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UNIVERSAL INEQUALITIES FOR EIGENVALUES OF A CLAMPED PLATE PROBLEM ON A HYPERBOLIC SPACE

QING-MING CHENG AND HONGCANG YANG

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Abstract. In this paper, we investigate universal inequalities for eigenvalues of a clamped plate problem on a bounded domain in an n-dimensional hyperbolic space. It is well known that, for a bounded domain in the n-dimensional Euclidean space, Payne, Pólya and Weinberger (1955), Hook (1990) and Chen and Qian (1990) studied universal inequalities for eigenvalues of the clamped plate problem. Recently, Cheng and Yang (2006) have derived the Yang-type universal inequality for eigenvalues of the clamped plate problem on a bounded domain in the n-dimensional Euclidean space, which is sharper than the other ones. For a domain in a unit sphere, Wang and Xia (2007) have also given a universal inequality for eigenvalues. For a bounded domain in the n-dimensional hyperbolic space, although many mathematicians want to obtain a universal inequality for eigenvalues of the clamped plate problem, there are no results on universal inequalities for eigenvalues. The main reason that one could not derive a universal inequality is that one cannot find appropriate trial functions. In this paper, by constructing "nice" trial functions, we obtain a universal inequality for eigenvalues of the clamped plate problem on a bounded domain in the hyperbolic space. Furthermore, we can prove that if the first eigenvalue of the clamped plate problem tends to $\frac{(n-1)^4}{16}$ when the domain tends to the hyperbolic space, then all of the eigenvalues tend to $\frac{(n-1)^4}{16}$.

1. Introduction

Let M and D denote an n-dimensional complete Riemannian manifold and a bounded domain with boundary ∂D in M, respectively. We consider the Dirichlet eigenvalue problem of the biharmonic operator, the so-called *clamped plate problem*, which describes vibrations of a clamped plate:

$$\begin{cases} \Delta^2 u = \Gamma u, & \text{in } D, \\ u|_{\partial D} = \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial D} = 0, \end{cases} \tag{1.1}$$

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where Δ^2 is the biharmonic operator in M and $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative on ∂D.

When $M = \mathbf{R}^n$, for the clamped plate problem, Payne, Pólya and Weinberger [14] and [15] established a universal inequality for eigenvalues. They obtained

$$\Gamma_{k+1} - \Gamma_k \leq \frac{8(n+2)}{n^2} \frac{1}{k} \sum_{i=1}^{k} \Gamma_i. \tag{1.2}$$

Hile and Yeh [10] improved the above result to

$$\sum_{i=1}^{k} \frac{\Gamma_i^{1/2}}{\Gamma_{k+1} - \Gamma_i} \geq \frac{n^2 k^{3/2}}{8(n+2)} \left(\sum_{i=1}^{k} \Gamma_i \right)^{-1/2}. \tag{1.3}$$

Furthermore, Hook [11] and Chen and Qian [3] proved the following inequality:

$$\frac{n^2 k^2}{8(n+2)} \leq \left[\sum_{i=1} \frac{\Gamma_i^{1/2}}{\Gamma_{k+1} - \Gamma_i} \right] \sum_{i=1}^{k} \Gamma_i^{1/2}. \tag{1.4}$$

Ashbaugh in [1] has pointed out whether one can establish inequalities for eigenvalues of the clamped plate problem which are analogs of the inequalities of Yang for eigenvalues of the Dirichlet eigenvalue problem of the Laplacian. In [6], Cheng and Yang have solved the problem of Ashbaugh affirmatively; that is, they have proved the following:

$$\Gamma_{k+1} - \frac{1}{k} \sum_{i=1}^{k} \Gamma_i \leq \left[\frac{8(n+2)}{n^2} \right]^{1/2} \frac{1}{k} \sum_{i=1}^{k} \left[\Gamma_i (\Gamma_{k+1} - \Gamma_i) \right]^{1/2}. \tag{1.5}$$

By making use of Chebyshev's inequality, it is not hard to prove that (1.5) implies (1.4).

