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Estimates for eigenvalues on Riemannian manifolds

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ABSTRACT

In this paper, we investigate eigenvalues of the Dirichlet eigenvalue problem of Laplacian on a bounded domain Ω in an n-dimensional complete Riemannian manifold M. When M is an n-dimensional Euclidean space $\mathbf{R}^n$, the conjecture of Pólya is well known: the kth eigenvalue λ_k of the Dirichlet eigenvalue problem of Laplacian satisfies

$$\lambda_k \geqslant \frac{4\pi^2}{(\omega_n \operatorname{vol} \Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \ldots.$$

Li and Yau [P. Li, S.T. Yau, On the Schrödinger equation and the eigenvalue problem, Comm. Math. Phys. 88 (1983) 309–318] (cf. Lieb [E. Lieb, The number of bound states of one-body Schrödinger operators and the Weyl problem, in: Proc. Sympos. Pure Math., vol. 36, 1980, pp. 241–252]) have given a partial solution for the conjecture of Pólya, that is, they have proved

$$\frac{1}{k}\sum_{i=1}^{k} \lambda_i \geqslant \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \operatorname{vol} \Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \ldots,$$

which is sharp in the sense of average. In this paper, we consider a general setting for complete Riemannian manifolds. We establish an analog of the Li and Yau's inequality for eigenvalues of the Dirichlet eigenvalue problem of Laplacian on a bounded domain in a complete Riemannian manifold. Furthermore, we obtain a universal inequality for eigenvalues of the Dirichlet eigenvalue problem of Laplacian on a bounded domain in a hyperbolic

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space $H^n(-1)$. From it, we prove that when the bounded domain Ω tends to $H^n(-1)$, all eigenvalues tend to $\frac{(n-1)^2}{4}$.

1. Introduction

Let M be an n-dimensional complete Riemannian manifold. We consider the following Dirichlet eigenvalue problem of Laplacian:

$$\begin{cases} \Delta u = -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in M with piecewise smooth boundary $\partial\Omega$ and Δ denotes the Laplacian on M. The eigenvalue problem (1.1) is also called a fixed membrane problem. It is well known that the spectrum of this eigenvalue problem (1.1) is real and discrete.

$$0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \to \infty,$$

where each λ_i has finite multiplicity which is repeated according to its multiplicity. Furthermore, the following Weyl's asymptotic formula holds (cf. [3]):

$$\lambda_k \sim \frac{4\pi^2}{(\omega_n \operatorname{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to \infty, \tag{1.2}$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$. From this asymptotic formula, it is not difficult to infer

$$\frac{1}{k}\sum_{i=1}^{k} \lambda_i \sim \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \operatorname{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to \infty. \tag{1.3}$$

In particular, when $M = \mathbf{R}^n$, Pólya [22] proved

$$\lambda_k \geqslant \frac{4\pi^2}{(\omega_n \operatorname{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \ldots, \tag{1.4}$$

if Ω is a tiling domain in $\mathbf{R}^n$ and he conjectured, for a general bounded domain,

Conjecture of Pólya. *If Ω is a bounded domain in $\mathbf{R}^n$, then eigenvalue λ_k of the eigenvalue problem* (1.1) *satisfies*

$$\lambda_k \geqslant \frac{4\pi^2}{(\omega_n \operatorname{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \textit{for } k = 1, 2, \ldots. \tag{1.5}$$

On the conjecture of Pólya, Li and Yau [18] (cf. Lieb [16]) attacked it and obtained

$$\frac{1}{k}\sum_{i=1}^{k} \lambda_i \geqslant \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \operatorname{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \ldots, \tag{1.6}$$

by making use of the Fourier transform. It is sharp in the sense of average according to (1.3). From this formula, we have

$$\lambda_k \geqslant \frac{n}{n+2}\frac{4\pi^2}{(\omega_n \operatorname{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}}, \quad \text{for } k=1,2,\dots, \tag{1.7}$$

which gives a partial solution for the conjecture of Pólya with a factor $\frac{n}{n+2}$.

