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SCALAR CURVATURE OF HYPERSURFACES WITH CONSTANT MEAN CURVATURE IN A SPHERE

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Abstract. Let M be an n-dimensional closed hypersurface with constant mean curvature H satisfying $|H| \leq \varepsilon(n)$ in a unit sphere S^{n+1}, $n \leq 7$, and S the square of the length of the second fundamental form of M. There exists a constant $\delta(n, H) > 0$, which depends only on n and H, such that if $S_0 \leq S \leq S_0 + \delta(n, H)$, then $S \equiv S_0$ and M is isometric to a Clifford hypersurface, where $\varepsilon(n)$ is a sufficiently small constant depending on n and $S_0 = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}$.

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1. Introduction. Let M be an n-dimensional closed hypersurface with constant mean curvature H in an (n+1)-dimensional unit sphere S^{n+1}. Denote by S and R the squared length of the second fundamental form and scalar curvature of M, respectively.

When $H \equiv 0$, a famous rigidity theorem due to Simons [**11**], Lawson [**5**], Chern, do Carmo and Kobayashi [**4**] says that if $S \leq n$, then $S \equiv 0$, or $S \equiv n$. That is, M is isometric to a totally geodesic sphere S^n or a Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$. These two kinds of hypersurfaces are the so-called isoparametric ones of types 1 and 2, respectively, where a hypersurface of S^n is called isoparametric of type g if it has g distinct constant principal curvatures of constant multiplicities. The following conjecture is proposed by Chern, we can find it in Yau [**13**]:

Chern Conjecture. For any $n \geq 3$, the set R_n of all the real numbers each of which can be realised as the constant scalar curvature of a closed minimally immersed hypersurface in S^{n+1} is discrete.

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There have been some works related to the Chern conjecture. In [**9**], Peng and Terng proved that if the scalar curvature of M is a constant, then there exists a positive constant $\delta(n)$ depending only on n such that if $n \leq S \leq n + \delta(n)$, then $S = n$. Furthermore, Cheng and Yang [**3**] improved the pinching constant $\delta(n)$ to $n/3$.

Without the assumption of constant scalar curvature, Peng and Terng [**10**] obtained the following important pinching theorem.

THEOREM 1.1 ([**10**]). *Let M be an n-dimensional closed minimal hypersurface in S^{n+1}, $n \leq 5$. Then there exists a positive constant $\delta(n)$ depending only on n such that if $n \leq S \leq n + \delta(n)$, then $S \equiv n$ and M is isometric to a Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$.*

Further, they proposed the following attractive problem:

OPEN PROBLEM. Let M be an n-dimensional closed minimal hypersurface in S^{n+1}, $n \geq 6$. Does there exist a positive constant $\delta(n)$ depending only on n such that if $n \leq S \leq n + \delta(n)$, then $S \equiv n$ and M is isometric to a Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$?

In [**2**], Cheng and Ishikawa have solved the above problem under a condition on Ricci curvature. Recently, Wei and Xu [**12**] have solved the open problem for $n = 6, 7$ through the following theorem.

THEOREM 1.2 ([**12**]). *Let M be an n-dimensional closed minimal hypersurface in S^{n+1}, $n = 6, 7$. Then there exists a positive constant $\delta(n)$ depending only on n such that if $n \leq S \leq n + \delta(n)$, then $S \equiv n$ and M is isometric to a Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$.*

When H is constant, that is, M is a hypersurface with constant mean curvature, a third author [**7**] extended Theorem 1.1 of Peng and Terng [**10**] for minimal hypersurfaces to the case of hypersurfaces with constant mean curvature H.

THEOREM 1.3 ([**7**]). *Let M be an n-dimensional closed hypersurface with constant mean curvature H satisfying $|H| \leq \varepsilon(n)$ in a unit sphere S^{n+1}, $n \leq 5$, and S the square of the length of the second fundamental form of M. Then there exists a constant $\delta(n, H) > 0$, which depends only on n and H, such that if $S_0 \leq S \leq S_0 + \delta(n, H)$, then $S \equiv S_0$ and M is isometric to a Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$ if $H = 0$; M is isometric to a Clifford hypersurface*

$$C_{1,n-1} = S^1\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times S^{n-1}\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)$$

if $H \neq 0$, where $\lambda = \frac{nH+\sqrt{n^2H^2+4(n-1)}}{2}$ and $\varepsilon(n)$ is a sufficiently small constant depending on n,

$$S_0 = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}. \tag{1.1}$$

In this paper, we study the case of $n = 6, 7$. We prove the following theorem.