When M is a unit sphere, Wang and Xia [16] have also given a universal inequality. They have proved

$$\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 \leq \frac{8(n+2)}{n^2} \sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)(\Gamma_i^{1/2} + \frac{n^2}{2n+4})(\Gamma_i^{1/2} + \frac{n^2}{4}). \tag{1.6}$$

When M is a hyperbolic space $H^n(-1)$, although many mathematicians want to derive a universal inequality for eigenvalues, there are no results on the universal inequalities for eigenvalues of the clamped plate problem (1.1). For a bounded domain in $H^n(-1)$, a main reason that one could not derive a universal inequality for eigenvalues is that one cannot find an appropriate trial function. In this paper, we find "nice" trial functions. By making use of them, we infer a universal inequality for eigenvalues of the eigenvalue problem (1.1).

Theorem 1.1. *Let Γ_i denote the i^{th} eigenvalue of the clamped plate problem* (1.1) *on a bounded domain D in $H^n(-1)$. Then, we have*

$$\sum_{i,j=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 \leq 24 \sum_{i,j=1}^{k} (\Gamma_{k+1} - \Gamma_i) \left\{ \Gamma_i^{\frac{1}{2}} - \frac{(n-1)^2}{4} \right\} \left\{ \Gamma_j^{\frac{1}{2}} - \frac{(n-1)^2}{6} \right\}. \tag{1.7}$$

Furthermore, we have the following Yang-type universal inequality for eigenvalues:

Corollary 1.2. *Let Γ_i denote the i^{th} eigenvalue of the clamped plate problem* (1.1) *on a bounded domain D in $H^n(-1)$. Then, we have*

$$\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \leq 24\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)(\Gamma_i-\frac{(n-1)^4}{16}). \tag{1.8}$$

Remark 1.3. For a buckling problem on a bounded domain in the hyperbolic space, a universal inequality for eigenvalues will be given in a forthcoming paper. Recently, Cheng, Ichikawa and Mametsuka [4] have obtained a universal inequality for eigenvalues of the clamped plate problem on a bounded domain in a complete Riemannian manifold. This occurred after we completed this paper.

For the Dirichlet eigenvalue problem of the Laplacian on a bounded domain in $H^n(-1)$, McKean [13] (cf. [2] and [9]) proved that the first eigenvalue $\lambda_1 \geq \frac{(n-1)^2}{4}$ and $\lim_{D\to H^n(-1)}\lambda_1 = \frac{(n-1)^2}{4}$. In [8], Cheng and Yang have proved that all of the eigenvalues of the Laplacian must tend to $\frac{(n-1)^2}{4}$ when the domain tends to $H^n(-1)$. From the Corollary 1.2 and the recursion formula in Cheng and Yang [7], we have the following:

Theorem 1.4. *Let Γ_i denote the i^{th} eigenvalue of the clamped plate problem* (1.1) *on a bounded domain D in $H^n(-1)$. If $\lim_{D\to H^n(-1)}\Gamma_1 = \frac{(n-1)^4}{16}$, then, for any k, we have*

$$\lim_{D\to H^n(-1)}\Gamma_k = \frac{(n-1)^4}{16}. \tag{1.9}$$

2. Proofs of the theorems

In this section, we shall prove our results.

For convenience, we will use the upper half-plane model of the hyperbolic space; that is,

$$H^n(-1) = \left\{\vec{x} = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n; x_n > 0\right\}$$

with the standard metric

$$ds^2 = \frac{(dx_1)^2 + (dx_2)^2 + \cdots + (dx_n)^2}{x_n^2}.$$

In this case, by a simple computation, we have the Laplacian in $H^n(-1)$:

$$\Delta = x_n^2\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j\partial x_j} + (2-n)x_n\frac{\partial}{\partial x_n}. \tag{2.1}$$

From the above formula, we have the following lemma:

Lemma 2.1. *Defining $f_i = x_i$, for $i = 1, 2, \cdots, n-1$, $f_n = \frac{1}{x_n}$ and $f = \log x_n$, we have*

$$\begin{aligned} &\Delta f_i = 0, \quad \text{for } i = 1, 2, \cdots, n-1,\\ &\Delta f_n = n f_n,\\ &\Delta f = 1 - n. \end{aligned} \tag{2.2}$$

