INEQUALITIES FOR EIGENVALUES OF LAPLACIAN WITH ANY ORDER*

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Abstract. In this paper we study eigenvalues of Laplacian with any order on a bounded domain in an n-dimensional Euclidean space and obtain estimates for eigenvalues, which are the Yang-type inequalities. In particular, the sharper result of Yang is included here. Furthermore, for lower order eigenvalues, we obtain two sharper inequalities. As a consequence, a proof of results announced by Ashbaugh [1] is also given.

1. introduction

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain in an n-dimensional Euclidian space $\mathbf{R}^n$. Assume that λ_i is the i^{th} eigenvalue of the Dirichlet eigenvalue problem of Laplacian with any order:

$$\begin{cases} (-\Delta)^l u = \lambda u & \text{in } \Omega \\ u = \dfrac{\partial u}{\partial \nu} = \cdots = \dfrac{\partial^{l-1} u}{\partial \nu^{l-1}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ is the Laplacian in $\mathbf{R}^n$ and ν denotes the outward unit normal. It is well known that this problem has a real and discrete spectrum

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \longrightarrow \infty.$$

When $l = 1$, the above problem is called a fixed membrane problem and it is called a clamped plate problem when $l = 2$.

When $l = 1$, Payne, Pólya and Weinberger [11] proved

$$\lambda_{k+1} - \lambda_k \le \frac{4}{kn} \sum_{i=1}^{k} \lambda_i, \quad k = 1, 2, \cdots. \tag{1.2}$$

Further, Hile and Protter [8] generalized the above result to

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \ge \frac{kn}{4}, \quad k = 1, 2, \cdots. \tag{1.3}$$

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In 1991, Yang [14] (cf. [7]) proved the following sharper result:

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i, \quad k=1,2,\cdots. \tag{1.4}$$

According to the inequality, we can infer

$$\lambda_{k+1} \le \frac{1}{k}(1+\frac{4}{n})\sum_{i=1}^{k}\lambda_i, \quad k=1,2,\cdots. \tag{1.5}$$

When $l=2$, one usually uses Γ_i to denote the i^{th} eigenvalue of (1.1). In this case, Payne, Pólya and Weinberger [11] proved

$$\Gamma_{k+1}-\Gamma_k \le \frac{8(n+2)}{n^2k}\sum_{i=1}^{k}\Gamma_i, \quad k=1,2,\cdots . \tag{1.6}$$

Chen and Qian [5] and Hook [9], independently, proved

$$\frac{n^2k^2}{8(n+2)} \le \sum_{i=1}^{k}\frac{\Gamma_i^{\frac{1}{2}}}{\Gamma_{k+1}-\Gamma_i}\sum_{i=1}^{k}\Gamma_i^{\frac{1}{2}}, \quad k=1,2,\cdots . \tag{1.7}$$

Recently, Cheng and Yang [6] have proved

$$\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i) \le (\frac{8(n+2)}{n^2})^{\frac{1}{2}}\sum_{i=1}^{k}(\Gamma_i(\Gamma_{k+1}-\Gamma_i))^{\frac{1}{2}}, \tag{1.8}$$

which is analogous to the inequality (1.4) of Yang.

For any integer l, Chen and Qian [5] and Hook [9], independently, proved

$$\frac{n^2k^2}{4l(n+2l-2)} \le \sum_{i=1}^{k}\frac{\lambda_i^{\frac{1}{l}}}{\lambda_{k+1}-\lambda_i}\sum_{i=1}^{k}\lambda_i^{\frac{l-1}{l}}, \quad k=1,2,\cdots . \tag{1.9}$$

By making use of the method of Cheng and Yang [6], Wu and Cao [13] have generalized the inequality (1.9) to

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\\ &\le \frac{\{4l(n+2l-2)\}^{\frac{1}{2}}}{n}\Big\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\lambda_i^{\frac{l-1}{l}}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\lambda_i^{\frac{1}{l}}\Big\}^{\frac{1}{2}}.\end{aligned} \tag{1.10}$$

We should notice that when $l=1$, the inequality (1.10) becomes

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i) \le \frac{2}{\sqrt{n}}\Big\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\lambda_i\Big\}^{\frac{1}{2}}.$$

From this inequality, we can only infer the inequality (1.5). But we can not derive the sharper inequality (1.4) of Yang. In this paper, one of our purposes is to derive an inequality for eigenvalues of the eigenvalue problem (1.1) for any l such that when $l = 1$, we have the sharper inequality (1.4) of Yang.