THEOREM 1.4. *Let M be an n-dimensional closed hypersurface with constant mean curvature H satisfying $|H| \leq \varepsilon(n)$ in a unit sphere S^{n+1}, $n \leq 7$, and S the squared length of the second fundamental form of M. There exists a constant $\delta(n, H) > 0$, which depends only on n and H, such that if $S_0 \leq S \leq S_0 + \delta(n, H)$, then $S \equiv S_0$ and M is isometric to a Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$ if $H = 0$; M is isometric to a Clifford hypersurface*

$$C_{1,n-1} = S^1\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times S^{n-1}\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)$$

if $H \neq 0$.

REMARK 1.1. When $H \equiv 0$, Theorem 1.3 reduces to Theorem 1.1 and Theorem 1.4 reduces to Theorem 1.2.

2. Preliminaries. Let M be a closed hypersurface with constant mean curvature H in S^{n+1}, and $e_1, \dots, e_n, e_{n+1}$ a local orthonormal frame field of S^{n+1} along M, such that $e_1, \dots, e_n$ are tangent to M. Let $\omega_1, \dots, \omega_n$ be the dual coframe field of $e_1, \dots, e_n$. We shall make use of the following convention on the range of indices:

$$1 \leq i, j, k, \dots \leq n.$$

We have the structure equation of M:

$$\begin{aligned} &dx = \textstyle\sum_i \omega_i e_i, \\ &de_i = \textstyle\sum_j \omega_{ij} e_j + \sum_j h_{ij} e_{n+1} - \omega_i x, \\ &de_{n+1} = -\textstyle\sum_{ij} h_{ij}\omega_j e_i, \\ &d\omega_{ij} = \textstyle\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{kl} R_{ijkl}\omega_k \wedge \omega_l, \end{aligned} \tag{2.1}$$

where $h_{ij} = h_{ji}$ and $R_{ijkl} = -R_{ijlk}$.

We have the Gauss equation (see, for example, [**1**])

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + h_{ik}h_{jl} - h_{il}h_{jk}. \tag{2.2}$$

Let R and $\mathbf{h}$ be the scalar curvature and the second fundamental form of M respectively. Denote by S the squared length of $\mathbf{h}$ and H the mean curvature of M. Then we have the following formulas:

$$\mathbf{h} = \sum_{ij} h_{ij}\omega_i \otimes \omega_j, \quad S = \sum_{ij} h_{ij}^2, \quad H = \frac{1}{n}\sum_i h_{ii}. \tag{2.3}$$

From the Gauss equations, we have

$$R = n(n-1) + n^2H^2 - S. \tag{2.4}$$

Denote by h_{ijk}, h_{ijkl} and h_{ijklm} components of the first, second and third covariant derivatives of the second fundamental form, respectively. Then (see [**6**])

$$\nabla\mathbf{h} = \sum_{ijk} h_{ijk}\omega_i \otimes \omega_j \otimes \omega_k, \quad h_{ijk} = h_{ikj}, \tag{2.5}$$

$$\nabla^2\mathbf{h} = \sum_{ijkl} h_{ijkl}\omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l, \tag{2.6}$$

$$h_{ijkl} = h_{ijlk} + \sum_m h_{mj} R_{mikl} + \sum_m h_{im} R_{mjkl}, \tag{2.7}$$

$$h_{ijklm} = h_{ijkml} + \sum_r h_{rjk} R_{rilm} + \sum_r h_{irk} R_{rjlm} + \sum_r h_{ijr} R_{rklm}. \tag{2.8}$$

For an arbitrary fixed point $p \in M$, we take orthonormal frames such that $h_{ij} = \lambda_i \delta_{ij}$ at p, for all i, j. Then at the point p, we have

$$\sum_i \lambda_i = nH, \quad \sum_i \lambda_i^2 = S. \tag{2.9}$$

We define f_3, f_4 to be

$$f_3 = \sum_{ijk} h_{ij} h_{jk} h_{ki}, \quad f_4 = \sum_{ijkl} h_{ij} h_{jk} h_{kl} h_{li}. \tag{2.10}$$

