k}{k+1}F_k \\ &= \left(1+\frac{2}{n}\right)\left[p_{k+1} + \left(1+\frac{2}{n}\frac{1}{1+k}\right)\Lambda_k\right]^2 - \frac{k}{k+1}\left(1+\frac{2}{n}\right)\Lambda_k^2 \\ &\quad -(k+1)\left[p_{k+1} + \left(1+\frac{2}{n}\right)\frac{1}{k+1}\Lambda_k\right]^2 + \frac{k}{k+1}F_k. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} F_{k+1} = &-\left(k - \frac{2}{n}\right)p_{k+1}^2 + 2\frac{2}{n}\frac{\left(1+\frac{2}{n}\right)}{k+1}p_{k+1}\Lambda_k \\ &+\frac{2}{n}\frac{\left(1+\frac{2}{n}\right)}{k+1}\Lambda_k^2 + \frac{4}{n^2}\frac{\left(1+\frac{2}{n}\right)}{(k+1)^2}\Lambda_k^2 + \frac{k}{k+1}F_k. \end{aligned} \tag{2.5}$$

From (2.1), we have

$$\left(\lambda_{k+1} - \left(1+\frac{2}{n}\right)\Lambda_k\right)^2 \leq \left(1+\frac{2}{n}\right)^2\Lambda_k^2 - \left(1+\frac{4}{n}\right)T_k. \tag{2.6}$$

Hence, (2.4) and (2.6) yield

$$(k+1)^2\left(\Lambda_{k+1}-\left(1+\frac{2}{n}\frac{1}{k+1}\right)\Lambda_k\right)^2 \le \left(1+\frac{2}{n}\right)^2\Lambda_k^2-\left(1+\frac{4}{n}\right)T_k.$$

From the definition of p_{k+1}, the above inequality and (2.2), we infer

$$0 \le -(k+1)^2 p_{k+1}^2-\frac{2}{n}\left(1+\frac{2}{n}\right)\Lambda_k^2+\left(1+\frac{4}{n}\right)F_k. \tag{2.7}$$

Multiplying (2.7) by $\left[\frac{1}{k+1}+\frac{2}{n}\left(\frac{1}{(k+1)^2}+\frac{\beta\left(1+\frac{2}{n}\right)}{(k+1)^3}\right)\right]$ and then adding it to (2.5), we infer

$$\begin{aligned}
F_{k+1} \le &\left(1+\frac{4}{n}\frac{1}{k+1}+\frac{2}{n}\frac{\left(1+\frac{4}{n}\right)}{(k+1)^2}+\frac{2\beta}{n}\frac{\left(1+\frac{2}{n}\right)\left(1+\frac{4}{n}\right)}{(k+1)^3}\right)F_k\\
&-\left(2k+1+\frac{2}{n}\frac{\left(1+\frac{2}{n}\right)\beta}{k+1}\right)p_{k+1}^2+2\frac{2}{n}\frac{\left(1+\frac{2}{n}\right)}{k+1}p_{k+1}\Lambda_k-\frac{4\beta}{n^2}\frac{\left(1+\frac{2}{n}\right)^2}{(k+1)^3}\Lambda_k^2\\
\le &\left(1+\frac{4}{n}\frac{1}{k+1}+\frac{2}{n}\frac{\left(1+\frac{4}{n}\right)}{(k+1)^2}+\frac{2\beta}{n}\frac{\left(1+\frac{2}{n}\right)\left(1+\frac{4}{n}\right)}{(k+1)^3}\right)F_k\\
&-\frac{4\beta}{n^2}\frac{\left(1+\frac{2}{n}\right)^2}{(k+1)^3}\Lambda_k^2+\frac{4}{n^2}\frac{\left(1+\frac{2}{n}\right)^2}{(k+1)^2(2k+1)}\Lambda_k^2\\
&-(2k+1)\left(p_{k+1}-\frac{2}{n}\frac{\left(1+\frac{2}{n}\right)}{(k+1)(2k+1)}\Lambda_k\right)^2.
\end{aligned}$$