On the other hand, for a complete Riemannian manifold M other than $\mathbf{R}^n$, is it possible for one to consider the same problem as the conjecture of Pólya? One of purposes in this paper is to study this problem by making use of a recursion formula of Cheng and Yang [10] (see Section 2) and Nash's theorem: each complete Riemannian manifold can be isometrically immersed in a Euclidean space. We prove the following:

Theorem 1.1. *Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Then, there exists a constant H_0^2, which only depends on M and Ω such that eigenvalues λ_i's of the eigenvalue problem* (1.1) *satisfy*

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{n^2}{4}H_0^2 \geqslant \frac{n}{\sqrt{(n+2)(n+4)}}\frac{4\pi^2}{(\omega_n \operatorname{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}}, \quad \textit{for } k=1,2,\dots. \tag{1.8}$$

Corollary 1.1. *Let Ω be a domain in the n-dimensional unit sphere $S^n(1)$. Then, eigenvalues λ_i's of the eigenvalue problem* (1.1) *satisfy*

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{n^2}{4} \geqslant \frac{n}{\sqrt{(n+2)(n+4)}}\frac{4\pi^2}{(\omega_n \operatorname{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}}, \quad \textit{for } k=1,2,\dots. \tag{1.9}$$

Corollary 1.2. *Let Ω be a bounded domain in an n-dimensional complete minimal submanifold M in a Euclidean space $\mathbf{R}^N$. Then, eigenvalues λ_i's of the eigenvalue problem* (1.1) *satisfy*

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i \geqslant \frac{n}{\sqrt{(n+2)(n+4)}}\frac{4\pi^2}{(\omega_n \operatorname{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}}, \quad \textit{for } k=1,2,\dots. \tag{1.10}$$

From the above results, we can propose the following:

The generalized conjecture of Pólya. *Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Then, there exists a constant $c(M,\Omega)$, which only depends on M and Ω such that eigenvalues λ_i's of the eigenvalue problem* (1.1) satisfy

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i + c(M,\Omega) \geqslant \frac{n}{n+2}\frac{4\pi^2}{(\omega_n \operatorname{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}}, \quad \textit{for } k=1,2,\dots, \tag{1.11}$$

$$\lambda_k + c(M,\Omega) \geqslant \frac{4\pi^2}{(\omega_n \operatorname{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}}, \quad \textit{for } k=1,2,\dots. \tag{1.12}$$

Remark 1.1. On the generalized conjecture of Pólya, we think that when M is the unit sphere $S^n(1)$, $c(M,\Omega)=\frac{n^2}{4}$, when M is the hyperbolic space $H^n(-1)$, $c(M,\Omega)=-\frac{(n-1)^2}{4}$ and when M is a complete minimal submanifold in $\mathbf{R}^N$, $c(M,\Omega)=0$.

Remark 1.2. Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. We can consider the so-called clamped plate problem:

$$\begin{cases} \Delta^2 u = \Gamma u \quad \text{in } \Omega, \\ u|_{\partial\Omega} = \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0 \end{cases} \tag{1.13}$$

and the so-called buckling problem:

$$\begin{cases} \Delta^2 u = -\Lambda \Delta u \quad \text{in } \Omega, \\ u|_{\partial\Omega} = \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \end{cases} \tag{1.14}$$

where Δ^2 is the biharmonic operator on M and ν denotes the unit outward normal vector on the boundary $\partial\Omega$ of Ω.

