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Calculus of Variations

Estimates for lower order eigenvalues of a clamped plate problem

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Abstract For a bounded domain Ω in a complete Riemannian manifold M^n, we study estimates for lower order eigenvalues of a clamped plate problem. We obtain universal inequalities for lower order eigenvalues. We would like to remark that our results are sharp.

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1 Introduction

Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M^n. Assume that Γ_i is the ith eigenvalue of *a clamped plate problem*, which describes characteristic vibrations of a clamped plate:

$$\begin{cases} \Delta^2 u = \Gamma u, & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ is the Laplacian on M^n and ν denotes the outward unit normal to the boundary $\partial\Omega$. It is well known that this problem has a real and discrete spectrum

$$0 < \Gamma_1 \leq \Gamma_2 \leq \cdots \leq \Gamma_k \leq \cdots \nearrow +\infty,$$

where each Γ_i has finite multiplicity which is repeated according to its multiplicity.

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In this paper, we do not introduce results on universal inequalities for higher order eigenvalues of the clamped plate problem. The readers who are interested in it can see the references [2,3–8,10,11]. Since for lower order eigenvalues of the clamped plate problem, one can obtain better universal inequalities for eigenvalues. We will focus our mind on the investigation of lower order eigenvalues of the clamped plate problem.

When M^n is an n-dimensional Euclidean space, for lower order eigenvalues of the clamped plate problem (1.1), Ashbaugh [1] announced the following two universal inequalities without proofs. Cheng, Ichikawa and Mametsuka [4] have given their proofs.

$$\sum_{i=1}^{n}(\Gamma_{i+1}^{\frac{1}{2}} - \Gamma_1^{\frac{1}{2}}) \leq 4\Gamma_1^{\frac{1}{2}}, \tag{1.2}$$

$$\sum_{i=1}^{n}(\Gamma_{i+1} - \Gamma_1) \leq 24\Gamma_1. \tag{1.3}$$

When M^n is a general complete Riemannian manifold other than the Euclidean space, it is natural to consider the following problem:

Problem Let M^n be an n-dimensional complete Riemannian manifold and Ω a bounded domain in M^n. Whether can one obtain a universal inequality for lower order eigenvalues, which are analogous to (1.2), of the clamped plate problem?

In this paper, we will answer the problem and prove the following results:

Theorem 1 *Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M^n. For the lower order eigenvalues of the clamped plate problem:*

$$\begin{cases} \Delta^2 u = \Gamma u, & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

we have

$$\sum_{i=1}^{n}(\Gamma_{i+1} - \Gamma_1)^{\frac{1}{2}} \leq \left(4\Gamma_1^{\frac{1}{2}} + n^2 H_0^2\right)^{\frac{1}{2}} \left\{(2n+4)\Gamma_1^{\frac{1}{2}} + n^2 H_0^2\right\}^{\frac{1}{2}}, \tag{1.4}$$

where H_0^2 is a nonnegative constant which only depends on M^n and Ω.

Corollary 1 *Under the assumption of the Theorem 1, we have*

$$\sum_{i=1}^{n}\left\{(\Gamma_{i+1} - \Gamma_1)^{\frac{1}{2}} - \Gamma_1^{\frac{1}{2}}\right\} \leq 4\Gamma_1^{\frac{1}{2}} + n^2 H_0^2. \tag{1.5}$$

Corollary 2 *When M^n is an n-dimensional complete minimal submanifold in a Euclidean space, we have*

$$\sum_{i=1}^{n}(\Gamma_{i+1} - \Gamma_1)^{\frac{1}{2}} \leq \{8(n+2)\Gamma_1\}^{\frac{1}{2}}. \tag{1.6}$$

Corollary 3 *When M^n is an n-dimensional unit sphere, we have*

$$\sum_{i=1}^{n}(\Gamma_{i+1} - \Gamma_1)^{\frac{1}{2}} \leq \left(4\Gamma_1^{\frac{1}{2}} + n^2\right)^{\frac{1}{2}} \left\{(2n+4)\Gamma_1^{\frac{1}{2}} + n^2\right\}^{\frac{1}{2}}. \tag{1.7}$$

Remark 1 For the unit sphere $S^n(1)$, by taking $\Omega = S^n(1)$, we know $\Gamma_1 = 0$ and $\Gamma_2 = \cdots = \Gamma_{n+1} = n^2$. Hence, our inequalities become equalities. Thus, our results are sharp.

