UNIVERSAL INEQUALITIES FOR EIGENVALUES OF A SYSTEM OF ELLIPTIC EQUATIONS

QING-MING CHENG AND HONGCANG YANG

Abstract. Let D be a bounded domain in an n-dimensional Euclidean space $\mathbf{R}^n$. Assume that

$$0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k \leq \cdots \to \infty$$

are eigenvalues of an eigenvalue problem of a system of n elliptic equations:

$$\begin{cases} \Delta \mathbf{u} + \alpha \mathrm{grad}(\mathrm{div}\mathbf{u}) = -\sigma \mathbf{u}, \text{ in } D, \\ \mathbf{u}|_{\partial D} = \mathbf{0}. \end{cases}$$

In particular, when $n = 3$, the eigenvalue problem describes the behavior of the elastic vibration. In this paper, we obtain universal inequalities for eigenvalues of the above eigenvalue problem by making use of a direct and explicit method, our results are sharper than one of Hook [7]. Furthermore, a universal inequality for lower order eigenvalues of the above eigenvalue problem is also derived.

1. Introduction

Let $\mathbf{R}^n$ denote an n-dimensional Euclidean space and D be a bounded domain in $\mathbf{R}^n$. For $n = 3$, we consider an eigenvalue problem for a system of equations of classical elasticity:

$$\begin{cases} \mu \Delta \mathbf{u} + (\lambda + \mu)\mathrm{grad}(\mathrm{div}(\mathbf{u})) = -\sigma \mathbf{u}, \text{ in } D, \\ \mathbf{u}|_{\partial D} = \mathbf{0}, \end{cases} \tag{1.1}$$

where Δ is the Lapalacian in $\mathbf{R}^3$, $\mathbf{u} = (u_1, u_2, u_3)$ denotes the elastic displacement vector, which is a vector-valued function from D into $\mathbf{R}^3$ and $\lambda, \mu > 0$ are the Lamé constants. Here, we have denoted the divergence of $\mathbf{u}$ by $\mathrm{div}(\mathbf{u})$ and the gradient of a function f by grad f, respectively. Eigenvalues of (1.1) satisfy

$$0 < \sigma_1 \leq \sigma_2 \leq \sigma_3 \leq \cdots \leq \sigma_k \leq \cdots \to \infty.$$

Here each eigenvalue is repeated according to its multiplicity. Pleijel [11] obtained an asymptotic formula for eigenvalues of (1.1):

$$\sigma_k \sim \mu \left(\frac{6\pi^2}{[2 + (2 + \lambda/\mu)^{-3/2}]} \right)^{\frac{2}{3}} (\frac{k}{V})^{\frac{2}{3}} \quad \text{as } k \to \infty,$$

Key words and phrases: eigenvalue, a universal inequality for eigenvalues, a system of equations of classical elasticity.

2001 *Mathematics Subject Classification*: 35P15.

where V denotes the volume of D. From the above formula, we have

$$\sum_{j=1}^{k}\sigma_j \sim \frac{3\mu}{5}\left(\frac{6\pi^2}{[2+(2+\lambda/\mu)^{-3/2}]V}\right)^{\frac{2}{3}}k^{\frac{5}{3}} \quad \text{as } k\to\infty.$$

Further, Levine and Protter [9] proved

$$\sum_{j=1}^{k}\sigma_j \geq \frac{3\mu}{5}\left(\frac{2\pi^2}{V}\right)^{\frac{2}{3}}k^{\frac{5}{3}} \quad \text{for } \ k=1,2,\cdots$$

and they conjectured

$$\sigma_k \geq \mu\left(\frac{3}{[2+(2+\lambda/\mu)^{-3/2}]}\right)^{\frac{2}{3}}(\frac{2\pi^2 k}{V})^{\frac{2}{3}} \quad \text{for } k=1,2,\cdots.$$

In fact, Levine and Protter studied the more general setting. Let D be a bounded domain in $\mathbf{R}^n$. They considered the following eigenvalue problem for a system of n elliptic equations:

$$\begin{cases} \mu\Delta\mathbf{u}+(\lambda+\mu)\mathrm{grad}(\mathrm{div}(\mathbf{u}))=-\sigma\mathbf{u}, \ \text{in } D,\\ \mathbf{u}|_{\partial D}=\mathbf{0}, \end{cases} \tag{1.2}$$

where Δ is the Lapalacian in $\mathbf{R}^n$, $\mathbf{u}=(u_1,u_2,\cdots,u_n)$ is a vector-valued function. They derived

$$\sum_{j=1}^{k}\sigma_j \geq \frac{4\pi^2 n\mu}{n+2}\frac{k^{1+2/n}}{(V\omega_{n-1})^{\frac{2}{n}}} \quad \text{for } \ k=1,2,\cdots,$$

where ω_{n-1} is the volume of the $n-1$-dimensional unit sphere.

For convenience, we rewrite (1.2) into

$$\begin{cases} \Delta\mathbf{u}+\alpha\mathrm{grad}(\mathrm{div}\mathbf{u})=-\sigma\mathbf{u}, \ \text{in } D,\\ \mathbf{u}|_{\partial D}=\mathbf{0}. \end{cases} \tag{1.3}$$

Here $\alpha=\dfrac{\lambda+\mu}{\mu}\geq 0$ and we still use the σ to denote the eigenvalue $\dfrac{\sigma}{\mu}$. Thus, the result of Levine and Protter becomes

$$\sum_{j=1}^{k}\sigma_j \geq \frac{4\pi^2 n}{n+2}\frac{k^{1+2/n}}{(V\omega_{n-1})^{\frac{2}{n}}} \quad \text{for } \ k=1,2,\cdots.$$

On the other hand, Hook [7] has studied universal inequalities for eigenvalues of (1.3). He has proved

$$\sum_{i=1}^{k}\frac{\sigma_i}{\sigma_{k+1}-\sigma_i}\geq\frac{n^2k}{4(n+\alpha)}, \ \text{for } k=1,2,\cdots. \tag{1.4}$$

In [7], Hook used an abstract method to prove the above inequality. In this paper, we will make use of a direct and explicit method to derive universal inequalities for eigenvalues of (1.3), which are sharper than one of Hook.