Then, at the point p, we have

$$f_3 = \sum_i \lambda_i^3, \quad f_4 = \sum_i \lambda_i^4. \tag{2.11}$$

We define A, B, μ_i, $\tilde{A}$, $\tilde{B}$ by

$$A = \sum_{ijk} h_{ijk}^2 \lambda_i^2, \quad B = \sum_{ijk} h_{ijk}^2 \lambda_i \lambda_j, \tag{2.12}$$

$$\mu_i = \lambda_i + nH, \quad \tilde{A} = \sum_{ijk} h_{ijk}^2 \mu_i^2, \quad \tilde{B} = \sum_{ijk} h_{ijk}^2 \mu_i \mu_j. \tag{2.13}$$

Then

$$\sum_i \mu_i^2 = S + n^2(n+2)H^2. \tag{2.14}$$

Since $H =$ constant, using (2.2), (2.5), (2.7) and (2.8), we easily get

$$\frac{1}{2}\triangle S = S(n-S) - n^2H^2 + nHf_3 + |\nabla \mathbf{h}|^2 \tag{2.15}$$

and

$$\begin{aligned}\frac{1}{2}\triangle(|\nabla \mathbf{h}|^2) = (2n+3-S)|\nabla \mathbf{h}|^2 - \frac{3}{2}(A-2B) - \frac{3}{2}(\tilde{A}-2\tilde{B}) \\ - \frac{3}{2}n^2H^2|\nabla \mathbf{h}|^2 + \frac{3}{2}|\nabla S|^2 + |\nabla^2 \mathbf{h}|^2.\end{aligned} \tag{2.16}$$

Further, the following formulas can be found in [**7**]:

LEMMA 2.1 ([**7**]). *Let M be a closed hypersurface with constant mean curvature H in S^{n+1}. Then*

$$|\nabla^2 \mathbf{h}|^2 \geq \frac{3}{2}[Sf_4 - f_3^2 - S^2 - S(S-n) - n^2H^2 + 2nHf_3], \tag{2.17}$$

$$\int_M A - 2B = \int_M Sf_4 - f_3^2 - S^2 + nHf_3 - \frac{|\nabla S|^2}{4}, \tag{2.18}$$

$$3(\tilde{A} - 2\tilde{B}) \le \frac{\sqrt{17}+1}{2}(S + n^2(n+2)H^2)|\nabla \mathbf{h}|^2. \tag{2.19}$$

Proof. From (2.7) and the Gauss equation (2.2) we have

$$h_{ijij} - h_{jiji} = (\lambda_i - \lambda_j)(1 + \lambda_i\lambda_j). \tag{2.20}$$

Thus,

$$\begin{aligned}
|\nabla^2 \mathbf{h}|^2 &\ge \sum_i h_{iiii}^2 + 3\sum_{i\neq j} h_{ijij}^2 \\
&= \sum_i h_{iiii}^2 + \frac{3}{4}\sum_{i\neq j}(h_{ijij}^2 + h_{jiji})^2 + \frac{3}{4}\sum_{i\neq j}(h_{ijij} - h_{jiji})^2 \\
&\ge \frac{3}{4}\sum_{ij}(\lambda_i - \lambda_j)^2(1 + \lambda_i\lambda_j)^2 \\
&= \frac{3}{2}[Sf_4 - f_3^2 - S^2 - S(S-n) - n^2H^2 + 2nHf_3].
\end{aligned}$$

This proves (2.17).