For the clamped plate problem (1.13), it is not hard to prove

$$\Gamma_k \geqslant \lambda_k^2,$$

by the variational principle. Hence, we derive, from Theorem 1.1,

$$\Gamma_k \geqslant \left\{ \frac{n}{\sqrt{(n+2)(n+4)}} \frac{4\pi^2}{(\omega_n \operatorname{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}} - \frac{n^2}{4} H_0^2 \right\}^2, \quad \text{for } k = 1, 2, \ldots.$$

In particular, when M is a minimal submanifold in a Euclidean space, we have

$$\Gamma_k \geqslant \frac{n^2}{(n+2)(n+4)} \frac{16\pi^4}{(\omega_n \operatorname{vol}\Omega)^{\frac{4}{n}}} k^{\frac{4}{n}}, \quad \text{for } k = 1, 2, \ldots$$

(cf. [17] for the case of the Euclidean space).

For the buckling problem (1.14), we have $\Lambda_k \geqslant \lambda_k$ by the variational principle. Hence, we can obtain the lower bound for Λ_k's similar to (1.8) and (1.10) from Theorem 1.1.

On universal estimates for eigenvalues of the clamped plate problem and the buckling problem, the readers can see [5,7] and [9].

The other purpose in this paper is to investigate estimates for eigenvalues of the eigenvalue problem (1.1) when M is the hyperbolic space $H^n(-1)$ with constant curvature -1.

When M is $\mathbf{R}^n$, universal inequalities for the eigenvalue λ_k of the eigenvalue problem (1.1) was studied by many mathematicians. The main contributions was obtained by Payne, Pólya and Weinberger [20,21] (cf. [24]), Hile and Protter [15] and Yang [25] (cf. [10]). Namely, Payne, Pólya and Weinberger [21] and Hile and Protter [15] proved, respectively,

$$\lambda_{k+1} - \lambda_k \leqslant \frac{4}{nk} \sum_{i=1}^{k} \lambda_i \tag{1.15}$$

and

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geqslant \frac{kn}{4}. \tag{1.16}$$

Furthermore, Yang [25] (cf. [10]) has proved a sharp universal inequality:

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leqslant \frac{4}{n}\sum_{i=1}^{k}\lambda_i(\lambda_{k+1}-\lambda_i), \tag{1.17}$$

which has been called the first inequality of Yang by Ashbaugh ([1] and [2] and so on).

For the Dirichlet eigenvalue problem of Laplacian on a domain in $S^n(1)$, Cheng and Yang [6] have proved the following Yang-type inequality:

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leqslant \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i+\frac{n^2}{4}\right), \tag{1.18}$$

which is optimal since the above inequality becomes an equality for any k when $\Omega = S^n(1)$.

When M is $H^n(-1)$, although many mathematicians want to derive a universal inequality for eigenvalues, there are no any results on universal inequalities for eigenvalues of the eigenvalue problem (1.1) excepting $n=2$. If $n=2$, by making use of estimates for eigenvalues of the eigenvalue problem of the Schrödinger like operator with a weight, Harrell and Michel [14] and Ashbaugh [2] have obtained several results. In fact, if $n=2$, the Laplacian on $H^2(-1)$ is like to the Laplacian on $\mathbf{R}^2$ with a weight (see a formula (3.1)). But, when $n>2$, this property does not hold again. For a bounded domain in $H^n(-1)$, main reason that one could not derive a universal inequality, is that one cannot find an appropriate trial function. It is our purpose to give a universal inequality for eigenvalues of the eigenvalue problem (1.1) when M is the hyperbolic space $H^n(-1)$.

Theorem 1.2. *For a bounded domain Ω in $H^n(-1)$, eigenvalues λ_i's of the eigenvalue problem* (1.1) *satisfy*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leqslant 4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i-\frac{(n-1)^2}{4}\right). \tag{1.19}$$

Let Ω be an n-disk of radius $r>0$ in $H^n(-1)$. McKean [19] (cf. [3] and [12]) has proved that the first eigenvalue $\lambda_1(r)$ of the eigenvalue problem (1.1) satisfies

$$\begin{gathered}\lambda_1(r) \geqslant \frac{(n-1)^2}{4},\\ \lim_{r\to\infty}\lambda_1(r) = \frac{(n-1)^2}{4}.\end{gathered} \tag{1.20}$$