Theorem 1.1. *Let σ_i denote the i^{th} eigenvalue of the eigenvalue problem* (1.3)*. Then we have*

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i) \le \frac{2\sqrt{n+\alpha}}{n}\left\{\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{\frac{1}{2}}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{\frac{1}{2}}\sigma_i\right\}^{1/2}. \tag{1.5}$$

From the above theorem 1.1, we can conclude a more explicit inequality:

Corollary 1.1. *Under the assumption of the theorem 1.1, we have*

$$\sigma_{k+1} \le \left(1+\frac{4(n+\alpha)}{n^2}\right)\frac{1}{k}\sum_{i=1}^{k}\sigma_i. \tag{1.6}$$

Proof of Corollary 1.1. From the Chebyshev's inequality, we have

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{\frac{1}{2}}\sigma_i \le \frac{1}{k}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{\frac{1}{2}}\sum_{i=1}^{k}\sigma_i.$$

Since

$$\left(\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{\frac{1}{2}}\right)^2 \le k\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i),$$

the theorem 1.1 and the above inequalities imply

$$\left(\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\right)^2 \le \frac{4(n+\alpha)}{n^2}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\sum_{i=1}^{k}\sigma_i,$$

which is a quadratic inequality of σ_{k+1}. Thus, we have

$$\sigma_{k+1} \le \left(1+\frac{4(n+\alpha)}{n^2}\right)\frac{1}{k}\sum_{i=1}^{k}\sigma_i.$$

Remark 1.1. It is obvious that the inequality (1.5) is sharper than the inequality (1.6). On the other hand, from the Chebyshev's inequality, we have

$$\frac{1}{k}\sum_{i=1}^{k}\frac{\sigma_i}{\sigma_{k+1}-\sigma_i} \ge \frac{\sum_{i=1}^{k}\sigma_i}{\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)}.$$

Hence, we know that our inequality (1.6) is sharper than the inequality (1.4) of Hook.

When one considers the Dirichlet eigenvalue problem of the Laplacian:

$$\begin{cases} \Delta u = -\lambda u, & \text{in } D, \\ u|_{\partial D} = 0, \end{cases}$$

Payne, Pólya and Weinberger [12], [13] proved

$$\lambda_{k+1}-\lambda_k \le \frac{2}{k}\sum_{i=1}^{k}\lambda_i, \ k=1,2,\cdots.$$

As a generalization of the above result, Hile and Protter [6] proved

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1}-\lambda_i} \geq \frac{nk}{4}, \text{ for } k = 1, 2, \cdots .$$

Further, Yang [14] (cf. [5]) obtained a very sharp inequality, that is, he derived

$$\sum_{i=1}^{k} \Big(\lambda_{k+1}-\lambda_i\Big)\Big(\lambda_{k+1}-(1+\frac{4}{n})\lambda_i\Big) \leq 0, \text{ for } k = 1, 2, \cdots ,$$

which is called Yang's first inequality. According to the Yang's first inequality, we can infer

$$\lambda_{k+1} \leq \frac{1}{k}(1+\frac{4}{n})\sum_{i=1}^{k}\lambda_i, \text{ for } k = 1, 2, \cdots ,$$

which is called Yang's second inequality. In [1] and [2], Ashbaugh has proved the following relation: the Yang's first inequality $\Longrightarrow$ the Yang's second inequality $\Longrightarrow$ the inequality of Hile and Protter $\Longrightarrow$ the inequality of Payne, Pólya and Weinberger.

For the eigenvalue problem (1.3), the result of Hook is corresponding to the result of Hile and Protter and our results are corresponding to the results of Yang.

For the investigation of universal inequalities for eigenvalues of the clamped plate problem or the buckling problem, see [3] and [4] in details.

Remark 1.2. *After we finished our paper, professor Parnovski told us their paper* [10]*, which is a very interesting paper. In* [10]*, Levitin and Parnovski have considered the gaps of any two consecutive eigenvalues, they have obtained*

$$\sigma_{k+1}-\sigma_k \leq \frac{\max\{4+\alpha^2; (n+2)\alpha+8\}}{n+\alpha}\frac{1}{k}\sum_{i=1}^{k}\sigma_i, \text{ for } k = 1, 2, \cdots .$$

For lower order eigenvalues of the eigenvalue problem (1.3), we have the following stronger result:

Theorem 1.2. *Let σ_i denote the i^{th} eigenvalue of the eigenvalue problem* (1.3)*, $i = 1, 2, \cdots, n+1$. Then, we have*

$$\frac{\sigma_2+\sigma_3+\cdots+\sigma_{n+1}}{\sigma_1} \leq n+4(1+\alpha).$$

2. Proof of Theorem 1.1

In this section, we shall prove the theorem 1.1.