Remark 2 After the first author and the third author have proved the above results, the second author tells them that he has also proved the same results. Hence, the authors decide to write this joint paper together.

2 Preliminaries

In order to prove Theorem 1, we need the following Nash's theorem.

Nash's Theorem *Each complete Riemannian manifold M^n can be isometrically immersed into a Euclidian space $\mathbb{R}^N$.*

Assume that M^n is an n-dimensional isometrically immersed submanifold in $\mathbb{R}^N$. Let $\Omega \subset M^n$ be a bounded domain of M^n and $p \in \Omega$. Let $(x^1, \cdots, x^n)$ be a local coordinate system in a neighborhood U of $p \in M$. Let $\mathbf{y}$ be the position vector of p in $\mathbb{R}^N$, which is defined by

$$\mathbf{y} = (y^1(x^1, \ldots, x^n), \ldots, y^N(x^1, \ldots, x^n)).$$

Since M^n is isometrically immersed in $\mathbb{R}^N$, we have

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \langle \sum_{\alpha=1}^{N} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha}, \sum_{\beta=1}^{N} \frac{\partial y^\beta}{\partial x^i} \frac{\partial}{\partial y^\beta} \rangle = \sum_{\alpha=1}^{N} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\alpha}{\partial x^j}, \tag{2.1}$$

where g denotes the induced metric of M^n from $\mathbb{R}^N$ and $<, >$ is the standard inner product in $\mathbb{R}^N$. The following lemma can be found in [2].

Lemma 1 *For any function $u \in C^\infty(M^n)$, we have*

$$\sum_{\alpha=1}^{N} (g(\nabla y^\alpha, \nabla u))^2 = |\nabla u|^2, \tag{2.2}$$

$$\sum_{\alpha=1}^{N} g(\nabla y^\alpha, \nabla y^\alpha) = \sum_{\alpha=1}^{N} |\nabla y^\alpha|^2 = n, \tag{2.3}$$

$$\sum_{\alpha=1}^{N} (\Delta y^\alpha)^2 = n^2 |H|^2, \tag{2.4}$$

$$\sum_{\alpha=1}^{N} \Delta y^\alpha \nabla y^\alpha = 0, \tag{2.5}$$

where ∇ denotes the gradient operator on M^n and $|H|$ is the mean curvature of M^n.

3 Proofs of results

Proof of Theorem 1 Since M^n is a complete Riemannian manifold, Nash's theorem implies that there exists an isometric immersion from M^n into a Euclidean space $\mathbb{R}^N$. Thus, M^n can be considered as an n-dimensional complete isometrically immersed submanifold in $\mathbb{R}^N$.

Let u_i be an eigenfunction corresponding to eigenvalue Γ_i such that $\{u_i\}_{i\in\mathbf{N}}$ becomes an orthonormal basis of $L^2(\Omega)$, that is,

$$\int_\Omega u_i u_j = \delta_{ij}, \quad \forall i, j \in \mathbf{N}.$$

We define an $N \times N$-matrix B as follows:

$$B := (b_{\alpha\beta})$$

where $b_{\alpha\beta} = \int_\Omega y^\alpha u_1 u_{\beta+1}$ and $\mathbf{y} = (y^\alpha)$ is the position vector of the immersion in $\mathbb{R}^N$. Using the orthogonalization of Gram and Schmidt, we know that there exist an upper triangle matrix $R = (R_{\alpha\beta})$ and an orthogonal matrix $Q = (q_{\alpha\beta})$ such that $R = QB$, i.e.,

$$R_{\alpha\beta} = \sum_{\gamma=1}^N q_{\alpha\gamma} b_{\gamma\beta} = \int_\Omega \sum_{\gamma=1}^N q_{\alpha\gamma} y^\gamma u_1 u_{\beta+1} = 0, \quad 1 \le \beta < \alpha \le N. \tag{3.1}$$

Defining $g^\alpha = \sum_{\gamma=1}^N q_{\alpha\gamma} y^\gamma$, we get

$$\int_\Omega g^\alpha u_1 u_{\beta+1} = \int_\Omega \sum_{\gamma=1}^N q_{\alpha\gamma} y^\gamma u_1 u_{\beta+1} = 0, \quad 1 \le \beta < \alpha \le N. \tag{3.2}$$

We put

$$\psi_\alpha := (g^\alpha - a^\alpha) u_1, \qquad a^\alpha := \int_\Omega g^\alpha u_1^2, \qquad 1 \le \alpha \le N, \tag{3.3}$$

then it follows that

$$\int_\Omega \psi_\alpha u_{\beta+1} = 0, \qquad 0 \le \beta < \alpha \le N. \tag{3.4}$$