Letting $\beta = k+1/2k+1$, we have

$$F_{k+1} \le \left(1+\frac{4}{n}\frac{1}{k+1}+\frac{2}{n}\frac{\left(1+\frac{4}{n}\right)}{(k+1)^2}+\frac{2}{n}\frac{\left(1+\frac{2}{n}\right)\left(1+\frac{4}{n}\right)}{(k+1)^2(2k+1)}\right)F_k. \tag{2.8}$$

Since

$$
\begin{aligned}
\left(\frac{k+1}{k}\right)^{\frac{4}{n}} &= \left(1-\frac{1}{k+1}\right)^{-\frac{4}{n}} \\
&= 1+\frac{4}{n}\frac{1}{k+1}+\frac{1}{2}\frac{4}{n}\frac{\left(1+\frac{4}{n}\right)}{(k+1)^2}+\frac{1}{6}\frac{4}{n}\frac{\left(1+\frac{4}{n}\right)\left(2+\frac{4}{n}\right)}{(k+1)^3} \\
&\quad +\frac{1}{24}\frac{4}{n}\frac{\left(1+\frac{4}{n}\right)\left(2+\frac{4}{n}\right)\left(3+\frac{4}{n}\right)}{(k+1)^4}+\cdots \\
&\geq 1+\frac{4}{n}\frac{1}{k+1}+\frac{1}{2}\frac{4}{n}\frac{\left(1+\frac{4}{n}\right)}{(k+1)^2}+\frac{1}{3}\frac{4}{n}\frac{\left(1+\frac{4}{n}\right)\left(1+\frac{2}{n}\right)}{(k+1)^3} \\
&\quad +\frac{1}{4}\frac{4}{n}\frac{\left(1+\frac{4}{n}\right)\left(1+\frac{2}{n}\right)}{(k+1)^4},
\end{aligned}
\tag{2.9}
$$

we have

$$
\begin{aligned}
F_{k+1} &\leq \left[\left(\frac{k+1}{k}\right)^{\frac{4}{n}}-\frac{k-1}{3(2k+1)}\frac{2}{n}\frac{\left(1+\frac{2}{n}\right)\left(1+\frac{4}{n}\right)}{(k+1)^3}-\frac{1}{n}\frac{\left(1+\frac{4}{n}\right)\left(1+\frac{2}{n}\right)}{(k+1)^4}\right]F_k \\
&\leq C(n,k)\left(\frac{k+1}{k}\right)^{\frac{4}{n}}F_k,
\end{aligned}
$$

where

$$
C(n,k)=\left[1-\frac{1}{3n}\left(\frac{k}{k+1}\right)^{\frac{4}{n}}\frac{\left(1+\frac{2}{n}\right)\left(1+\frac{4}{n}\right)}{(k+1)^3}\right]<1.
$$

□

Corollary 2.1 *Under the assumptions in Theorem* 2.1*, we have*

$$
\lambda_{k+1}\leq\left(1+\frac{4}{n}\right)k^{2/n}\lambda_1. \tag{2.10}
$$

Proof By making use of the formula (2.3) in Theorem 2.1, we have

$$
F_k\leq C(n,k-1)\left(\frac{k}{k-1}\right)^{\frac{4}{n}}F_{k-1}\leq k^{\frac{4}{n}}F_1=\frac{2}{n}k^{\frac{4}{n}}\lambda_1^2. \tag{2.11}
$$

we infer from (2.6)

$$
\left[\lambda_{k+1}-\left(1+\frac{2}{n}\right)\Lambda_k\right]^2\leq\left(1+\frac{4}{n}\right)F_k-\frac{2}{n}\left(1+\frac{2}{n}\right)\Lambda_k^2.
$$

Hence, we have

$$\frac{\frac{2}{n}}{\left(1+\frac{4}{n}\right)}\lambda_{k+1}^2+\frac{1+\frac{2}{n}}{1+\frac{4}{n}}\left(\lambda_{k+1}-\left(1+\frac{4}{n}\right)\Lambda_k\right)^2\leq\left(1+\frac{4}{n}\right)F_k.$$

Thus, we derive

$$\lambda_{k+1}^2\leq\frac{n}{2}\left(1+\frac{4}{n}\right)^2F_k\leq\left(1+\frac{4}{n}\right)^2k^{\frac{4}{n}}\lambda_1^2. \tag{2.12}$$