From the domain monotonicity of eigenvalues, we have, for any bounded domain Ω in $H^n(-1)$,

$$\begin{gathered}\lambda_1(\Omega) \geqslant \frac{(n-1)^2}{4},\\ \lim_{\Omega\to H^n(-1)}\lambda_1(\Omega) = \frac{(n-1)^2}{4},\end{gathered} \tag{1.21}$$

where $\Omega \to H^n(-1)$ means that Ω includes an n-disk of radius $r>0$ and $r\to\infty$. It is obvious that, for any $k>1$,

$$\lambda_k(\Omega) > \lambda_1(\Omega) \geqslant \frac{(n-1)^2}{4}.$$

It is important to study the behaviors of $\lambda_k(\Omega)$, for $k \geqslant 2$, when Ω tends to $H^n(-1)$. By making use of the recursion formula of Cheng and Yang [10] (see Section 2) and the universal inequality (1.19) in Theorem 1.2, we prove that all eigenvalues tend to $\frac{(n-1)^2}{4}$ if Ω tends to $H^n(-1)$.

Corollary 1.3. *Let Ω be a bounded domain in $H^n(-1)$. Then, the eigenvalue $\lambda_k(\Omega)$ of the eigenvalue problem* (1.1) *satisfies*

$$\lim_{\Omega \to H^n(-1)} \lambda_k(\Omega) = \frac{(n-1)^2}{4}.$$

2. Lower bounds for eigenvalues

In this section, we will give a proof of Theorem 1.1. In order to prove Theorem 1.1, the following recursion formula of Cheng and Yang [10] plays an important role.

Theorem 2.1. *Let $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_{k+1}$ be any non-negative real numbers satisfying*

$$\sum_{i=1}^{k} (\mu_{k+1} - \mu_i)^2 \leqslant \frac{4}{t} \sum_{i=1}^{k} \mu_i(\mu_{k+1} - \mu_i). \tag{2.1}$$

Define

$$G_k = \frac{1}{k}\sum_{i=1}^{k} \mu_i, \qquad T_k = \frac{1}{k}\sum_{i=1}^{k} \mu_i^2, \qquad F_k = \left(1 + \frac{2}{t}\right) G_k^2 - T_k. \tag{2.2}$$

Then, we have

$$F_{k+1} \leqslant C(t,k)\left(\frac{k+1}{k}\right)^{\frac{4}{t}} F_k, \tag{2.3}$$

where t is any positive real number and

$$C(t,k) = 1 - \frac{1}{3t}\left(\frac{k}{k+1}\right)^{\frac{4}{t}} \frac{(1+\frac{2}{t})(1+\frac{4}{t})}{(k+1)^3} < 1.$$

Proof of Theorem 1.1. Since M is a complete Riemannian manifold, from Nash's theorem, we know that M can be isometrically immersed into a Euclidean space $\mathbf{R}^N$, that is, there exists an isometric immersion:

$$\varphi : M \to \mathbf{R}^N.$$

We denote mean curvature of the immersion φ by $|H|$. Thus, M can be seen as a complete submanifold isometrically immersed into $\mathbf{R}^N$. From Theorem 1.1 in [4] (cf. [13] and [11]), we have

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leqslant \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\left(\lambda_i + \frac{n^2}{4} \sup_{\Omega} |H|^2\right). \tag{2.4}$$

Since eigenvalues are invariants of isometries, we know that the above inequality holds for any isometric immersion from M into a Euclidean space. We define

$$\Phi = \{\varphi;\ \varphi \text{ is an isometric immersion from } M \text{ into a Euclidean space}\}.$$

Putting

$$H_0^2 = \inf_{\varphi \in \Phi} \sup_{\Omega} |H|^2,$$

from the formula (2.4), we infer

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leqslant \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\left(\lambda_i + \frac{n^2}{4} H_0^2\right). \tag{2.5}$$