Proof of Theorem 1.1. Let u_i be the i^{th} orthonormal eigenfunction corresponding to the eigenvalue Γ_i, $i = 1, 2, \cdots, k$; that is, u_i satisfies

$$\begin{cases} \Delta^2 u_i = \Gamma_i u_i, & \text{in } D, \\ u_i|_{\partial D} = \left.\dfrac{\partial u_i}{\partial \nu}\right|_{\partial D} = 0, \\ \displaystyle\int_D u_i u_j = \delta_{ij}, \text{ for any } i, j. \end{cases} \tag{2.3}$$

For the function $f = \log x_n$, we have

$$|\nabla f|^2 = \nabla f \cdot \nabla f = 1, \quad \Delta f = 1 - n. \tag{2.4}$$

We define functions

$$\varphi_i = f u_i - \sum_{j=1}^{n} a_{ij} u_j,$$

with $a_{ij} = \int_D f u_i u_j$. Then, we have

$$\varphi_i|_{\partial D} = \left.\frac{\partial \varphi_i}{\partial \nu}\right|_{\partial D} = 0,$$

$$\int_D u_j \varphi_i = 0, \quad \text{for any } i, j = 1, \cdots, k. \tag{2.5}$$

Thus, φ_i's are trial functions. Hence, from the Rayleigh-Ritz inequality we have

$$\Gamma_{k+1} \leq \frac{\int_D (\Delta \varphi_i)^2}{\int_D (\varphi_i)^2}. \tag{2.6}$$

From (2.3), (2.4) and (2.5), we obtain

$$\begin{aligned} \Delta^2 \varphi_i &= \Delta^2 (f u_i - \sum_{j=1}^{k} a_{ij} u_j) \\ &= \Delta(\Delta f u_i + 2\nabla f \cdot \nabla u_i + f \Delta u_i) - \sum_{j=1}^{k} a_{ij} \Gamma_j u_j \\ &= (1-n)\Delta u_i + 2\Delta(\nabla f \cdot \nabla u_i) + \Delta f \Delta u_i \\ &\quad + 2\nabla f \cdot \nabla(\Delta u_i) + f \Delta^2 u_i - \sum_{j=1}^{k} a_{ij} \Gamma_j u_j \\ &= 2(1-n)\Delta u_i + 2\Delta(\nabla f \cdot \nabla u_i) + 2\nabla f \cdot \nabla(\Delta u_i) + \Gamma_i f u_i - \sum_{j=1}^{k} a_{ij} \Gamma_j u_j. \end{aligned}$$

Hence, we infer

$$\begin{aligned} \int_D (\Delta \varphi_i)^2 &= \int_D \varphi_i \Delta^2 \varphi_i \\ &= \Gamma_i \parallel \varphi_i \parallel^2 + 2 \int_D \varphi_i \Big\{ (1-n)\Delta u_i + \Delta(\nabla f \cdot \nabla u_i) + \nabla f \cdot \nabla(\Delta u_i) \Big\}. \end{aligned}$$

Thus,

$$
\begin{aligned}
&(\Gamma_{k+1}-\Gamma_i) \parallel \varphi_i \parallel^2 \\
&\leq 2\int_D \varphi_i\Big\{(1-n)\Delta u_i + \Delta(\nabla f\cdot\nabla u_i) + \nabla f\cdot\nabla(\Delta u_i)\Big\} \\
&= 2\int_D f u_i\Big\{(1-n)\Delta u_i + \Delta(\nabla f\cdot\nabla u_i) + \nabla f\cdot\nabla(\Delta u_i)\Big\} \\
&- 2\sum_{j=1}^{k} a_{ij}\int_D \Big\{(1-n)\Delta u_i + \Delta(\nabla f\cdot\nabla u_i) + \nabla f\cdot\nabla(\Delta u_i)\Big\}u_j.
\end{aligned} \tag{2.7}
$$