□

3 Eigenvalues of Dirichlet Laplacian

In this section, we shall study the following eigenvalue problem of Dirichlet Laplacian on $\Omega\subset\mathbf{R}^n$:

$$\begin{cases}\triangle u=-\lambda u, & \text{in}\quad\Omega,\\ u=0, & \text{on}\quad\partial\Omega.\end{cases} \tag{3.1}$$

First of all, we define several constants, which depend only on either n or k. Define

$$\begin{aligned}a_1(n)&=\frac{n(1+\frac{4}{n})\left(1+\frac{8}{n+1}+\frac{8}{(n+1)^2}\right)^{\frac{1}{2}}}{(n+1)^{\frac{2}{n}}}-n,\\ a_2(k,n)&=\frac{n}{k^{\frac{2}{n}}}\left(1+\frac{4(n+k+4)}{n^2+5n-4(k-1)}\right)-n,\\ a_2(k)&=\max\{a(n,k),k\leq n\leq 400\},\\ a_3(k)&=\frac{4}{1-\frac{k}{400}}-2\log k,\\ a(k)&=\max\{a_1(k),a_2(k+1)),a_3(k+1)\}.\end{aligned}$$

Theorem 3.1 *Let λ_{k+1} be the $(k+1)$th eigenvalue of the eigenvalue problem* (3.1). *Then, we have*

(1) *for $n\geq 41$ and $k\geq 41$,*

$$\lambda_{k+1}\leq k^{2/n}\lambda_1;$$

(2) *for any n and k,*

$$\lambda_{k+1}\leq C_0(n,k)k^{2/n}\lambda_1,$$

where

$$C_0(n,k) = \begin{cases} \dfrac{j_{n/2,1}^2}{j_{n/2-1,1}^2}, & \text{for } k = 1 \\ 1 + \dfrac{a(\min\{n, k-1\})}{n}, & \text{for } k \geq 2 \end{cases}$$

and $a(1) \leq 2.64, a(2) \leq 2.27$ *and* $a(p) \leq 2.2 - 4\log(1 + \frac{p-3}{50})$ *for* $p \geq 3$.

Proposition 3.1 *Under the assumption of Theorem* 3.1*, we have, for* $k \geq n+1$,

$$\lambda_{k+1} \leq \frac{\left(1+\frac{4}{n}\right)\left(1+\frac{8}{n+1}+\frac{8}{(n+1)^2}\right)^{\frac{1}{2}}}{(n+1)^{\frac{2}{n}}} k^{\frac{2}{n}}\lambda_1 = \left(1 + \frac{a_1(n)}{n}\right) k^{\frac{2}{n}}\lambda_1, \quad (3.2)$$

where $a_1(n) \leq 2.31$.

Proof Since λ_{k+1} is the $(k+1)$th eigenvalue of the eigenvalue problem (3.1), we know that the λ_{k+1} satisfies Yang's inequality (1.7). Hence, the conditions in Theorem 2.1 are satisfied. By making use of the formula (2.3), we have, from (2.11),

$$\lambda_{k+1}^2 \leq \frac{n}{2}\left(1+\frac{4}{n}\right)^2 F_k \leq \frac{n}{2}\left(1+\frac{4}{n}\right)^2 \left(\frac{k}{n+1}\right)^{\frac{4}{n}} F_{n+1}. \quad (3.3)$$

On the other hand,

$$\begin{aligned} F_{n+1} &= \frac{2}{n}\Lambda_{n+1}^2 - \sum_{i=1}^{n+1}\frac{(\lambda_i - \Lambda_{n+1})^2}{n+1} \\ &\leq \frac{2}{n}\Lambda_{n+1}^2 - \frac{(\lambda_1 - \Lambda_{n+1})^2 + \frac{1}{n}(\lambda_1 - \Lambda_{n+1})^2}{n+1} \\ &= \frac{2}{n}\left(\Lambda_{n+1}^2 - \frac{(\lambda_1 - \Lambda_{n+1})^2}{2}\right). \end{aligned} \quad (3.4)$$