Thus, ψ_α, $1 \le \alpha \le N$, are trial functions. From the Rayleigh-Ritz inequality, we have

$$\Gamma_{\alpha+1} \le \frac{\int_\Omega \psi_\alpha \Delta^2 \psi_\alpha}{\int_\Omega \psi_\alpha^2}, \quad 1 \le \alpha \le N. \tag{3.5}$$

By a direct calculation, we have

$$\begin{aligned}\int_\Omega \psi_\alpha \Delta^2 \psi_\alpha &= \int_\Omega \psi_\alpha \Delta^2 (g^\alpha u_1 - a^\alpha u_1)\\ &= \int_\Omega \psi_\alpha \{u_1 \Delta^2 g^\alpha + 2\nabla(\Delta g^\alpha)\cdot\nabla u_1 + 2\Delta g^\alpha \Delta u_1\\ &\quad +2\Delta(\nabla g^\alpha \cdot \nabla u_1) + 2\nabla g^\alpha \cdot \nabla(\Delta u_1) + \Gamma_1 g^\alpha u_1\}.\end{aligned} \tag{3.6}$$

Then by (3.4) and (3.5), we conclude

$$(\Gamma_{\alpha+1} - \Gamma_1)\|\psi_\alpha\|^2 \le \int_\Omega r_\alpha \psi_\alpha := \omega_\alpha, \quad 1 \le \alpha \le N, \tag{3.7}$$

where

$$r_\alpha = u_1 \Delta^2 g^\alpha + 2\nabla(\Delta g^\alpha)\cdot\nabla u_1 + 2\Delta g^\alpha \Delta u_1 + 2\Delta(\nabla g^\alpha\cdot\nabla u_1) + 2\nabla g^\alpha\cdot\nabla(\Delta u_1).$$

By making use of Stokes' formula, it is easy to get

$$\int_\Omega r_\alpha a^\alpha u_1 = 0$$

and

$$\omega_\alpha = \int_\Omega r_\alpha \psi_\alpha = \int_\Omega r_\alpha (g^\alpha u_1 - a^\alpha u_1) = \int_\Omega r_\alpha g^\alpha u_1.$$

We also obtain the following equations from Stokes' theorem

$$2\int_\Omega g^\alpha u_1 \nabla(\Delta g^\alpha)\cdot\nabla u_1 = \int_\Omega \{2u_1\Delta g^\alpha \nabla u_1\cdot\nabla g^\alpha + u_1^2(\Delta g^\alpha)^2 - g^\alpha u_1^2 \Delta^2 g^\alpha\},$$

$$2\int_\Omega g^\alpha u_1 \Delta(\nabla g^\alpha\cdot\nabla u_1) = \int_\Omega \{2u_1\Delta g^\alpha \nabla g^\alpha\cdot\nabla u_1 + 4(\nabla g^\alpha\cdot\nabla u_1)^2 + 2g^\alpha\Delta u_1\nabla g^\alpha\cdot\nabla u_1\},$$

$$2\int_\Omega g^\alpha u_1 \nabla g^\alpha\cdot\nabla(\Delta u_1) = -2\int_\Omega \{|\nabla g^\alpha|^2 u_1\Delta u_1 + g^\alpha\Delta u_1 \nabla g^\alpha\cdot\nabla u_1 + g^\alpha\Delta g^\alpha u_1\Delta u_1\}.$$

Consequently, we get

$$\begin{aligned}\omega_\alpha &= \int_\Omega \{(\Delta g^\alpha)^2 u_1^2 + 4(\nabla g^\alpha\cdot\nabla u_1)^2 - 2|\nabla g^\alpha|^2 u_1\Delta u_1 + 4u_1\Delta g^\alpha\nabla g^\alpha\cdot\nabla u_1\}\\ &= \|u_1\Delta g^\alpha + 2\nabla g^\alpha\cdot\nabla u_1\|^2 - 2\int_\Omega |\nabla g^\alpha|^2 u_1\Delta u_1. \end{aligned}\tag{3.8}$$

(3.7) and (3.8) imply

$$(\Gamma_{\alpha+1} - \Gamma_1)\|\psi_\alpha\|^2 \le \|u_1\Delta g^\alpha + 2\nabla g^\alpha\cdot\nabla u_1\|^2 - 2\int_\Omega |\nabla g^\alpha|^2 u_1\Delta u_1, \quad 1\le\alpha\le N. \tag{3.9}$$