Defining b_{ij} by

$$b_{ij} = \int_D \Big\{(1-n)\Delta u_i + \Delta(\nabla f\cdot\nabla u_i) + \nabla f\cdot\nabla(\Delta u_i)\Big\}u_j,$$

we have

$$2b_{ij} = -(\Gamma_i - \Gamma_j)a_{ij} = -2b_{ji}. \tag{2.8}$$

In fact,

$$
\begin{aligned}
b_{ij} &= \int_D (1-n)\Delta u_i u_j + \int_D \nabla f\cdot\nabla u_i\Delta u_j - \int_D (\nabla u_j\cdot\nabla f\Delta u_i + u_j\Delta f\Delta u_i) \\
&= \int_D \nabla f\cdot\nabla u_i\Delta u_j - \int_D \nabla u_j\cdot\nabla f\Delta u_i.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_D \nabla f\cdot\nabla u_i\Delta u_j \\
&= -\int_D \Delta f\Delta u_j u_i - \int_D \nabla f\cdot\nabla(\Delta u_j)u_i \\
&= (n-1)\int_D \Delta u_j u_i + \int_D f\nabla u_i\cdot\nabla(\Delta u_j) + \int_D f u_i\Delta^2 u_j \\
&= (n-1)\int_D \Delta u_j u_i - \int_D \nabla f\cdot\nabla u_i\Delta u_j - \int_D f\Delta u_i\Delta u_j + \Gamma_j\int_D f u_i u_j,
\end{aligned}
$$

we have

$$
\begin{aligned}
&2\int_D \nabla f\cdot\nabla u_i\Delta u_j \\
&= (n-1)\int_D \Delta u_j u_i - \int_D f\Delta u_i\Delta u_j + \Gamma_j\int_D f u_i u_j.
\end{aligned}
$$

Furthermore, we know that

$$
\begin{aligned}
&2\int_D \nabla f\cdot\nabla u_j\Delta u_i \\
&= (n-1)\int_D \Delta u_i u_j - \int_D f\Delta u_j\Delta u_i + \Gamma_i\int_D f u_j u_i.
\end{aligned}
$$

Hence, we infer that

$$2b_{ij} = -(\Gamma_i - \Gamma_j)a_{ij}.$$

From (2.7) and (2.8), we have

$$(2.9)\quad \begin{aligned}&(\Gamma_{k+1}-\Gamma_i)\parallel \varphi_i \parallel^2\\ &\leq 2\int_D fu_i\Big\{(1-n)\Delta u_i+\Delta(\nabla f\cdot\nabla u_i)+\nabla f\cdot\nabla(\Delta u_i)\Big\}+\sum_{j=1}^k(\Gamma_i-\Gamma_j)a_{ij}^2.\end{aligned}$$

Since

$$\int_D fu_i\Delta u_i=(1-n)+2\int_D\nabla f\cdot\nabla u_iu_i+\int_D fu_i\Delta u_i,$$

we infer that

$$(2.10)\quad \int_D u_i\nabla f\cdot\nabla u_i=\frac{n-1}{2}.$$

By a direct computation, we have

$$(2.11)\quad \begin{aligned}&\int_D fu_i\Delta(\nabla f\cdot\nabla u_i)\\ &=\int_D\Big\{\Delta fu_i+2\nabla f\cdot\nabla u_i+f\Delta u_i\Big\}\nabla f\cdot\nabla u_i\\ &=(1-n)\int_D u_i\nabla f\cdot\nabla u_i+2\int_D(\nabla f\cdot\nabla u_i)^2+\int_D f\Delta u_i\nabla f\cdot\nabla u_i,\end{aligned}$$

$$(2.12)\quad \begin{aligned}&\int_D fu_i\nabla f\cdot\nabla(\Delta u_i)\\ &=-\int_D\Delta u_i\nabla f\cdot\nabla(fu_i)-\int_D f\Delta fu_i\Delta u_i\\ &=-\int_D u_i\Delta u_i-\int_D f\Delta u_i\nabla f\cdot\nabla u_i-(1-n)\int_D fu_i\Delta u_i.\end{aligned}$$

Therefore, we derive

$$(2.13)\quad \begin{aligned}&2\int_D fu_i\Big\{(1-n)\Delta u_i+\Delta(\nabla f\cdot\nabla u_i)+\nabla f\cdot\nabla(\Delta u_i)\Big\}\\ &=2\int_D|\nabla u_i|^2+4\int_D(\nabla f\cdot\nabla u_i)^2-(n-1)^2\\ &\leq 6\int_D|\nabla u_i|^2-(n-1)^2.\end{aligned}$$