It is obvious that $\Lambda_{n+1}^2 - (\lambda_1 - \Lambda_{n+1})^2/2$ is an increasing function of Λ_{n+1}. From the result of Ashbaugh and Benguria [4], we have

$$\lambda_{n+1} + \cdots + \lambda_2 \leq (n+4)\lambda_1. \quad (3.5)$$

Thus, we derive

$$\Lambda_{n+1} \leq \left(1 + \frac{4}{n+1}\right)\lambda_1. \quad (3.6)$$

Hence, we have

$$\frac{n}{2}F_{n+1} \leq \left(1 + \frac{8}{n+1} + \frac{8}{(n+1)^2}\right)\lambda_1^2. \tag{3.7}$$

From (3.3) and (3.7), we complete the proof of Proposition 3.1. □

Proposition 3.2 *Under the assumption of Theorem* 3.1*, we have, for* $k \geq 55$ *and* $n \geq 54$*,*

$$\lambda_{k+1} \leq k^{\frac{2}{n}}\lambda_1. \tag{3.8}$$

Proof If $k \geq n+1$, from Proposition 3.1, we have

$$\lambda_{k+1} \leq \frac{1}{(n+1)^{\frac{2}{n}}}\left(1 + \frac{4}{n}\right)^2 k^{\frac{2}{n}}\lambda_1.$$

Since

$$\begin{aligned}(n+1)^{\frac{2}{n}} &= \exp\left(\frac{2}{n}\log(n+1)\right)\\ &\geq 1 + \frac{2}{n}\log(n+1) + \frac{2}{n^2}(\log(n+1))^2\\ &\geq \left(1 + \frac{1}{n}\log(n+1)\right)^2,\end{aligned} \tag{3.9}$$

we have

$$\lambda_{k+1} \leq \left(\frac{1+\frac{4}{n}}{1+\frac{1}{n}\log(n+1)}\right)^2 k^{\frac{2}{n}}\lambda_1. \tag{3.10}$$

Then, when $n \geq 54, \log(n+1) \geq 4$, we have

$$\lambda_{k+1} \leq k^{\frac{2}{n}}\lambda_1.$$

On the other hand, if $k \leq n$, then $\Lambda_k \leq \Lambda_{n+1}$. Since

$$\begin{aligned}\frac{n}{2}F_k &= \Lambda_k^2 - \frac{n}{2}\frac{\sum_{i=1}^k(\lambda_i - \Lambda_k)^2}{k}\\ &\leq \Lambda_k^2 - \frac{n}{2}\frac{(\lambda_1 - \Lambda_k)^2 + \frac{\left\{\sum_{i=2}^k(\lambda_i - \Lambda_k)\right\}^2}{k-1}}{k}\\ &\leq \Lambda_k^2 - \frac{(\lambda_1 - \Lambda_k)^2}{2}\\ &\leq \Lambda_{n+1}^2 - \frac{(\lambda_1 - \Lambda_{n+1})^2}{2} \leq \left(1 + \frac{4}{n}\right)^2\lambda_1^2,\end{aligned}$$

from (2.11), we have

$$\lambda_{k+1} \leq \left(1+\frac{4}{n}\right)\sqrt{\frac{n}{2}F_k} \leq \frac{1}{k^{\frac{2}{n}}}\left(1+\frac{4}{n}\right)^2 k^{\frac{2}{n}}\lambda_1 \leq \left(\frac{1+\frac{4}{n}}{1+\frac{\log k}{n}}\right)^2 k^{\frac{2}{n}}\lambda_1.$$

Here we used $k^{\frac{2}{n}} \geq (1+\frac{\log k}{n})^2$. By the same assertion as above, when $k \geq 55$, we also have

$$\lambda_{k+1} \leq k^{\frac{2}{n}}\lambda_1.$$

□

Proof of Theorem 3.1 From Propositions 3.1 and 3.2, we know, for $n \geq 54$ and $k \geq 55, \lambda_{k+1} \leq k^{\frac{2}{n}}\lambda_1$ and for any n, if $k \geq n+1$, then $\lambda_{k+1} \leq (1+\frac{a_1(n)}{n})k^{\frac{2}{n}}\lambda_1$. Hence, we only need to prove the case that $k \leq 54$ and $k \leq n$. Because of $k \leq n$ and $k \leq 54$, from (3.5), we derive,

$$\lambda_{k+1} \leq \frac{1}{n-k+1}\{(n+5)\lambda_1 - k\Lambda_k\}. \tag{3.11}$$