Letting $\mu_i = \lambda_i + \frac{n^2}{4} H_0^2$, we have

$$\sum_{i=1}^{k} (\mu_{k+1} - \mu_i)^2 \leqslant \frac{4}{n} \sum_{i=1}^{k} (\mu_{k+1} - \mu_i)\mu_i. \tag{2.6}$$

From Theorem 2.1 with $t = n$ of Cheng and Yang [10], we have

$$F_{k+1} \leqslant C(n,k)\left(\frac{k+1}{k}\right)^{\frac{4}{n}} F_k \leqslant \left(\frac{k+1}{k}\right)^{\frac{4}{n}} F_k.$$

Therefore, we infer

$$\frac{F_{k+1}}{(k+1)^{\frac{4}{n}}} \leqslant \frac{F_k}{k^{\frac{4}{n}}}.$$

For any positive integers l and k, we have

$$\frac{F_{k+l}}{(k+l)^{\frac{4}{n}}} \leqslant \frac{F_k}{k^{\frac{4}{n}}}. \tag{2.7}$$

From Weyl's asymptotic formula (1.2)

$$\lim_{l \to \infty} \frac{\lambda_l}{l^{\frac{2}{n}}} = \frac{4\pi^2}{(\omega_n \operatorname{vol} \Omega)^{\frac{2}{n}}},$$

by making use of an elementary computation, we infer

$$\lim_{l \to \infty} \frac{\frac{1}{l}\sum_{i=1}^{l} \lambda_i}{l^{\frac{2}{n}}} = \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \operatorname{vol} \Omega)^{\frac{2}{n}}}$$

and

$$\lim_{l \to \infty} \frac{\frac{1}{l}\sum_{i=1}^{l} \lambda_i^2}{l^{\frac{4}{n}}} = \frac{n}{n+4} \frac{16\pi^4}{(\omega_n \operatorname{vol} \Omega)^{\frac{4}{n}}}.$$

Hence, we obtain, from the definitions of F_k and μ_i,

$$\lim_{l\to\infty}\frac{F_{k+l}}{(k+l)^{\frac{4}{n}}}=\frac{2n}{(n+2)(n+4)}\frac{16\pi^4}{(\omega_n \operatorname{vol}\Omega)^{\frac{4}{n}}}.$$

According to (2.7), we have, for any positive integer k,

$$\frac{F_k}{k^{\frac{4}{n}}}\geqslant\frac{2n}{(n+2)(n+4)}\frac{16\pi^4}{(\omega_n \operatorname{vol}\Omega)^{\frac{4}{n}}}.$$

Since

$$F_k=\left(1+\frac{2}{n}\right)G_k^2-T_k=\frac{2}{n}G_k^2-\frac{1}{k}\sum_{i=1}^{k}(\mu_i-G_k)^2\leqslant\frac{2}{n}G_k^2,$$

we derive

$$\frac{2}{n}\frac{G_k^2}{k^{\frac{4}{n}}}\geqslant\frac{F_k}{k^{\frac{4}{n}}}\geqslant\frac{2n}{(n+2)(n+4)}\frac{16\pi^4}{(\omega_n \operatorname{vol}\Omega)^{\frac{4}{n}}}.$$

Thus, we have proved, from the definition of μ_i,

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i+\frac{n^2}{4}H_0^2\geqslant\frac{n}{\sqrt{(n+2)(n+4)}}\frac{4\pi^2}{(\omega_n \operatorname{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}},\quad \text{for } k=1,2,\ldots.$$

This finishes the proof of Theorem 1.1. □

Proof of Corollary 1.1. Since $S^n(1)$ can be seen as a compact hypersurface in $\mathbf{R}^{n+1}$ with the mean curvature 1, from Theorem 1.1, we have the inequality (1.9). □

Proof of Corollary 1.2. Since M is a complete minimal submanifold in $\mathbf{R}^N$, the mean curvature $|H|=0$. From Theorem 1.1, we have the inequality (1.10). □