Proof of Theorem 1.1. Let $g_p = x^p$, $p = 1, \cdots, n$, where $x^1, x^2, \cdots, x^n$ are the standard Euclidean coordinate functions and $\mathbf{e}_1 = (1, 0, \cdots, 0), \mathbf{e}_2 = (0, 1, \cdots, 0), \mathbf{e}_n = (0, 0, \cdots, 1)$. Assume that $\{\mathbf{u}_i\}_{i=1}^{\infty}$ is a sequence of orthonormal vector-valued eigenfunctions corresponding to the i^{th} eigenvalue σ_i, that is, $\mathbf{u}_i$ satisfies

$$\begin{cases} \Delta \mathbf{u}_i + \alpha \mathrm{grad}(\mathrm{div}(\mathbf{u}_i)) = -\sigma_i \mathbf{u}_i, & \text{in } D, \\ \mathbf{u}_i|_{\partial D} = \mathbf{0}, \\ \displaystyle\int_D \langle \mathbf{u}_i, \mathbf{u}_j \rangle = \delta_{ij}, \ \text{for any } i, j. \end{cases} \tag{2.1}$$

Defining a vector-valued function $\mathbf{v}_i$ by

$$\mathbf{v}_i = g_p \mathbf{u}_i - \sum_{j=1}^{k} a_{ij} \mathbf{u}_j, \tag{2.2}$$

where $a_{ij} = \int_D g_p \langle \mathbf{u}_i, \mathbf{u}_j \rangle = a_{ji}$, then we have

$$\mathbf{v}_i|_{\partial D} = \mathbf{0}, \quad \int_D \langle \mathbf{u}_j, \mathbf{v}_i \rangle = 0, \quad \text{for any } i, j = 1, \cdots, k. \tag{2.3}$$

Hence, $\mathbf{v}_i$ is a trial vector-valued function. From the Rayleigh-Ritz inequality (cf. [8]), we have

$$\sigma_{k+1} \leq \frac{\int_D -\langle \Delta \mathbf{v}_i, \mathbf{v}_i \rangle + \alpha (\mathrm{div}(\mathbf{v}_i))^2}{\int_D |\mathbf{v}_i|^2}, \tag{2.4}$$

where $|\mathbf{v}_i|^2 = \langle \mathbf{v}_i, \mathbf{v}_i \rangle$. From the definition of g_p, we have

$$\mathrm{grad} g_p = \mathbf{e}_p. \tag{2.5}$$

Putting $\mathbf{u}_{i,p} = \partial_p \mathbf{u}_i$, from the definition of $\mathbf{v}_i$, we derive

$$\begin{aligned} \Delta \mathbf{v}_i &= \Delta(g_p \mathbf{u}_i) - \sum_{j=1}^{k} a_{ij} \Delta \mathbf{u}_j \\ &= g_p \Delta \mathbf{u}_i + 2\mathbf{u}_{i,p} - \sum_{j=1}^{k} a_{ij} \Delta \mathbf{u}_j \\ &= g_p \Big(-\sigma_i \mathbf{u}_i - \alpha \mathrm{grad}(\mathrm{div}(\mathbf{u}_i)) \Big) + 2\mathbf{u}_{i,p} \\ &- \sum_{j=1}^{k} a_{ij} \Big(-\sigma_j \mathbf{u}_j - \alpha \mathrm{grad}(\mathrm{div}(\mathbf{u}_j)) \Big) \\ &= -\sigma_i g_p \mathbf{u}_i + \sum_{j=1}^{k} a_{ij} \sigma_j \mathbf{u}_j + 2\mathbf{u}_{i,p} \\ &- \alpha g_p \mathrm{grad}(\mathrm{div}(\mathbf{u}_i)) + \alpha \sum_{j=1}^{k} a_{ij} \mathrm{grad}(\mathrm{div}(\mathbf{u}_j)). \end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&\int_D -\langle \Delta \mathbf{v}_i, \mathbf{v}_i\rangle \\
=&\int_D \Big\{ -\langle -\sigma_i g_p \mathbf{u}_i + \sum_{j=1}^k a_{ij}\sigma_j \mathbf{u}_j + 2\mathbf{u}_{i,p} \\
&- \alpha g_p \mathrm{grad}(\mathrm{div}(\mathbf{u}_i)) + \alpha \sum_{j=1}^k a_{ij}\mathrm{grad}(\mathrm{div}(\mathbf{u}_j)), \mathbf{v}_i\rangle \Big\} \qquad (2.6)\\
=&\sigma_i \parallel \mathbf{v}_i \parallel^2 - 2\int_D \langle \mathbf{u}_{i,p}, \mathbf{v}_i\rangle \\
&+ \alpha\Big(\int_D \langle g_p \mathrm{grad}(\mathrm{div}(\mathbf{u}_i)), \mathbf{v}_i\rangle - \sum_{j=1}^k a_{ij}\int_D \langle \mathrm{grad}(\mathrm{div}(\mathbf{u}_j)), \mathbf{v}_i\rangle \Big),
\end{aligned}
$$

where

$$\|\mathbf{v}_i\|^2 = \int_D |\mathbf{v}_i|^2.$$

Define

$$b_{ij} = \int_D \langle \mathbf{u}_{i,p}, \mathbf{u}_j\rangle = -b_{ji}. \qquad (2.7)$$

By a simple calculation, we have, from (2.2),

$$
\begin{aligned}
&\int_D \langle \mathbf{u}_{i,p}, \mathbf{v}_i\rangle \\
&= \int_D \langle \mathbf{u}_{i,p}, g_p\mathbf{u}_i - \sum_{j=1}^k a_{ij}\mathbf{u}_j\rangle \\
&= \int_D g_p \langle \mathbf{u}_{i,p}, \mathbf{u}_i\rangle - \sum_{j=1}^k a_{ij} b_{ij}.
\end{aligned}
$$