From Yang's inequality (1.8), we have

$$\lambda_{k+1} \leq \left(1+\frac{4}{n}\right)\Lambda_k. \tag{3.12}$$

Since the right hand side of (3.10) is a decreasing function of Λ_k and the right hand side of (3.11) is an increasing function of Λ_k, for $\frac{1}{n-k+1}\{(n+5)\lambda_1 - k\Lambda_k\} = \left(1+\frac{4}{n}\right)\Lambda_k$, we infer

$$\begin{aligned}\lambda_{k+1} &\leq \frac{1}{k^{\frac{2}{n}}}\left(1+\frac{4(n+k+4)}{n^2+5n-4(k-1)}\right)k^{\frac{2}{n}}\lambda_1 \\ &= \left(1+\frac{a_2(k,n)}{n}\right)k^{\frac{2}{n}}\lambda_1. \end{aligned} \tag{3.13}$$

From the definition of $a_2(k) = \max\{a(n,k), k \leq n \leq 400\}$, when $n \leq 400$, we obtain

$$\lambda_{k+1} \leq \left(1+\frac{a_2(k)}{n}\right)k^{\frac{2}{n}}\lambda_1. \tag{3.14}$$

When $n > 400$ holds, from (3.10), we have

$$\lambda_{k+1} \leq \left(1+\frac{4}{n-k}\right)\lambda_1.$$

Since $n > 400$ and $k \le 54$, we know $\frac{2}{n} \log k < \frac{1}{50}$. Hence, we have

$$k^{-\frac{2}{n}} = e^{-\frac{2}{n}\log k} = 1 - \frac{2}{n}\log k + \frac{1}{2}\left(\frac{2}{n}\log k\right)^2 - \cdots$$
$$\le 1 - \frac{2}{n}\log k + \frac{1}{2}\left(\frac{2}{n}\log k\right)^2.$$

Therefore, we obtain

$$\left(1+\frac{4}{n-k}\right)k^{-\frac{2}{n}} \le \left(1+\frac{4}{n-k}\right)\left(1 - \frac{2}{n}\log k + \frac{1}{2}\left(\frac{2}{n}\log k\right)^2\right)$$
$$\le 1 + \frac{\left(4/(1-\frac{k}{400}) - 2\log k\right)}{n}.$$

Hence, we infer

$$\begin{aligned}\lambda_{k+1} &\le \left(1+\frac{4}{n-k}\right)k^{-\frac{2}{n}}k^{\frac{2}{n}}\lambda_1 \\ &\le \left(1 + \frac{\left(4/(1-\frac{k}{400}) - 2\log k\right)}{n}\right)k^{\frac{2}{n}}\lambda_1 \\ &= \left(1+\frac{a_3(k)}{n}\right)k^{\frac{2}{n}}\lambda_1. \qquad (3.15)\end{aligned}$$

By Table 1 of the values of $a_1(k), a_2(k+1)$ and $a_3(k+1)$ which are calculated by using Mathematica and are listed up in the next page, we have $a_1(1) \le a_2(2) \le a_3(2) = a(1) \le 2.64$ and, for $k \ge 2$,

$$a_3(k+1) \le a_2(k+1) \le a_1(k).$$

Hence, $a(k) = a_1(k)$ for $k \ge 2$. Further, for $k \ge 41$, we know $a(k) < 0$. Hence, for $k \ge 2$, we derive

$$\lambda_{k+1} \le \left(1+\frac{a(\min\{n, k-1\})}{n}\right)k^{\frac{2}{n}}\lambda_1$$