Proposition 2.1. *Let Ω be a domain in the n-dimensional complex projective space $CP^n(4)$ of the holomorphic sectional curvature* 1*. Then, eigenvalues λ_i's of the eigenvalue problem* (1.1) *satisfy*

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i+2n(n+1)\geqslant\frac{n}{\sqrt{(n+1)(n+2)}}\frac{4\pi^2}{(\omega_{2n} \operatorname{vol}\Omega)^{\frac{1}{n}}}k^{\frac{1}{n}},\quad \textit{for } k=1,2,\ldots. \tag{2.8}$$

Proof. From the formula (3.21) in Cheng and Yang [8], we infer that the eigenvalues of the eigenvalue problem (1.1) satisfy

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\leqslant\frac{2}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\bigl(\lambda_i+2n(n+1)\bigr). \tag{2.9}$$

From Weyl's asymptotic formula and the same proof as in Theorem 1.1, we can prove Proposition 2.1. □

3. Universal inequality for eigenvalues

In this section, we will give a proof of Theorem 1.2. For convenience, we will use the upper half-plane model of the hyperbolic space, that is,

$$H^n(-1) = \{\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n;\ x_n > 0\}$$

with the standard metric

$$ds^2 = \frac{(dx_1)^2 + (dx_2)^2 + \cdots + (dx_n)^2}{x_n^2}.$$

In this case, by a simple computation, we have the Laplacian on $H^n(-1)$

$$\Delta = x_n^2 \sum_{j=1}^n \frac{\partial^2}{\partial x_j \partial x_j} + (2-n) x_n \frac{\partial}{\partial x_n}. \tag{3.1}$$

From the above formula, we have the following lemma:

Lemma 3.1. *Defining $f_i = x_i$, for $i = 1, 2, \ldots, n-1$, $f_n = \frac{1}{x_n}$ and $f = \log x_n$, then we have*

$$\begin{aligned} \Delta f_i &= 0, \quad \text{for } i = 1, 2, \ldots, n-1, \\ \Delta f_n &= n f_n, \\ \Delta f &= 1 - n. \end{aligned} \tag{3.2}$$

We define a function

$$\varphi_i = f u_i - \sum_{j=1}^n a_{ij} u_j,$$

with $a_{ij} = \int_\Omega f u_i u_j$, where u_i is the eigenfunction corresponding to the eigenvalue λ_i such that $\{u_i\}_{i \in \mathbb{N}}$ becomes an orthonormal basis of $L^2(\Omega)$. It is easy to check

$$\varphi_i = 0 \quad \text{on } \partial\Omega, \qquad \int \varphi_i u_j = 0, \quad \text{for } j = 1, 2, \ldots, k.$$

Hence, φ_i is a trial function. By making use of the Rayleigh–Ritz inequality and the standard assertion on estimates for eigenvalues, we may have the following theorem which has been proved by Cheng and Yang [8]:

Theorem CY. *Let λ_i be the ith eigenvalue of the Dirichlet eigenvalue problem on an n-dimensional compact Riemannian manifold $\bar{\Omega} = \Omega \cup \partial\Omega$ with boundary $\partial\Omega$ and u_i be the orthonormal eigenfunction corresponding to λ_i. Then, for any function $f \in C^2(\Omega) \cap C^1(\partial\Omega)$ and any integer k, we have*

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \|u_i \nabla f\|^2 \leqslant \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \|2 \nabla f \cdot \nabla u_i + u_i \Delta f\|^2,$$

where $\|f\|^2 = \int_M f^2$ and $\nabla f \cdot \nabla u_i = g(\nabla f, \nabla u_i)$.