Consider the smooth function $\sum_{ij} h_{ij}(f_3)_{ij}$. Since M is closed and H is constant, from Stokes' theorem,

$$\int_M \sum_{ij} h_{ij}(f_3)_{ij} = -\int_M \sum_{ij} h_{ijj}(f_3)_i = 0. \tag{2.21}$$

Also,

$$\begin{aligned}
&\frac{1}{3}\sum_{ij} h_{ij}(f_3)_{ij} \\
&\quad= \frac{1}{3}\sum_k \lambda_k (f_3)_{kk} \\
&\quad= \sum_k \lambda_k \left(\sum_i h_{iikk}\lambda_i^2 + 2\sum_{ij} h_{ijk}^2\lambda_i\right) \\
&\quad= \sum_{ik} h_{iikk}\lambda_k\lambda_i^2 + 2\sum_{ijk} h_{ijk}^2\lambda_i\lambda_k \\
&\quad= \sum_{ik}[h_{kkii} + (\lambda_i - \lambda_k)(1 + \lambda_i\lambda_k)]\lambda_k\lambda_i^2 + 2B \\
&\quad= \sum_i \left(\frac{S_{ii}}{2} - \sum_{jk} h_{ijk}^2\right)\lambda_i^2 + \sum_{ik}(\lambda_i - \lambda_k)(1 + \lambda_i\lambda_k)\lambda_k\lambda_i^2 + 2B \\
&\quad= \frac{1}{2}\sum_{ijk} h_{ik}h_{kj}S_{ij} + Sf_4 - f_3^2 - S^2 + nHf_3 - (A - 2B).
\end{aligned}$$

Integrating both sides and using (2.21) yields

$$\begin{aligned}\int_M (A-2B) &= \int_M \left[\frac{1}{2}\sum_{ijk} h_{ik}h_{kj}S_{ij} + Sf_4 - f_3^2 - S^2 + nHf_3\right]\\ &= \int_M \left[-\frac{1}{2}\sum_{ijk}(h_{ik}h_{kj})_j S_i + Sf_4 - f_3^2 - S^2 + nHf_3\right]\\ &= \int_M \left[-\frac{1}{2}\sum_{ijk} h_{ikj}h_{kj}S_i + Sf_4 - f_3^2 - S^2 + nHf_3\right]\\ &= \int_M \left[-\frac{|\nabla S|^2}{4} + Sf_4 - f_3^2 - S^2 + nHf_3\right].\end{aligned}$$

This proves (2.18).

From (2.13), we have

$$\begin{aligned}&3(\tilde{A}-2\tilde{B})\\ &\quad= \sum_{ijk} h_{ijk}^2(\mu_i^2+\mu_j^2+\mu_k^2-2\mu_i\mu_j-2\mu_i\mu_k-2\mu_j\mu_k)\\ &\quad= \sum_{i\neq j, i\neq k, j\neq k} h_{ijk}^2[2(\mu_i^2+\mu_j^2+\mu_k^2)-(\mu_i+\mu_j+\mu_k)^2]\\ &\qquad+3\sum_{i\neq k} h_{iik}^2(\mu_k^2-4\mu_i\mu_k)+\sum_i h_{iii}^2(-3\mu_i^2)\\ &\quad\leq 2\sum_l \mu_l^2 \sum_{i\neq j, i\neq k, j\neq k} h_{ijk}^2 + \frac{\sqrt{17}+1}{2}\sum_l \mu_l^2 \sum_{ik} h_{iik}^2, \end{aligned} \tag{2.22}$$

where we have used

$$\begin{aligned}\mu_k^2-4\mu_i\mu_k &\leq \mu_k^2 + \frac{\sqrt{17}-1}{2}\mu_k^2 + \frac{\sqrt{17}+1}{2}\mu_i^2\\ &= \frac{\sqrt{17}+1}{2}(\mu_i^2+\mu_k^2)\\ &\leq \frac{\sqrt{17}+1}{2}\sum_l \mu_l^2.\end{aligned}$$

Inequality (2.19) follows from (2.14) and (2.22). □

3. Proof of Theorem 1.4. In order to prove our Theorem 1.4, the following lemma (Lemma 3.1) plays an important role.

If $n = 6$ or 7, we know that $t = 2.428$ and $s = 1.62$ satisfy the following inequalities:

$$\begin{cases} \dfrac{t-s}{n-1} + t > \dfrac{\sqrt{17}+1}{2}, \\ 3s > 2t, \\ s > 2 - \dfrac{t}{2} + 4\sqrt{2-\dfrac{t}{2}}\sqrt{1-\dfrac{2t}{\sqrt{17}+1}}. \end{cases} \tag{3.1}$$