and for $n \ge 41$ and $k \ge 41$, we have

$$\lambda_{k+1} \le k^{\frac{2}{n}}\lambda_1.$$

Table 1 The values of $a_1(k), a_2(k+1)$ and $a_3(k+1)$

k	1	2	3	4	5	6	7	8	9	10	
$a_1(k) \leq$	2.31	2.27	2.2	2.12	2.03	1.94	1.86	1.77	1.69	1.61	
$a_2(k+1) \leq$	2.62	2.05	2.00	1.96	1.90	1.84	1.77	1.70	1.63	1.56	
$a_3(k+1) \leq$	2.64	1.84	1.27	0.84	0.48	0.18	−0.07	−0.30	−0.50	−0.68	
k	11	12	13	14	15	16	17	18	19	20	
$a_1(k) \leq$	1.53	1.46	1.39	1.32	1.25	1.18	1.12	1.06	1.00	0.94	
$a_2(k+1) \leq$	1.49	1.42	1.35	1.29	1.22	1.16	1.10	1.04	0.98	0.92	
$a_3(k+1) \leq$	−0.84	−0.99	−1.13	−1.26	−1.37	−1.48	−1.59	−1.68	−1.78	−1.86	
k	21	22	23	24	25	26	27	28	29	30	
$a_1(k) \leq$	0.89	0.83	0.78	0.72	0.67	0.62	0.58	0.53	0.48	0.44	
$a_2(k+1) \leq$	0.87	0.82	0.76	0.71	0.66	0.61	0.57	0.52	0.47	0.43	
$a_3(k+1) \leq$	−1.94	−2.02	−2.10	−2.17	−2.23	−2.30	−2.36	−2.42	−2.47	−2.53	
k	31	32	33	34	35	36	37	38	39	40	41
$a_1(k) \leq$	0.39	0.35	0.31	0.27	0.23	0.19	0.15	0.11	0.07	0.03	−0.00
$a_2(k+1) \leq$	0.38	0.34	0.30	0.26	0.22	0.18	0.14	0.10	0.07	0.03	−0.01
$a_3(k+1) \leq$	−2.58	−2.63	−2.68	−2.72	−2.77	−2.81	−2.85	−2.89	−2.93	−2.97	−3.00

When $k = 1$, from the solution of the conjecture of Payne, Pólya and Weinberger (cf. [3]), we know

$$\lambda_2 \leq \frac{j^2_{n/2,1}}{j^2_{n/2-1,1}} \lambda_1.$$

It is easy to check that, when $k \geq 3$, by a simple calculation,

$$a(k) \leq 2.2 - 4 \log \left(1 + \frac{k-3}{50}\right).$$

This completes the proof of Theorem 3.1. □

Remark 3.1 According to Theorem 3.1, we would like to propose that, for any $n \geq 2$, there exists a positive integer $N(n)$ such that, if $k \geq N(n)$, then

$$\lambda_{k+1} \leq k^{\frac{2}{n}} \lambda_1$$

is satisfied.

Acknowledgements We would like to express our gratitude to the editor and the referee for their valuable comments and suggestions. This work has been, in part, done when the second author visited to ICTP and University of Calafornia, Berkeley. The second author would like to take the opportunity to express his indebtedness to Professor J. Eells and Professor H. Wu for their kindly support and hospitality.

Appendix: A proof of Yang's inequality (1.7)

In this section, we shall give a proof of Yang's inequality (1.7).
Proof of Yang's inequality (1.7). Let u_k be the orthonormal eigenfunction corresponding to the kth eigenvalue λ_k, i.e. u_k satisfies

$$\begin{cases} \Delta u_k = -\lambda_k u_k, & \text{in } \Omega \\ u_k|_{\partial\Omega} = 0, \\ \int_\Omega u_i u_j = \delta_{ij}. \end{cases} \tag{4.1}$$

Let $x^1, \ldots, x^n$ be the standard coordinate functions in $\mathbf{R}^n$. For any fixed $p = 1, \ldots, n$, putting $g = x^p$ and defining a trial function φ_i by

$$\varphi_i = gu_i - \sum_{j=1}^{k} a_{ij}u_j, \quad a_{ij} = \int_\Omega gu_iu_j = a_{ji}, \tag{4.2}$$

we have

$$\int_\Omega \varphi_i u_j = 0, \quad \text{for} \quad i, j = 1, \ldots, k.$$

Letting

$$b_{ij} = \int_\Omega u_j \nabla g \cdot \nabla u_i, \tag{4.3}$$

from Green's formula, we derive

$$\lambda_j a_{ij} = \int_\Omega g(-\Delta u_j)u_i = \int_\Omega (-2u_j \nabla u_i \cdot \nabla g - gu_j \Delta u_i) = -2b_{ij} + \lambda_i a_{ij},$$

namely,

$$2b_{ij} = (\lambda_i - \lambda_j)a_{ij}. \tag{4.4}$$

By a simple calculation, we have

$$\Delta\varphi_i = -\lambda_i g u_i + 2\nabla g \cdot \nabla u_i + \sum_{j=1}^{k} a_{ij}\lambda_j u_j.$$