On the other hand,

$$\begin{aligned}&\int_\Omega \psi_\alpha(u_1\Delta g^\alpha + 2\nabla u_1\cdot\nabla g^\alpha)\\ &\quad= \int_\Omega (g^\alpha u_1 - u_1 a^\alpha)(u_1\Delta g^\alpha + 2\nabla u_1\cdot\nabla g^\alpha)\\ &\quad= \int_\Omega g^\alpha u_1(u_1\Delta g^\alpha + 2\nabla u_1\cdot\nabla g^\alpha)\\ &\quad= \int_\Omega \left(g^\alpha u_1^2\Delta g^\alpha + \frac{1}{2}\nabla u_1^2\cdot\nabla(g^\alpha)^2\right). \end{aligned}\tag{3.10}$$

By using of Stokes' formula, we know

$$\int_\Omega g^\alpha u_1^2 \Delta g^\alpha = -\int_\Omega |u_1 \nabla g^\alpha|^2 - \frac{1}{2}\int_\Omega \nabla u_1^2 \cdot \nabla (g^\alpha)^2. \tag{3.11}$$

Substituting (3.11) into (3.10), we infer

$$-\int_\Omega \psi_\alpha (u_1 \Delta g^\alpha + 2\nabla u_1 \cdot \nabla g^\alpha) = \int_\Omega |u_1 \nabla g^\alpha|^2. \tag{3.12}$$

From (3.12) and (3.9), we have, for any positive constant δ,

$$\begin{aligned}
&(\Gamma_{\alpha+1} - \Gamma_1)^{\frac{1}{2}} \int_\Omega |u_1 \nabla g^\alpha|^2 \\
&= -(\Gamma_{\alpha+1} - \Gamma_1)^{\frac{1}{2}} \int_\Omega \psi_\alpha (u_1 \Delta g^\alpha + 2\nabla u_1 \cdot \nabla g^\alpha) \\
&\le \frac{\delta}{2}(\Gamma_{\alpha+1} - \Gamma_1)\|\psi_\alpha\|^2 + \frac{1}{2\delta}\|u_1 \Delta g^\alpha + 2\nabla u_1 \cdot \nabla g^\alpha\|^2 \\
&\le \left(\frac{\delta}{2} + \frac{1}{2\delta}\right)\|u_1 \Delta g^\alpha + 2\nabla u_1 \cdot \nabla g^\alpha\|^2 - \delta \int_\Omega |\nabla g^\alpha|^2 u_1 \Delta u_1.
\end{aligned} \tag{3.13}$$

According to the lemma in the Sect. 2 and the definition of g^α, we then have

$$\begin{aligned}
&\sum_{\alpha=1}^N \|u_1 \Delta g^\alpha + 2\nabla g^\alpha \cdot \nabla u_1\|^2 \\
&= \sum_{\alpha=1}^N \int_\Omega \{u_1^2 (\Delta g^\alpha)^2 + 4(\nabla u_1 \cdot \nabla g^\alpha)^2 + 2(\Delta g^\alpha \nabla g^\alpha) \cdot \nabla u_1^2\} \\
&= n^2 \int_\Omega |H|^2 u_1^2 + 4\int_\Omega |\nabla u_1|^2 \\
&\le 4\Gamma_1^{\frac{1}{2}} + n^2 \sup_\Omega |H|^2.
\end{aligned} \tag{3.14}$$

For any point p, by a transformation of coordinates if necessary, we have, for any α,