Since

$$\int_D g_p\langle \mathbf{u}_{i,p}, \mathbf{u}_i\rangle = -\int_D \langle \mathbf{u}_i, \partial_p(g_p\mathbf{u}_i)\rangle = -\int_D \langle \mathbf{u}_i, \mathbf{u}_i\rangle - \int_D g_p\langle \mathbf{u}_{i,p}, \mathbf{u}_i\rangle,$$

we have

$$\int_D g_p\langle \mathbf{u}_{i,p}, \mathbf{u}_i\rangle = -\frac{1}{2}.$$

Thus, we derive

$$-2\int_D g_p\langle \mathbf{u}_{i,p}, \mathbf{v}_i\rangle = 1 + 2\sum_{j=1}^k a_{ij} b_{ij}. \qquad (2.8)$$

From

$$\mathrm{div}(g_p\mathbf{v}_i) = g_p\mathrm{div}(\mathbf{v}_i) + \langle \mathbf{v}_i, \mathbf{e}_p\rangle, \quad \mathrm{div}(g_p\mathbf{u}_i) = g_p\mathrm{div}(\mathbf{u}_i) + \langle \mathbf{u}_i, \mathbf{e}_p\rangle,$$

we infer

$$
\begin{aligned}
&\int_D \langle g_p \mathrm{grad}(\mathrm{div}(\mathbf{u}_i)), \mathbf{v}_i\rangle - \sum_{j=1}^{k} a_{ij}\int_D \langle \mathrm{grad}(\mathrm{div}(\mathbf{u}_j)), \mathbf{v}_i\rangle \\
&= -\int_D \mathrm{div}(\mathbf{u}_i)\mathrm{div}(g_p\mathbf{v}_i) + \sum_{j=1}^{k} a_{ij}\int_D \mathrm{div}(\mathbf{u}_j)\mathrm{div}(\mathbf{v}_i) \\
&= -\int_D \mathrm{div}(\mathbf{u}_i)(g_p\mathrm{div}(\mathbf{v}_i) + \langle \mathbf{v}_i, \mathbf{e}_p\rangle) + \sum_{j=1}^{k} a_{ij}\int_D \mathrm{div}(\mathbf{u}_j)\mathrm{div}(\mathbf{v}_i) \\
&= -\int_D \mathrm{div}(g_p\mathbf{u}_i)\mathrm{div}(\mathbf{v}_i) + \sum_{j=1}^{k} a_{ij}\int_D \mathrm{div}(\mathbf{u}_j)\mathrm{div}(\mathbf{v}_i) \\
&+ \int_D \Big(\mathrm{div}(\mathbf{v}_i)\langle \mathbf{u}_i, \mathbf{e}_p\rangle - \mathrm{div}(\mathbf{u}_i)\langle \mathbf{v}_i, \mathbf{e}_p\rangle\Big) \\
&= -\int_D (\mathrm{div}(\mathbf{v}_i))^2 + \int_D \Big(\mathrm{div}(\mathbf{v}_i)\langle \mathbf{u}_i, \mathbf{e}_p\rangle - \mathrm{div}(\mathbf{u}_i)\langle \mathbf{v}_i, \mathbf{e}_p\rangle\Big).
\end{aligned}
$$

Then, according to (2.4), (2.6) and (2.8), we obtain

$$
(\sigma_{k+1} - \sigma_i) \parallel \mathbf{v}_i \parallel^2 \leq 1 + 2\sum_{j=1}^{k} a_{ij}b_{ij} + \alpha\int_D \Big(\mathrm{div}(\mathbf{v}_i)\langle \mathbf{u}_i, \mathbf{e}_p\rangle - \mathrm{div}(\mathbf{u}_i)\langle \mathbf{v}_i, \mathbf{e}_p\rangle\Big). \quad (2.9)
$$

From (2.1), we derive

$$
\begin{aligned}
&\sigma_j a_{ij} = \int_D \sigma_j g_p \langle \mathbf{u}_i, \mathbf{u}_j\rangle \\
&= -\int_D \langle g_p\mathbf{u}_i, \Delta\mathbf{u}_j + \alpha\mathrm{grad}(\mathrm{div}(\mathbf{u}_j))\rangle \\
&= -\int_D \langle \Delta(g_p\mathbf{u}_i), \mathbf{u}_j\rangle + \alpha\int_D \mathrm{div}(g_p\mathbf{u}_i)\mathrm{div}(\mathbf{u}_j) \\
&= -\int_D \langle g_p\Delta\mathbf{u}_i, \mathbf{u}_j\rangle - 2\int_D \langle \mathbf{u}_{i,p}, \mathbf{u}_j\rangle + \alpha\int_D \Big(g_p\mathrm{div}(\mathbf{u}_i)\mathrm{div}(\mathbf{u}_j) + \langle \mathbf{u}_i, \mathbf{e}_p\rangle\mathrm{div}(\mathbf{u}_j)\Big) \\
&= \int_D \langle g_p(\sigma_i\mathbf{u}_i + \alpha\mathrm{grad}(\mathrm{div}(\mathbf{u}_i))), \mathbf{u}_j\rangle - 2\int_D \langle \mathbf{u}_{i,p}, \mathbf{u}_j\rangle \\
&+ \alpha\int_D \Big(g_p\mathrm{div}(\mathbf{u}_i)\mathrm{div}(\mathbf{u}_j) + \langle \mathbf{u}_i, \mathbf{e}_p\rangle\mathrm{div}(\mathbf{u}_j)\Big) \\
&= \sigma_i a_{ij} - 2\int_D \langle \mathbf{u}_{i,p}, \mathbf{u}_j\rangle \\
&+ \alpha\int_D \Big(\langle g_p\mathrm{grad}(\mathrm{div}(\mathbf{u}_i)), \mathbf{u}_j\rangle + g_p\mathrm{div}(\mathbf{u}_i)\mathrm{div}(\mathbf{u}_j) + \langle \mathbf{u}_i, \mathbf{e}_p\rangle\mathrm{div}(\mathbf{u}_j)\Big) \\
&= \sigma_i a_{ij} - 2b_{ij} + \alpha\int_D \Big(\langle \mathbf{u}_i, \mathbf{e}_p\rangle\mathrm{div}(\mathbf{u}_j) - \langle \mathbf{u}_j, \mathbf{e}_p\rangle\mathrm{div}(\mathbf{u}_i)\Big).
\end{aligned}
$$