We define $\Phi(i, k)$ by

$$\Phi(i, k) = \begin{cases} \mu_k^2 - 4\mu_i\mu_k, & k \neq i, \\ s(S + n^2(n+2)H^2), & k = i. \end{cases} \tag{3.2}$$

LEMMA 3.1. *Let M be an n-dimensional closed hypersurface with constant mean curvature H in S^{n+1}, for $n = 6, 7$. Then*

$$\sum_i h_{iik}^2 \Phi(i, k) \leq t(S + n^2(n+2)H^2) \sum_i h_{iik}^2, \quad 1 \leq k \leq n. \tag{3.3}$$

Proof. Without loss of generality, we can assume $k = 1$. If $\Phi(i, 1) \leq t(S + n^2(n + 2)H^2)$ for any i, it is obvious that (3.3) holds. Otherwise, if there exists an i such that $\Phi(i, 1) > t(S + n^2(n + 2)H^2)$, without loss of generality, we can suppose $\Phi(2, 1) > t(S + n^2(n + 2)H^2)$; then, according to

$$\frac{\sqrt{17}+1}{2}(\mu_1^2 + \mu_2^2) \geq \mu_1^2 - 4\mu_1\mu_2 = \Phi(2, 1) > t(S + n^2(n+2)H^2),$$

we have

$$\Phi(2, 1) \leq \frac{\sqrt{17}+1}{2}(S + n^2(n+2)H^2) \tag{3.4}$$

and, for $3 \leq m \leq n$,

$$\mu_m^2 \leq S + n^2(n+2)H^2 - (\mu_1^2 + \mu_2^2) < \left(1 - \frac{2t}{\sqrt{17}+1}\right)(S + n^2(n+2)H^2). \tag{3.5}$$

On the other hand, since

$$\mu_1^2 + (\mu_1^2 + 4\mu_2^2) \geq \mu_1^2 - 4\mu_1\mu_2 > t(S + n^2(n+2)H^2), \tag{3.6}$$

we infer

$$\begin{aligned} \mu_1^2 &\leq (S + n^2(n+2)H^2) - \mu_2^2 \\ &< (S + n^2(n+2)H^2) - \tfrac{t(S+n^2(n+2)H^2)}{2} + (\mu_1^2 + \mu_2^2) \\ &\leq (2 - \tfrac{t}{2})(S + n^2(n+2)H^2). \end{aligned} \tag{3.7}$$

From (3.5) and (3.7), we have

$$\Phi(m, 1) = \mu_1^2 - 4\mu_1\mu_m < s(S + n^2(n+2)H^2). \tag{3.8}$$

Since $\sum_i h_{ii}$ =constant, we have

$$\sum_{i\neq 2} h_{ii1}^2 \geq \frac{h_{221}^2}{n-1}. \tag{3.9}$$

From (3.4), (3.8) and (3.9), we obtain

$$\begin{aligned}
&t(S+n^2(n+2)H^2)\sum_i h_{ii1}^2 \\
&= t(S+n^2(n+2)H^2)\{h_{221}^2+\sum_{i\neq 2} h_{ii1}^2\} \\
&\geq t(S+n^2(n+2)H^2)h_{221}^2+\sum_{i\neq 2} h_{ii1}^2\Phi(i,1) \\
&\quad +(t-s)(S+n^2(n+2)H^2)\frac{h_{221}^2}{n-1} \\
&\geq \sum_i h_{ii1}^2\Phi(i,1),
\end{aligned} \tag{3.10}$$

since t and s satisfy (3.1). This finishes the proof of Lemma 3.1. □

Proof of Theorem 1.4. For $n \leq 5$ or $H = 0$, Theorem 1.4 has been proved in [**7**] and [**12**]. Thus, we only consider the case of $|H| > 0$ and $n = 6, 7$.