Thus, from $\int_D|\nabla u_i|^2\leq\Gamma_i^{\frac{1}{2}}$, we derive

$$(2.14)\quad \begin{aligned}&(\Gamma_{k+1}-\Gamma_i)\parallel\varphi_i\parallel^2\\ &\leq 6\Gamma_i^{\frac{1}{2}}-(n-1)^2+\sum_{j=1}^k(\Gamma_i-\Gamma_j)a_{ij}^2.\end{aligned}$$

Defining

$$c_{ij}=\int_D(\nabla f\cdot\nabla u_i-\frac{n-1}{2}u_i)u_j,$$

since

$$\int_D\nabla f\cdot\nabla u_iu_j=(n-1)\int_D u_iu_j-\int_D\nabla f\cdot\nabla u_ju_i,$$

we have

$$c_{ij} = -\int_D (\nabla f \cdot \nabla u_j - \frac{n-1}{2} u_j) u_i = -c_{ji}. \tag{2.15}$$

According to $|\nabla f|^2 = 1$ and

$$\int_D f u_i \nabla f \cdot \nabla u_i = -\int_D \Delta f f u_i^2 - \int_D u_i^2 |\nabla f|^2 - \int_D f u_i \nabla f \cdot \nabla u_i,$$

we have

$$2\int_D f u_i \nabla f \cdot \nabla u_i = -1 + (n-1)\int_D f u_i^2.$$

Hence, we infer that

$$\begin{aligned} & -2\int_D \varphi_i (\nabla f \cdot \nabla u_i - \frac{n-1}{2} u_i) \\ & = -2\int_D f u_i \nabla f \cdot \nabla u_i + 2\sum_{j=1}^k a_{ij} \int_D u_j \nabla f \cdot \nabla u_i \\ & = 1 + 2\sum_{j=1}^k a_{ij} \int_D (\nabla f \cdot \nabla u_i - \frac{n-1}{2} u_i) u_j \\ & = 1 + 2\sum_{j=1}^k a_{ij} c_{ij}. \end{aligned} \tag{2.16}$$

On the other hand, for any positive constant α_i, we have

$$\begin{aligned} & 1 + 2\sum_{j=1}^k a_{ij} c_{ij} \\ & = -2\int_D \varphi_i (\nabla f \cdot \nabla u_i - \frac{n-1}{2} u_i) \\ & = -2\int_D \varphi_i (\nabla f \cdot \nabla u_i - \frac{n-1}{2} u_i - \sum_{j=1}^k c_{ij} u_j) \\ & \le \alpha_i \|\varphi_i\|^2 + \frac{1}{\alpha_i} \|\nabla f \cdot \nabla u_i - \frac{n-1}{2} u_i - \sum_{j=1}^k c_{ij} u_j\|^2 \\ & = \alpha_i \|\varphi_i\|^2 + \frac{1}{\alpha_i} \Big\{ \|\nabla f \cdot \nabla u_i - \frac{n-1}{2} u_i\|^2 - \sum_{j=1}^k c_{ij}^2 \Big\}. \end{aligned} \tag{2.17}$$

If $\Gamma_{k+1} - \Gamma_i > 0$, we define

$$\alpha_i = (\Gamma_{k+1} - \Gamma_i)\beta_i.$$

Hence, for any i and for $\beta_i > 0$, we infer that

$$\begin{aligned} & (\Gamma_{k+1} - \Gamma_i)^2 (1 + 2\sum_{j=1}^k a_{ij} c_{ij}) \\ & \le (\Gamma_{k+1} - \Gamma_i)^3 \beta_i \|\varphi_i\|^2 + \frac{1}{\beta_i} (\Gamma_{k+1} - \Gamma_i) \Big\{ \|\nabla f \cdot \nabla u_i - \frac{n-1}{2} u_i\|^2 - \sum_{j=1}^k c_{ij}^2 \Big\}. \end{aligned}$$