Hence, we infer

$$\int_\Omega |\nabla\varphi_i|^2 = \lambda_i \int_\Omega \varphi_i^2 - 2\int_\Omega \varphi_i \nabla g \cdot \nabla u_i.$$

On the other hand, from the definition of φ_i, (4.3) and (4.4), we derive

$$-2\int_\Omega \varphi_i \nabla g\cdot\nabla u_i = -2\int_\Omega g\nabla g\cdot u_i\nabla u_i + 2\sum_{j=1}^k a_{ij}\int_\Omega u_j\nabla g\cdot\nabla u_i$$
$$= 1+\sum_{j=1}^k(\lambda_i-\lambda_j)a_{ij}^2. \tag{4.5}$$

From the Rayleigh–Ritz inequality, we obtained

$$(\lambda_{k+1}-\lambda_i)\int_\Omega \varphi_i^2 \le 1+\sum_{j=1}^k(\lambda_i-\lambda_j)a_{ij}^2. \tag{4.6}$$

Multiplying (4.5) by $(\lambda_{k+1}-\lambda_i)^2$ and then taking sum on i from 1 through k, we obtain

$$\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2+\sum_{i,j=1}^k(\lambda_i-\lambda_j)(\lambda_{k+1}-\lambda_i)^2a_{ij}^2$$
$$= -2\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_\Omega \varphi_i\nabla g\cdot\nabla u_i.$$

By the symmetry of a_{ij} and the anti-symmetry of b_{ij}, we have

$$-2\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_\Omega \varphi_i\nabla g\cdot\nabla u_i$$
$$= \sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2-4\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)b_{ij}^2 \equiv f. \tag{4.7}$$

Similarly, multiplying (4.6) by $(\lambda_{k+1}-\lambda_i)^2$ and taking sum on i from 1 through k, we infer

$$\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^3\int_\Omega\varphi_i^2 \le \sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2-4\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)b_{ij}^2 = f. \tag{4.8}$$

Since $\int_\Omega u_i\varphi_j = 0$ for all $i,j=1,\ldots,k$, we have, for arbitrary constants d_{ij},

$$f^2 = \left\{-2\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_\Omega\varphi_i\nabla g\cdot\nabla u_i\right\}^2$$

$$\leq 4\sum_{i=1}^{k}\int_{\Omega}(\lambda_{k+1}-\lambda_i)^3\varphi_i^2\sum_{i=1}^{k}\int_{\Omega}\left[(\lambda_{k+1}-\lambda_i)^{1/2}\nabla g\cdot\nabla u_i-\sum_{j=1}^{k}d_{ij}u_j\right]^2$$

$$\leq 4f\sum_{i=1}^{k}\int_{\Omega}\left[(\lambda_{k+1}-\lambda_i)|\nabla g\cdot\nabla u_i|^2\right.$$

$$\left.-2\sum_{j=1}^{k}d_{ij}(\lambda_{k+1}-\lambda_i)^{1/2}u_j\nabla g\cdot\nabla u_i+\left(\sum_{j=1}^{k}d_{ij}u_j\right)^2\right].$$

Then we have

$$f\leq 4\sum_{i=1}^{k}\int_{\Omega}(\lambda_{k+1}-\lambda_i)|\nabla_p u_i|^2+4\left[-2\sum_{i,j=1}^{k}d_{ij}(\lambda_{k+1}-\lambda_i)^{1/2}b_{ij}+\sum_{i,j=1}^{k}d_{ij}^2\right].$$

Putting $d_{ij}=(\lambda_{k+1}-\lambda_i)^{1/2}b_{ij}$, we obtain

$$f\leq 4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_{\Omega}|\nabla_p u_i|^2-4\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)b_{ij}^2.$$

From (4.7), we infer

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\leq 4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_{\Omega}|\nabla_p u_i|^2.$$