Integrating equation (2.15), $S\times$(2.15) and (2.16) gives

$$\int_M S(n-S) = \int_M n^2H^2 - nHf_3 - |\nabla \mathbf{h}|^2, \tag{3.11}$$

$$\int_M \frac{1}{2}|\nabla S|^2 = \int_M S^2(S-n) + n^2H^2S - nHSf_3 - S|\nabla \mathbf{h}|^2, \tag{3.12}$$

$$\begin{aligned}
\int_M |\nabla^2\mathbf{h}|^2 = \int_M &(S-2n-3+\frac{3}{2}n^2H^2)|\nabla \mathbf{h}|^2 + \frac{3}{2}(A-2B) \\
&+\frac{3}{2}(\tilde{A}-2\tilde{B})+\frac{3}{2}|\nabla S|^2.
\end{aligned} \tag{3.13}$$

From (2.17) and (3.11), we derive

$$\int_M |\nabla^2\mathbf{h}|^2 \geq \int_M \frac{3}{2}(Sf_4 - f_3^2 - S^2 + nHf_3 - |\nabla \mathbf{h}|^2). \tag{3.14}$$

From (2.18) and (3.12), we infer

$$\begin{aligned}
\int_M \left\{\frac{3}{2}(A-2B)+\frac{3}{2}|\nabla S|^2\right\} = \int_M \Big\{&\frac{3}{2}(Sf_4 - f_3^2 - S^2 + nHf_3 - |\nabla \mathbf{h}|^2) \\
&+\frac{9}{4}\Big[S^2(S-n)+n^2H^2S-nHSf_3-S|\nabla \mathbf{h}|^2\Big]+\frac{3}{2}|\nabla \mathbf{h}|^2\Big\}.
\end{aligned} \tag{3.15}$$

Substituting (3.15) into (3.13) and using (3.14), we obtain

$$\int_M \left\{ (S - 2n - \frac{3}{2} + \frac{3}{2}n^2H^2)|\nabla \mathbf{h}|^2 + \frac{3}{2}(\tilde{A} - 2\tilde{B}) + \frac{9}{4}\left[S\left(S(S-n) + n^2H^2 - nHf_3 \right) - S|\nabla \mathbf{h}|^2 \right] \right\} \geq 0. \tag{3.16}$$

It is not difficult to prove the following elementary inequality (cf. [**8**]):

$$\left| \sum_i (\lambda_i - H)^3 \right| \leq \frac{n-2}{\sqrt{n(n-1)}}(S - nH^2)^{\frac{3}{2}}.$$

Since $S \geq S_0$ and $S \geq S_0$ is equivalent to

$$\sqrt{n + \frac{n^3H^2}{4(n-1)}} - \sqrt{S - nH^2} + \frac{n(n-2)|H|}{2\sqrt{n(n-1)}} \leq 0,$$

where S_0 is defined by (1.1), we have

$$\begin{aligned} & S(S-n) + n^2H^2 - nHf_3 \\ & = -(S - nH^2)\{n + nH^2 - (S - nH^2)\} - nH\sum_i (\lambda_i - H)^3 \\ & \geq -(S - nH^2)\left\{ n + nH^2 - (S - nH^2) + \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\sqrt{S - nH^2} \right\} \\ & = -(S - nH^2)\left[\sqrt{n + \frac{n^3H^2}{4(n-1)}} + \sqrt{S - nH^2} - \frac{n(n-2)|H|}{2\sqrt{n(n-1)}} \right] \\ & \quad \times \left[\sqrt{n + \frac{n^3H^2}{4(n-1)}} - \sqrt{S - nH^2} + \frac{n(n-2)|H|}{2\sqrt{n(n-1)}} \right] \\ & \geq 0. \end{aligned} \tag{3.17}$$

According to $S_0 \leq S \leq S_0 + \delta(n, H)$, we derive, from (3.11) and (3.17),

$$\begin{aligned} & \int_M S\left[S(S-n) + n^2H^2 - nHf_3 \right] \\ & \quad \leq (S_0 + \delta(n, H)) \int_M S(S-n) + n^2H^2 - nHf_3 \\ & \quad = (S_0 + \delta(n, H)) \int_M |\nabla \mathbf{h}|^2. \end{aligned} \tag{3.18}$$

From (3.16) and (3.18), we obtain

$$\int_M \{(S - 2n - \frac{3}{2} + \frac{3}{2}n^2H^2)|\nabla \mathbf{h}|^2 + \frac{3}{2}(\tilde{A} - 2\tilde{B}) + \frac{9}{4}[S_0 + \delta(n, H) - S]|\nabla \mathbf{h}|^2\} \geq 0. \tag{3.19}$$