Hence,

$$2b_{ij} = (\sigma_i - \sigma_j)a_{ij} + \alpha \int_D \Big(\langle \mathbf{u}_i, \mathbf{e}_p \rangle \mathrm{div}(\mathbf{u}_j) - \langle \mathbf{u}_j, \mathbf{e}_p \rangle \mathrm{div}(\mathbf{u}_i) \Big), \tag{2.10}$$

$$\begin{aligned} &\mathrm{div}(\mathbf{v}_i)\langle \mathbf{u}_i, \mathbf{e}_p \rangle - \mathrm{div}(\mathbf{u}_i)\langle \mathbf{v}_i, \mathbf{e}_p \rangle \\ &= \mathrm{div}(g_p \mathbf{u}_i - \sum_{j=1}^k a_{ij}\mathbf{u}_j)\langle \mathbf{u}_i, \mathbf{e}_p \rangle - \mathrm{div}(\mathbf{u}_i)\langle g_p \mathbf{u}_i - \sum_{j=1}^k a_{ij}\mathbf{u}_j, \mathbf{e}_p \rangle \\ &= \langle \mathbf{u}_i, \mathbf{e}_p \rangle^2 - \sum_{j=1}^k a_{ij} \Big(\langle \mathbf{u}_i, \mathbf{e}_p \rangle \mathrm{div}(\mathbf{u}_j) - \langle \mathbf{u}_j, \mathbf{e}_p \rangle \mathrm{div}(\mathbf{u}_i) \Big). \end{aligned} \tag{2.11}$$

We infer, from (2.9), (2.10) and (2.11),

$$(\sigma_{k+1} - \sigma_i) \parallel \mathbf{v}_i \parallel^2 \leq 1 + \sum_{j=1}^k (\sigma_i - \sigma_j)a_{ij}^2 + \alpha \int_D \langle \mathbf{u}_i, \mathbf{e}_p \rangle^2. \tag{2.12}$$

On the other hand, for any constant $A_i > 0$, we have, from (2.3), (2.7) and (2.8),

$$\begin{aligned} 1 + 2\sum_{j=1}^k a_{ij}b_{ij} &= -2\int_D \langle \mathbf{u}_{i,p}, \mathbf{v}_i \rangle \\ &= \int_D \langle -2\mathbf{u}_{i,p} + 2\sum_{j=1}^k b_{ij}\mathbf{u}_j, \mathbf{v}_i \rangle \\ &\leq A_i \int_D |\mathbf{v}_i|^2 + \frac{1}{A_i}\Big(\int_D |\mathbf{u}_{i,p}|^2 - \sum_{j=1}^k b_{ij}^2 \Big). \end{aligned} \tag{2.13}$$

Multiplying (2.13) by $(\sigma_{k+1} - \sigma_i)$, we infer, from (2.12),

$$\begin{aligned} &(1 + 2\sum_{j=1}^k a_{ij}b_{ij})(\sigma_{k+1} - \sigma_i) \\ \leq &(\sigma_{k+1} - \sigma_i) \left\{ A_i \|\mathbf{v}_i\|^2 + \frac{1}{A_i}\Big(\|\mathbf{u}_{i,p}\|^2 - \sum_{j=1}^k b_{ij}^2 \Big) \right\} \\ \leq &A_i \Big(1 + \sum_{j=1}^k (\sigma_i - \sigma_j)a_{ij}^2 + \alpha \int_D \langle \mathbf{u}_i, \mathbf{e}_p \rangle^2 \Big) \\ &+ \frac{\sigma_{k+1} - \sigma_i}{A_i} \left(\|\mathbf{u}_{i,p}\|^2 - \sum_{j=1}^k b_{ij}^2 \right). \end{aligned} \tag{2.14}$$

Putting $A_i = (\sigma_{k+1} - \sigma_i)^{1/2} A$, we have

$$\begin{aligned}
&(\sigma_{k+1} - \sigma_i) + 2\sum_{j=1}^{k} (\sigma_{k+1} - \sigma_i) a_{ij} b_{ij} \\
&\leq A(\sigma_{k+1} - \sigma_i)^{1/2} \left(1 + \sum_{j=1}^{k} (\sigma_i - \sigma_j) a_{ij}^2 + \alpha \int_D \langle \mathbf{u}_i, \mathbf{e}_p \rangle^2 \right) \\
&+ \frac{(\sigma_{k+1} - \sigma_i)^{1/2}}{A} \left(\|\mathbf{u}_{i,p}\|^2 - \sum_{j=1}^{k} b_{ij}^2 \right).
\end{aligned} \tag{2.15}$$