$$|\nabla g^\alpha|^2 = g(\nabla g^\alpha, \nabla g^\alpha) \le 1. \tag{3.15}$$

From (3.15), we infer

$$
\begin{aligned}
&\sum_{\alpha=1}^{N}(\Gamma_{\alpha+1}-\Gamma_1)^{\frac{1}{2}}|\nabla g^{\alpha}|^2\\
&\quad\geq \sum_{i=1}^{n}(\Gamma_{i+1}-\Gamma_1)^{\frac{1}{2}}|\nabla g^{i}|^2+(\Gamma_{n+1}-\Gamma_1)^{\frac{1}{2}}\sum_{A=n+1}^{N}|\nabla g^{A}|^2\\
&\quad= \sum_{i=1}^{n}(\Gamma_{i+1}-\Gamma_1)^{\frac{1}{2}}|\nabla g^{i}|^2+(\Gamma_{n+1}-\Gamma_1)^{\frac{1}{2}}\left(n-\sum_{j=1}^{n}|\nabla g^{j}|^2\right)\\
&\quad= \sum_{i=1}^{n}(\Gamma_{i+1}-\Gamma_1)^{\frac{1}{2}}|\nabla g^{i}|^2+(\Gamma_{n+1}-\Gamma_1)^{\frac{1}{2}}\sum_{j=1}^{n}(1-|\nabla g^{j}|^2)\\
&\quad\geq \sum_{i=1}^{n}(\Gamma_{i+1}-\Gamma_1)^{\frac{1}{2}}|\nabla g^{i}|^2+\sum_{j=1}^{n}(\Gamma_{j+1}-\Gamma_1)^{\frac{1}{2}}(1-|\nabla g^{j}|^2)\\
&\quad= \sum_{j=1}^{n}(\Gamma_{j+1}-\Gamma_1)^{\frac{1}{2}}.
\end{aligned}
$$

For (3.13), taking sum on α from 1 to N and using of (3.14) and the above inequality, we obtain

$$
\sum_{i=1}^{n}(\Gamma_{i+1}-\Gamma_1)^{\frac{1}{2}}\leq\left(\frac{\delta}{2}+\frac{1}{2\delta}\right)\left(4\Gamma_1^{\frac{1}{2}}+n^2\sup_{\Omega}|H|^2\right)+n\delta\Gamma_1^{\frac{1}{2}}.
$$

Taking

$$
\delta=\sqrt{\frac{4\Gamma_1^{\frac{1}{2}}+n^2\sup_{\Omega}|H|^2}{4\Gamma_1^{\frac{1}{2}}+n^2\sup_{\Omega}|H|^2+2n\Gamma_1^{\frac{1}{2}}}},
$$

we obtain

$$
\sum_{i=1}^{n}(\Gamma_{i+1}-\Gamma_1)^{\frac{1}{2}}\leq\left(4\Gamma_1^{\frac{1}{2}}+n^2\sup_{\Omega}|H|^2\right)^{\frac{1}{2}}\left\{(2n+4)\Gamma_1^{\frac{1}{2}}+n^2\sup_{\Omega}|H|^2\right\}^{\frac{1}{2}}. \tag{3.16}
$$

Since the spectrum of the clamped plate problem is an invariant of isometries, we know that (3.16) holds for any isometric immersion from M^n into a Euclidean space.

Now we define Φ as follows:

$$
\Phi:=\{\psi\,|\,\psi \text{ is an isometric immersion from } M \text{ into a Euclidean space}\}.
$$

Defining

$$
H_0^2:=\inf_{\psi\in\Phi}\sup_{\Omega}|H|^2,
$$

we obtain

$$
\sum_{i=1}^{n}(\Gamma_{i+1}-\Gamma_1)^{\frac{1}{2}}\leq\left(4\Gamma_1^{\frac{1}{2}}+n^2H_0^2\right)^{\frac{1}{2}}\left\{(2n+4)\Gamma_1^{\frac{1}{2}}+n^2H_0^2\right\}^{\frac{1}{2}}. \tag{3.17}
$$

This completes the proof of Theorem 1.

Proof of Corollary 1 Since

$$(4\Gamma_1^{\frac{1}{2}} + n^2 H_0^2)^{\frac{1}{2}} \left\{(2n+4)\Gamma_1^{\frac{1}{2}} + n^2 H_0^2\right\}^{\frac{1}{2}} \leq 4\Gamma_1^{\frac{1}{2}} + n^2 H_0^2 + n\Gamma_1^{\frac{1}{2}}.$$

from (3.17), we then obtain

$$\sum_{i=1}^{n} \left\{(\Gamma_{i+1} - \Gamma_1)^{\frac{1}{2}} - \Gamma_1^{\frac{1}{2}}\right\} \leq 4\Gamma_1^{\frac{1}{2}} + n^2 H_0^2. \tag{3.18}$$

This finishes the proof of Corollary 1.

Proof of Corollary 2 For a complete minimal submanifold in a Euclidean space, we have $|H| = 0$. From the proof of the Theorem 1, the Corollary 2 is clear.

Proof of Corollary 3 Since an n-dimensional unit sphere can be seen as a compact hypersurface with constant mean curvature 1 in the Euclidean space $\mathbb{R}^{n+1}$, we have $|H| = 1$. From the proof of the Theorem 1, the Corollary 3 is obvious.

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