On the other hand, since t and s satisfy (3.1), we have, from Lemma 3.1,

$$\begin{aligned}
&3\sum_{i\neq k} h_{iik}^2(\mu_k^2 - 4\mu_i\mu_k) + \sum_i h_{iii}^2(-3\mu_i^2) \\
&= 3\sum_k\sum_i h_{iik}^2\Phi(i,k) - 3\sum_k h_{kkk}^2 s(S + n^2(n+2)H^2) + \sum_i h_{iii}^2(-3\mu_i^2) \\
&\leq t(S + n^2(n+2)H^2)\sum_{ik} 3h_{iik}^2 - t(S + n^2(n+2)H^2)\sum_i 2h_{iii}^2 \\
&= t(S + n^2(n+2)H^2)\left(\sum_{i\neq k} 3h_{iik}^2 + \sum_i h_{iii}^2\right). \qquad (3.20)
\end{aligned}$$

Hence, we infer

$$\begin{aligned}
&3(\tilde{A} - 2\tilde{B}) \\
&= \sum_{i\neq j, j\neq k, k\neq i} h_{ijk}^2(2(\mu_i^2 + \mu_j^2 + \mu_k^2) - (\mu_i + \mu_j + \mu_k)^2) \\
&\quad + 3\sum_{i\neq k} h_{iik}^2(\mu_k^2 - 4\mu_i\mu_k) + \sum_i h_{iii}^2(-3\mu_i^2) \\
&\leq 2(S + n^2(n+2)H^2)\sum_{i\neq j, j\neq k, k\neq i} h_{ijk}^2 \\
&\quad + t(S + n^2(n+2)H^2)\left(3\sum_{i\neq k} h_{iik} + \sum_i h_{iii}^2\right) \\
&\leq t(S + n^2(n+2)H^2)|\nabla \mathbf{h}|^2. \qquad (3.21)
\end{aligned}$$

From (3.19) and (3.21), we have

$$\int_M \left[\frac{2t-5}{4}S - 2n - \frac{3}{2} + \frac{3}{2}n^2H^2 + \frac{t}{2}n^2(n+2)H^2 + \frac{9}{4}(S_0 + \delta(n,H))\right]|\nabla \mathbf{h}|^2 \geq 0.$$

Since $S_0 \leq S \leq S_0 + \delta(n,H)$, we have

$$\int_M \left[\frac{2t+4}{4}S_0 - 2n - \frac{3}{2} + \frac{3}{2}n^2H^2 + \frac{t}{2}n^2(n+2)H^2 + \frac{9}{4}\delta(n,H)\right]|\nabla \mathbf{h}|^2 \geq 0. \qquad (3.22)$$

From Definition (1.1) of S_0, we have

$$\begin{aligned}
&\frac{2t+4}{4}S_0 - 2n - \frac{3}{2} + \frac{3}{2}n^2H^2 + \frac{t}{2}n^2(n+2)H^2 + \frac{9}{4}\delta(n,H) \\
&= \frac{nt}{2} - n - \frac{3}{2} + \left\{\frac{n(t+2)}{4(n-1)} + \frac{(n+2)t(n)}{2} + \frac{3}{2}\right\}n^2H^2 \\
&\quad + \frac{n(n-2)(t+2)}{4(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2} + \frac{9}{4}\delta(n,H).
\end{aligned}$$

Since $\frac{nt}{2} - n - \frac{3}{2} < 0$ and $|H| \leq \varepsilon(n)$, if $\varepsilon(n)$ is small enough, we can choose $\delta(n, H) > 0$ such that

$$\frac{2t+4}{4}S_0 - 2n - \frac{3}{2} + \frac{3}{2}n^2H^2 + \frac{t}{2}n^2(n+2)H^2 + \frac{9}{4}\delta(n, H) < 0. \tag{3.23}$$

According to (3.22) and the above inequality, we infer $|\nabla \mathbf{h}| \equiv 0$. Hence, all of the above inequalities are equalities. From (3.17), we have $S \equiv S_0$ and M is isometric to a Clifford hypersurface. This completes the proof of Theorem 1.4. □

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