Taking sum on p from 1 through n, we obtain

$$n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\leq 4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_{\Omega}|\nabla u_i|^2=4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i.$$

□

References

1. Ashbaugh, M.S.: Isoperimetric and universal inequalities for eigenvalues, in spectral theory and geometry (Edinburgh, 1998). In: Davies, E.B., Safalov, Y. (eds.) London Math. Soc. Lecture Notes, vol. 273, pp. 95–139. Cambridge University Press, Cambridge (1999)
2. Ashbaugh, M.S.: Universal eigenvalue bounds of Payne-Pólya-Weinberger, Hile-Prottter and H.C. Yang. Proc. Indian Acad. Sci. Math. Sci. **112**, 3–30 (2002)
3. Ashbaugh, M.S., Benguria, R.D.: A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions. Ann. of Math. **135**, 601–628 (1992)

4. Ashbaugh, M.S., Benguria, R.D.: More bounds on eigenvalue ratios for Dirichlet Laplacians in *n* dimensions. SIAM J. Math Anal. **24**, 1622–1651 (1993)
5. Chavel, I.: Eigenvalues in Riemannian geometry. Academic, New York (1984)
6. Cheng, Q.-M., Yang, H.C.: Estimates on eigenvalues of Laplacian. Math. Ann. **331**, 445–460 (2005)
7. Cheng, Q.-M., Yang, H.C.: Inequalities for eigenvalues of a clamped plate problem. Trans. Am. Math. Soc. **358**, 2625–2635 (2006)
8. Cheng, Q.-M., Yang, H.C.: Inequalities for eigenvalues of Laplacian on domains and compact hypersurfaces in complex projective spaces. J. Math. Soc. Jpn **58**, 545–561 (2006)
9. Harrell, E.M. II.: Some geometric bounds on eigenvalue gaps. Comm. Partial. Differ. Equ. **18**, 179–198 (1993)
10. Harrell, E.M. II., Michel, P.L.: Commutator bounds for eigenvalues with applications to spectral geometry. Comm. Partial. Differ. Equ. **19**, 2037–2055 (1994)
11. Harrell, E.M. II., Stubbe, J.: On trace identities and universal eigenvalue estimates for some partial differential operators. Trans. Am. Math. Soc. **349**, 1797–1809 (1997)
12. Hile, G.N., Protter, M.H.: Inequalities for eigenvalues of the Laplacian. Indiana Univ. Math. J. **29**, 523–538 (1980)
13. Kac, M.: Can one hear the shape of a drum? Am. Math. Mon. **73**, 1–23 (1966)
14. Li, P.: Eigenvalue estimates on homogeneous manifolds. Comment. Math. Helv. **55**, 347–363 (1980)
15. Li, P., Yau, S.T.: On the Schrödinger equation and the eigenvalue problem. Comm. Math. Phys. **88**, 309–318 (1983)
16. Payne, L.E., Polya, G., Weinberger, H.F.: Sur le quotient de deux fréquences propres consécutives. Comptes Rendus Acad. Sci. Paris **241**, 917–919 (1955)
17. Payne, L.E., Polya, G., Weinberger, H.F.: On the ratio of consecutive eigenvalues. J. Math. Phis. **35**, 289–298 (1956)
18. Protter, M.H.: Can one hear the shape of a drum? SIAM Rev. **29**, 185–197 (1987)
19. Stewartson, K., Waechter, R.T.: On hearing the shape of a drum: further results. Proc. Camb. Philol. Soc. **69**, 353–363 (1971)
20. Sun, H.J., Cheng, Q.-M., Yang, H.C.: Lower order eigenvalues of Dirichlet Laplacian (Preprint) (2006)
21. Thompson, C.J.: On the ratio of consecutive eigenvalues in *n*-dimensions. Stud. Appl. Math. **48**, 281–283 (1969)
22. Yang, H.C.: An estimate of the difference between consecutive eigenvalues. Preprint IC/91/60 of ICTP, Trieste (1991)
23. Yang, P.C., Yau, S.T.: Eigenvalues of the Laplacian of compact Riemannian surfaces and minimal submanifolds. Ann. Sc. Norm. Sup. Pisa CI. Sci. **7**, 55–63 (1980)