From (2.10) and

$$\|\nabla f\cdot\nabla u_i-\frac{n-1}{2}u_i\|^2=\|\nabla f\cdot\nabla u_i\|^2-\frac{(n-1)^2}{4}\le\|\nabla u_i\|^2-\frac{(n-1)^2}{4},$$

we have

$$\begin{aligned}&(\Gamma_{k+1}-\Gamma_i)^2(1+2\sum_{j=1}^k a_{ij}c_{ij})\\&\le(\Gamma_{k+1}-\Gamma_i)^2\beta_i\Big\{6\Gamma_i^{\frac12}-(n-1)^2+\sum_{j=1}^k(\Gamma_i-\Gamma_j)a_{ij}^2\Big\}\\&+\frac{1}{\beta_i}(\Gamma_{k+1}-\Gamma_i)\Big\{\Gamma_i^{\frac12}-\frac{(n-1)^2}{4}\Big\}-\frac{1}{\beta_i}(\Gamma_{k+1}-\Gamma_i)\sum_{j=1}^k c_{ij}^2.\end{aligned}\tag{2.18}$$

Since $c_{ij}=-c_{ji}$, we have

$$\sum_{i,j=1}^k(\Gamma_{k+1}-\Gamma_i)^2a_{ij}c_{ij}=-\sum_{i,j=1}^k(\Gamma_{k+1}-\Gamma_i)(\Gamma_i-\Gamma_j)a_{ij}c_{ij}.\tag{2.19}$$

Thus,

$$\begin{aligned}2\sum_{i,j=1}^k(\Gamma_{k+1}-\Gamma_i)(\Gamma_i-\Gamma_j)a_{ij}c_{ij}-\sum_{i,j=1}^k(\Gamma_{k+1}-\Gamma_j)(\Gamma_i-\Gamma_j)^2\beta_ia_{ij}^2\\-\sum_{i,j=1}^k\frac{1}{\beta_i}(\Gamma_{k+1}-\Gamma_i)c_{ij}^2\le 0.\end{aligned}$$

According to (2.18), (2.19) and the above inequality, we derive

$$\begin{aligned}&\sum_{i,j=1}^k(\Gamma_{k+1}-\Gamma_i)^2\\&\le\sum_{i,j=1}^k(\Gamma_{k+1}-\Gamma_i)^2\beta_i\Big\{6\Gamma_i^{\frac12}-(n-1)^2\Big\}\\&+\sum_{i,j=1}^k(\Gamma_{k+1}-\Gamma_i)^2\beta_i(\Gamma_i-\Gamma_j)a_{ij}^2+\sum_{i,j=1}^k(\Gamma_{k+1}-\Gamma_i)\beta_i(\Gamma_i-\Gamma_j)^2a_{ij}^2\\&+\sum_{i,j=1}^k\frac{1}{\beta_i}(\Gamma_{k+1}-\Gamma_i)\Big\{\Gamma_i^{\frac12}-\frac{(n-1)^2}{4}\Big\}.\end{aligned}\tag{2.20}$$

From the variational principle, we can prove that

$$\Gamma_i\ge\lambda_i^2,$$

where λ_i denotes the i^{th} eigenvalue of the Dirichlet eigenvalue problem of the Laplacian on the same domain D. Since $\lambda_1\ge\dfrac{(n-1)^2}{4}$, putting

$$\beta_i=\frac{1}{\Gamma_i^{\frac12}-\dfrac{(n-1)^2}{6}}\beta>0,$$

we have

$$\begin{aligned}
&\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\beta_i(\Gamma_i-\Gamma_j)a_{ij}^2+\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)\beta_i(\Gamma_i-\Gamma_j)^2a_{ij}^2\\
&=\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)(\Gamma_{k+1}-\Gamma_j)\beta_i(\Gamma_i-\Gamma_j)a_{ij}^2\\
&=\frac{1}{2}\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)(\Gamma_{k+1}-\Gamma_j)(\Gamma_i-\Gamma_j)(\beta_i-\beta_j)a_{ij}^2\\
&=-\frac{\beta}{2}\sum_{i,j=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)(\Gamma_{k+1}-\Gamma_j)(\Gamma_i^{\frac{1}{2}}+\Gamma_j^{\frac{1}{2}})(\Gamma_i^{\frac{1}{2}}-\Gamma_j^{\frac{1}{2}})^2}{(\Gamma_i^{\frac{1}{2}}-\frac{(n-1)^2}{6})(\Gamma_j^{\frac{1}{2}}-\frac{(n-1)^2}{6})}a_{ij}^2\\
&\leq 0.
\end{aligned}\tag{2.21}$$