Taking sum on i from 1 to k for (2.15), we obtain

$$\begin{aligned}
&\sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i) + 2\sum_{i,j=1}^{k} (\sigma_{k+1} - \sigma_i) a_{ij} b_{ij} \\
&\leq A \sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)^{1/2} \left(1 + \alpha \int_D \langle \mathbf{u}_i, \mathbf{e}_p \rangle^2 \right) + \frac{1}{A} \sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)^{1/2} \|\mathbf{u}_{i,p}\|^2 \\
&+ A \sum_{i,j=1}^{k} (\sigma_{k+1} - \sigma_i)^{1/2} (\sigma_i - \sigma_j) a_{ij}^2 - \frac{1}{A} \sum_{i,j=1}^{k} (\sigma_{k+1} - \sigma_i)^{1/2} b_{ij}^2.
\end{aligned} \tag{2.16}$$

Since a_{ij} is symmetric and b_{ij} is anti-symmetric, we have

$$\begin{aligned}
&2\sum_{i,j=1}^{k} (\sigma_{k+1} - \sigma_i) a_{ij} b_{ij} \\
&= \sum_{i,j=1}^{k} (\sigma_{k+1} - \sigma_i) a_{ij} b_{ij} + \sum_{i,j=1}^{k} (\sigma_{k+1} - \sigma_j) a_{ji} b_{ji} \\
&= -\sum_{i,j=1}^{k} (\sigma_i - \sigma_j) a_{ij} b_{ij} \\
&\geq -\frac{1}{2} \Bigg\{ A \sum_{i,j=1}^{k} \frac{(\sigma_i - \sigma_j)^2 a_{ij}^2}{(\sigma_{k+1} - \sigma_i)^{1/2} + (\sigma_{k+1} - \sigma_j)^{1/2}} \\
&+ \frac{1}{2A} \sum_{i,j=1}^{k} \left((\sigma_{k+1} - \sigma_i)^{1/2} + (\sigma_{k+1} - \sigma_j)^{1/2} \right) b_{ij}^2 \Bigg\}.
\end{aligned}$$

$$\frac{1}{A} \sum_{i,j=1}^{k} (\sigma_{k+1} - \sigma_i)^{1/2} b_{ij}^2 = \frac{1}{2A} \sum_{i,j=1}^{k} \left((\sigma_{k+1} - \sigma_i)^{1/2} + (\sigma_{k+1} - \sigma_j)^{1/2} \right) b_{ij}^2,$$

$$\begin{aligned}
&\sum_{i,j=1}^{k}(\sigma_{k+1}-\sigma_i)^{1/2}(\sigma_i-\sigma_j)a_{ij}^2\\
&=\frac{1}{2}\left(\sum_{i,j=1}^{k}(\sigma_{k+1}-\sigma_i)^{1/2}(\sigma_i-\sigma_j)a_{ij}^2+\sum_{i,j=1}^{k}(\sigma_{k+1}-\sigma_j)^{1/2}(\sigma_j-\sigma_i)a_{ij}^2\right)\\
&=-\frac{1}{2}\sum_{i,j=1}^{k}\frac{(\sigma_i-\sigma_j)^2a_{ij}^2}{(\sigma_{k+1}-\sigma_i)^{1/2}+(\sigma_{k+1}-\sigma_j)^{1/2}}.
\end{aligned}$$

From (2.16), we have

$$\begin{aligned}
&\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\\
&\le A\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{1/2}\left(1+\alpha\int_D\langle\mathbf{u}_i,\mathbf{e}_p\rangle^2\right)+\frac{1}{A}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{1/2}\|\mathbf{u}_{i,p}\|^2.
\end{aligned}$$

Taking sum on p from 1 to n, we obtain

$$\begin{aligned}
&n\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\\
&\le nA\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{1/2}+\alpha A\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{1/2}\sum_{p=1}^{n}\int_D\langle\mathbf{u}_i,\mathbf{e}_p\rangle^2\\
&+\frac{1}{A}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{1/2}\sum_{p=1}^{n}\|\mathbf{u}_{i,p}\|^2.
\end{aligned}$$

Since

$$\sum_{p=1}^{n}\int_D\langle\mathbf{u}_i,\mathbf{e}_p\rangle^2=1,\quad \sum_{p=1}^{n}\|\mathbf{u}_{i,p}\|^2=\sigma_i-\alpha\int_D(\mathrm{div}(\mathbf{u}_i))^2\le\sigma_i,$$

we infer

$$\begin{aligned}
&n\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\\
&\le(n+\alpha)A\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{1/2}+\frac{1}{A}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{1/2}\sigma_i.
\end{aligned}$$

Putting

$$\frac{1}{A}=\left\{\frac{(n+\alpha)\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{1/2}}{\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{1/2}\sigma_i}\right\}^{\frac{1}{2}},$$

we have

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\le\frac{2(n+\alpha)^{1/2}}{n}\left\{\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{1/2}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{1/2}\sigma_i\right\}^{\frac{1}{2}}.$$

This completes the proof of the theorem 1.1.

3. Proof of Theorem 1.2

In this section, we shall give a proof of the theorem 1.2.