Proof of Theorem 1.2. Let u_i be the eigenfunction corresponding to the eigenvalue λ_i such that $\{u_i\}_{i\in\mathbb{N}}$ becomes an orthonormal basis of $L^2(\Omega)$. Put $f=\log x_n$. Since $H^n(-1)$ is complete and Ω is a bounded domain, we know that $\bar{\Omega}$ is a compact Riemannian manifold with boundary. From Theorem CY of Cheng and Yang, we infer

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\|u_i\nabla f\|^2\leqslant\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\|2\nabla f\cdot\nabla u_i+u_i\Delta f\|^2.$$

It is not difficult to prove that $|\nabla f|^2=1$. Thus, we have

$$\|u_i\nabla f\|^2=1,$$

and

$$\begin{aligned}\|2\nabla f\cdot\nabla u_i+u_i\Delta f\|^2&=4\int_\Omega(\nabla f\cdot\nabla u_i)^2+4\int_\Omega\nabla f\cdot\nabla u_i(u_i\Delta f)+\int_\Omega(u_i\Delta f)^2\\&=4\int_\Omega(\nabla f\cdot\nabla u_i)^2+4(1-n)\int_\Omega u_i\nabla f\cdot\nabla u_i+(n-1)^2,\end{aligned}$$

according to Lemma 3.1. Since

$$\int_\Omega u_i\nabla f\cdot\nabla u_i=-\int_\Omega u_i\nabla f\cdot\nabla u_i-\int_\Omega(u_i)^2\Delta f,$$

we have

$$\int_\Omega u_i\nabla f\cdot\nabla u_i=\frac{n-1}{2}.$$

From the Cauchy–Schwarz inequality, we have

$$(\nabla f\cdot\nabla u_i)^2\leqslant|\nabla f|^2|\nabla u_i|^2=|\nabla u_i|^2.$$

Hence, we infer

$$\|2\nabla f\cdot\nabla u_i+u_i\Delta f\|^2\leqslant4\|\nabla u_i\|^2-(n-1)^2=4\lambda_i-(n-1)^2.$$

Therefore, we obtain

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\leqslant4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i-\frac{(n-1)^2}{4}\right).$$

This finishes the proof of Theorem 1.2. $\square$

Proof of Corollary 1.3. From Theorem 1.2 and putting $\mu_i=\lambda_i-\frac{(n-1)^2}{4}\geqslant0$, we have

$$\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)^2\leqslant4\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)\mu_i.$$

Thus, Theorem 2.1 holds with $t = 1$. By making use of the recursion formula in Section 2, we have

$$\mu_{k+1} \leqslant 5k^2 \mu_1$$

(cf. Cheng and Yang [10]). Since $\mu_1 \to 0$ when $\Omega \to H^n(-1)$ from (1.21), we have, for a fixed k,

$$\lim_{\Omega \to H^n(-1)} \mu_{k+1} = 0,$$

namely,

$$\lim_{\Omega \to H^n(-1)} \lambda_{k+1} = \frac{(n-1)^2}{4}.$$

This completes the proof of Corollary 1.3. $\square$

4. A remark on a conjecture of Yau

For a compact Riemann surface M_g with genus g, we can consider a closed eigenvalue problem:

$$\Delta u = -\lambda u.$$

By making use of branched conformal maps from M_g to $S^2(1)$, Yang and Yau [26] proved

$$\lambda_1 \leqslant \frac{8\pi(1+g)}{\mathrm{Area}(M_g)}.$$

Furthermore, Yau conjectured the following (see [23]):

Conjecture of Yau. *For a Riemann surface M_g with genus g, there is an absolute constant c such that for any metric on M_g,*

$$\frac{\lambda_k}{k} \leqslant \frac{c(1+g)}{\mathrm{Area}(M_g)}.$$

From Nash's theorem, we know that M_g with a metric can be isometrically immersed into a Euclidean space $\mathbf{R}^N$. By the same proof as in Chen and Cheng [4] and using the recursion formula of Cheng and Yang [10], we infer

$$\frac{\lambda_k}{k} \leqslant 3\left(\lambda_1 + H_0^2\right) \leqslant \frac{24\pi(1+g)}{\mathrm{Area}(M_g)} + 3H_0^2,$$

where H_0 only depends on the M_g.

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