From (2.20) and the above inequality, we obtain

$$\begin{aligned}
&\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\leq 6\beta\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\\
&+\frac{1}{\beta}\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left\{\Gamma_i^{\frac{1}{2}}-\frac{(n-1)^2}{4}\right\}\left\{\Gamma_j^{\frac{1}{2}}-\frac{(n-1)^2}{6}\right\}.
\end{aligned}\tag{2.22}$$

Taking

$$\beta^2=\frac{\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left\{\Gamma_i^{\frac{1}{2}}-\frac{(n-1)^2}{4}\right\}\left\{\Gamma_j^{\frac{1}{2}}-\frac{(n-1)^2}{6}\right\}}{6\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2},$$

we derive

$$\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\leq 24\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left\{\Gamma_i^{\frac{1}{2}}-\frac{(n-1)^2}{4}\right\}\left\{\Gamma_j^{\frac{1}{2}}-\frac{(n-1)^2}{6}\right\}.$$

This finishes the proof of Theorem 1.1. □

Proof of Corollary 1.2. Since $\Gamma_i^{\frac{1}{2}}\geq\frac{(n-1)^2}{4}$, we have

$$\left\{\Gamma_i^{\frac{1}{2}}-\frac{(n-1)^2}{4}\right\}\left\{\Gamma_j^{\frac{1}{2}}-\frac{(n-1)^2}{6}\right\}\leq\Gamma_i-\frac{(n-1)^4}{16}.$$

From Theorem 1.1, Corollary 1.2 is proved. □

Proof of Theorem 1.4. According to the following recursion formula of Cheng and Yang [7] with $\mu_i=\Gamma_i-\frac{(n-1)^4}{16}$ and $t=\frac{1}{6}$, we have, by making use of the same assertion as in Cheng and Yang [7], that

$$\mu_{k+1}\leq 25k^{12}\mu_1;$$

that is,

$$\Gamma_{k+1}-\frac{(n-1)^4}{16}\leq 25k^{12}(\Gamma_1-\frac{(n-1)^4}{16}).$$

Hence, if $\lim_{D\to H^n(-1)}\Gamma_1 = \dfrac{(n-1)^4}{16}$, then, for any fixed k, we have

$$\lim_{D\to H^n(-1)}\Gamma_k = \frac{(n-1)^4}{16}. \tag{2.23}$$

This completes the proof of Theorem 1.4. ∎

Recursive Formula (Cheng and Yang [7])**.** *Let* $\mu_1 \le \mu_2 \le \cdots \le \mu_{k+1}$ *be any non-negative real numbers satisfying*

$$\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)^2 \le \frac{4}{t}\sum_{i=1}^{k}\mu_i(\mu_{k+1}-\mu_i). \tag{2.24}$$

Define

$$G_k = \frac{1}{k}\sum_{i=1}^{k}\mu_i, \qquad T_k = \frac{1}{k}\sum_{i=1}^{k}\mu_i^2, \quad F_k = \left(1+\frac{2}{t}\right)G_k^2 - T_k. \tag{2.25}$$

Then, we have

$$F_{k+1} \le C(t,k)\left(\frac{k+1}{k}\right)^{\frac{4}{t}} F_k, \tag{2.26}$$

where t *is any positive real number and*

$$C(t,k) = 1 - \frac{1}{3t}\left(\frac{k}{k+1}\right)^{\frac{4}{t}}\frac{\left(1+\frac{2}{t}\right)\left(1+\frac{4}{t}\right)}{(k+1)^3} < 1.$$

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Department of Mathematics, Faculty of Science and Engineering, Saga University, Saga 840-8502, Japan
E-mail address: cheng@ms.saga-u.ac.jp

Academy of Mathematics and Systematical Sciences, The Chinese Academy of Sciences, Beijing 100080, People's Republic of China
E-mail address: yanghc@math03.math.ac.cn