Proof of Theorem 1.2. Defining $c_{ij} = \int_D x^i \langle \mathbf{u}_1, \mathbf{u}_{j+1}\rangle$, $i, j = 1, 2, \cdots, n$, we consider an $n \times n$-matrix $C = (c_{ij})$. From the orthogonalization of Gram and Schmidt, there exist an upper triangle matrix $T = (T_{ij})$ and an orthogonal matrix $P = (p_{ij})$ such that

$$T = PC.$$

Thus, we have

$$T_{ij} = \sum_{k=1}^{n} p_{ik} c_{kj} = \sum_{k=1}^{n} p_{ik} \int_D x^k \langle \mathbf{u}_1, \mathbf{u}_{j+1}\rangle = 0, \text{ for } i > j.$$

Putting $y_i = \sum_{k=1}^{n} p_{ik} x^k$, we have

$$\int_D y_i \langle \mathbf{u}_1, \mathbf{u}_{j+1}\rangle = 0, \text{ for } i > j.$$

We define a vector-valued function $\mathbf{w}_i$ by

$$\mathbf{w}_i = (y_i - a_i)\mathbf{u}_1, \quad a_i = \int_D y_i \langle \mathbf{u}_1, \mathbf{u}_1\rangle.$$

We infer

$$\mathbf{w}_i|_{\partial D} = \mathbf{0}, \quad \int_D \langle \mathbf{u}_j, \mathbf{w}_i\rangle = 0, \quad \text{for any } j = 1, \cdots, i.$$

Hence, $\mathbf{w}_i$ is a trial vector-valued function. From the Rayleigh-Ritz inequality, we have

$$\sigma_{i+1} \leq \frac{\int_D -\langle \Delta \mathbf{w}_i, \mathbf{w}_i\rangle + \alpha(\mathrm{div}(\mathbf{w}_i))^2}{\int_D |\mathbf{w}_i|^2}. \tag{3.1}$$

From the definition of y_i, we have

$$\mathrm{grad} y_i = \mathbf{p}_i = (p_{i1}, p_{i2}, \cdots, p_{in}).$$

$$\begin{aligned} \Delta \mathbf{w}_i &= \Delta\{(y_i - a_i)\mathbf{u}_1\} \\ &= (y_i - a_i)\Delta \mathbf{u}_1 + 2\sum_{k=1}^{n} p_{ik}\mathbf{u}_{1,k} \\ &= (y_i - a_i)(-\sigma_1 \mathbf{u}_1 - \alpha \mathrm{grad}(\mathrm{div}(\mathbf{u}_1)) + 2\sum_{k=1}^{n} p_{ik}\mathbf{u}_{1,k}. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\int_D -\langle \Delta \mathbf{w}_i, \mathbf{w}_i\rangle \\ =&\int_D -\langle (y_i - a_i)(-\sigma_1 \mathbf{u}_1 - \alpha \mathrm{grad}(\mathrm{div}(\mathbf{u}_1)) + 2\sum_{k=1}^{n} p_{ik}\mathbf{u}_{1,k}, \mathbf{w}_i\rangle \\ =&\sigma_1 \parallel \mathbf{w}_i \parallel^2 - 2\int_D \langle \sum_{k=1}^{n} p_{ik}\mathbf{u}_{1,k}, \mathbf{w}_i\rangle + \alpha \int_D \langle (y_i - a_i)\mathrm{grad}(\mathrm{div}(\mathbf{u}_1)), \mathbf{w}_i\rangle. \end{aligned} \tag{3.2}$$

Since

$$\begin{aligned}
&\int_D \sum_{k=1}^n p_{ik}\langle \mathbf{u}_{1,k}, \mathbf{w}_i\rangle \\
&= \int_D \sum_{k=1}^n p_{ik}\langle \mathbf{u}_{1,k}, (y_i - a_i)\mathbf{u}_1\rangle \\
&= -\int_D \sum_{k=1}^n p_{ik}\langle \mathbf{u}_1, (y_i - a_i)\mathbf{u}_{1,k}\rangle - \int_D \sum_{k=1}^n p_{ik}^2\langle \mathbf{u}_1, \mathbf{u}_1\rangle,
\end{aligned}$$

we have

$$-2\int_D \sum_{k=1}^n p_{ik}\langle \mathbf{u}_{1,k}, \mathbf{w}_i\rangle = 1. \tag{3.3}$$

$$\begin{aligned}
&\int_D \langle (y_i - a_i)\mathrm{grad}(\mathrm{div}(\mathbf{u}_1)), \mathbf{w}_i\rangle \\
&= -\int_D \mathrm{div}(\mathbf{u}_1)\mathrm{div}((y_i - a_i)\mathbf{w}_i) \\
&= -\int_D \mathrm{div}(\mathbf{u}_1)\{(y_i - a_i)\mathrm{div}(\mathbf{w}_i) + \langle \mathbf{p}_i, \mathbf{w}_i\rangle\} \qquad (3.4)\\
&= -\int_D (\mathrm{div}(\mathbf{w}_i))^2 + \int_D \Big(\mathrm{div}(\mathbf{w}_i)\langle \mathbf{u}_1, \mathbf{p}_i\rangle - \mathrm{div}(\mathbf{u}_1)\langle \mathbf{w}_i, \mathbf{p}_i\rangle\Big) \\
&= -\int_D (\mathrm{div}(\mathbf{w}_i))^2 + \int_D \langle \mathbf{u}_1, \mathbf{p}_i\rangle^2.
\end{aligned}$$

Thus, we obtain from (3.1), (3.2), (3.3) and (3.4)

$$(\sigma_{i+1} - \sigma_1) \parallel \mathbf{w}_i \parallel^2 \le 1 + \alpha \int_D \langle \mathbf{u}_1, \mathbf{p}_i\rangle^2. \tag{3.5}$$

On the other hand, for any constant $B_i > 0$, we have

$$\begin{aligned}
1 &= -2\int_D \sum_{k=1}^n p_{ik}\langle \mathbf{u}_{1,k}, \mathbf{w}_i\rangle \\
&\le B_i \int_D |\mathbf{w}_i|^2 + \frac{1}{B_i}\int_D |\sum_{k=1}^n p_{ik}\mathbf{u}_{1,k}|^2.
\end{aligned}$$

If $\sigma_{i+1} - \sigma_1 = 0$, it is obvious that

$$\sigma_{i+1} - \sigma_1 \le \frac{1}{2(1+\alpha)}(\sigma_{i+1} - \sigma_1)^2 \int_D |\mathbf{w}_i|^2 + 2(1+\alpha)\int_D |\sum_{k=1}^n p_{ik}\mathbf{u}_{1,k}|^2.$$

If $\sigma_{i+1} - \sigma_1 > 0$, taking $B_i = \dfrac{1}{2(1+\alpha)}(\sigma_{i+1} - \sigma_1)$, we also have

$$\sigma_{i+1} - \sigma_1 \le \frac{1}{2(1+\alpha)}(\sigma_{i+1} - \sigma_1)^2 \int_D |\mathbf{w}_i|^2 + 2(1+\alpha)\int_D |\sum_{k=1}^n p_{ik}\mathbf{u}_{1,k}|^2.$$

Hence, for $i = 1, 2, \cdots, n$, we derive from (3.5) and the above inequality

$$\sigma_{i+1} - \sigma_1 \leq \frac{1}{2(1+\alpha)}(\sigma_{i+1} - \sigma_1)(1 + \alpha \int_D \langle \mathbf{u}_1, \mathbf{p}_i \rangle^2) + 2(1+\alpha) \int_D |\sum_{k=1}^n p_{ik} \mathbf{u}_{1,k}|^2.$$

Since $\int_D \langle \mathbf{u}_1, \mathbf{p}_i \rangle^2 \leq 1$, we have

$$\sigma_{i+1} - \sigma_1 \leq 4(1+\alpha) \int_D |\sum_{k=1}^n p_{ik} \mathbf{u}_{1,k}|^2.$$

Taking sum on i from 1 to n, we infer

$$\sum_{i=1}^n (\sigma_{i+1} - \sigma_1) \leq 4(1+\alpha) \sum_{i=1}^n \int_D |\sum_{k=1}^n p_{ik} \mathbf{u}_{1,k}|^2.$$

$$\sum_{i=1}^n \int_D |\sum_{k=1}^n p_{ik} \mathbf{u}_{1,k}|^2 = \sum_{k=1}^n \int_D |\mathbf{u}_{1,k}|^2 \leq \sigma_1.$$

Therefore,

$$\frac{\sigma_2 + \sigma_3 + \cdots + \sigma_{n+1}}{\sigma_1} \leq n + 4(1+\alpha).$$

This finishes the proof of the theorem 2.

Acknowledgements

The first author is partly supported by a Grant-in-Aid for Scientific Research from JSPS and the second author is partly supported by NSF of China and Fund of CAS.

References

[1] M. S. Ashbaugh, Isoperimetric and universal inequalities for eigenvalues, in Spectral theory and geometry (Edinburgh, 1998), E B Davies and Yu Safalov eds., London Math. Soc. Lecture Notes, vol. **273,** Cambridge Univ. Press, Cambridge, 1999, pp. 95-139.

[2] M. S. Ashbaugh, Universal eigenvalue bounds of Payne-Pólya-Weinberger, Hile-Prottter and H.C. Yang, Proc. Indian Acad. Sci. Math. Sci. vol. **112**(2002), pp. 3-30.

[3] Q. -M. Cheng and H. C. Yang, Inequalities for eigenvalues of a clamped plate problem, Trans. Amer. Math. Soc., **358** (2006), 2625-2635.

[4] Q. -M. Cheng and H. C. Yang, Universal bounds for eigenvalues of a buckling problem, Commun. Math. Phys., **262** (2006), 663-675.

[5] Q. -M. Cheng and H. C. Yang, Bounds on eigenvalues of Dirichlet Laplacian, Math. Ann., **337** (2007), 159-175.

[6] G. N. Hile and M. H. Protter, Inequalities for eigenvalues of the Laplacian, Indiana Univ. Math. J. **29** (1980), 523-538.

[7] S. M. Hook, Domain independent upper bounds for eigenvalues of elliptic operator, Trans. Amer. Math. Soc. **318** (1990), 615-642.

[8] B. Kawohl and G. Sweers, Remarks on eigenvalues and eigenfunctions of a special elliptic system, Z. Angew. Math. Phys. **38** (1987), 730-740.

[9] H. A. Levine and M. H. Protter, Unrestricted lower bounds for eigenvalues of elliptic equations and systems of equations with applications to problems in elasticity, Math. Methods Appl. Sci. **7** (1985), 210-222.

[10] M. Levitin and L. Parnovski, Commutators, spectral trance identities, and universal estimates for eigenvalues, J. Functional Analysis, **192** (2002), 425-445.

[11] A. Pleijel, Proprietés asymptotique des fonctions fondamentales du problems des vibrations dans un corps élastique, Arkiv f. Mat. Astr. o. Fysik **26** (1939), 1-9.

[12] L. E. Payne, G. Polya and H. F. Weinberger, Sur le quotient de deux fréquences propres consécutives, Comptes Rendus Acad. Sci. Paris **241** (1955), 917-919.

[13] L. E. Payne, G. Polya and H. F. Weinberger, On the ratio of consecutive eigenvalues, J. Math. and Phys. **35** (1956),289–298.

[14] H. C. Yang, An estimate of the differance between consecutive eigenvalues, preprint IC/91/60 of ICTP, Trieste, 1991.

Qing-Ming Cheng
Department of Mathematics
Faculty of Science and Engineering
Saga University
Saga 840-8502, Japan
e-mail: cheng@ms.saga-u.ac.jp

Hongcang Yang
Academy of Mathematics and Systematical Sciences
CAS, Beijing 100080, China
e-mail: yanghc@math03.math.ac.cn