Theorem 1. *Let Ω be a bounded domain in an n-dimensional Euclidian space $\mathbf{R}^n$. Assume that $\lambda_i, i = 1, 2, \cdots$ is the i^{th} eigenvalue of the eigenvalue problem* (1.1)*. Then, we have*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4l(n+2l-2)}{n^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i. \tag{1.11}$$

Remark 1. *For any l, our result is the Yang-type inequality and it is simple. In particular, when $l = 1$, it becomes the inequality* (1.4) *of Yang.*

Next, we consider the lower order eigenvalues of (1.1). When $l = 1$, Payne, Pólya, and Weinberger [11] obtained

$$\frac{\lambda_2+\lambda_3}{\lambda_1} \leq 6, \quad \text{for} \ \ \Omega \subset \mathbf{R}^2$$

and they conjectured

$$\frac{\lambda_2+\lambda_3}{\lambda_1} \leq \left.\frac{\lambda_2+\lambda_3}{\lambda_1}\right|_{disk} \approx 5.077, \quad \text{for} \ \ \Omega \subset \mathbf{R}^2. \tag{1.12}$$

The above conjecture was researched by many authors (cf. [1], [3], [10]). Ashbaugh and Benguria [2] proved, for any dimension n,

$$\frac{\lambda_2+\lambda_3+\cdots+\lambda_{n+1}}{\lambda_1} \leq n\left(1+\frac{4}{n}\right), \quad \text{for} \ \ \Omega \subset \mathbf{R}^n. \tag{1.13}$$

When Ω is a bounded domain in a complete Riemannian manifold, Chen and Cheng [4] have derived analogous results. In particular, when Ω is a domain in the unit sphere, a sharper result has been obtained by Sun, Cheng and Yang [12]. When $l = 2$, for the clamped plate problem, Ashbaugh has announced two inequalities which are analogous to (1.13) for any dimension n in [1]:

$$\sum_{\alpha=1}^{n}(\Gamma_{\alpha+1}^{\frac{1}{2}}-\Gamma_1^{\frac{1}{2}}) \leq 4\Gamma_1^{\frac{1}{2}}, \tag{1.14}$$

$$\sum_{\alpha=1}^{n}(\Gamma_{\alpha+1}-\Gamma_1) \leq 24\Gamma_1. \tag{1.15}$$

Here one should remember that we replace the eigenvalue λ with Γ for $l = 2$. However, in his paper, Ashbaugh didn't give the proofs of his results. Our second purpose is to derive universal inequalities for lower order eigenvalues of the eigenvalue problem (1.1) for any l. From our results, one can infer the results of Ashbaugh.

Theorem 2. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain in an n-dimensional Euclidian space $\mathbf{R}^n$. Assume that λ_i is the i^{th}eigenvalue of the eigenvalue problem* (1.1). *Then, we have*

$$\sum_{\alpha=1}^{n}(\lambda_{\alpha+1}^{\frac{1}{l}} - \lambda_1^{\frac{1}{l}})^{l-1} \leq (2l)^{l-1}\lambda_1^{\frac{l-1}{l}}, \quad \text{for} \quad l \geq 2. \tag{1.16}$$

Theorem 3. *Under the same assumption as in the theorem 2, we have*

$$\sum_{\alpha=1}^{n}(\lambda_{\alpha+1} - \lambda_1) \leq 4l(2l-1)\lambda_1. \tag{1.17}$$

Remark 2. *When $l = 2$, our inequality* (1.16) *becomes the inequality* (1.14) *of Ashbaugh. For $l = 1$ and $l = 2$, our results in the theorem* 3 *become the inequalities* (1.13) *and* (1.15), *respectively.*

2. Proof of Theorem 1

At first , we prove an algebraic lemma.

Lemma 1. *Let A_i, B_i and C_i, for $i = 1, 2, \cdots, k$, satisfy $A_1 \geq A_2 \geq \cdots \geq A_k \geq 0$, $0 \leq B_1 \leq B_2 \leq \cdots \leq B_k$ and $0 \leq C_1 \leq C_2 \leq \cdots \leq C_k$, respectively. Then, we have*

$$\sum_{i=1}^{k} A_i^2 B_i \sum_{i=1}^{k} A_i C_i \leq \sum_{i=1}^{k} A_i^2 \sum_{i=1}^{k} A_i B_i C_i. \tag{2.1}$$

Proof. When $k = 1$, we have

$$A_1^2 B_1 A_1 C_1 - A_1^2 A_1 B_1 C_1 = 0.$$

Suppose that the inequality (2.1) holds for $k-1$, then for any integer k,

$$\begin{aligned}
&\sum_{i=1}^{k} A_i^2 B_i \sum_{i=1}^{k} A_i C_i - \sum_{i=1}^{k} A_i^2 \sum_{i=1}^{k} A_i B_i C_i \\
&= \sum_{i=1}^{k-1} A_i^2 B_i \sum_{i=1}^{k-1} A_i C_i - \sum_{i=1}^{k-1} A_i^2 \sum_{i=1}^{k-1} A_i B_i C_i \\
&+ A_k^3 B_k C_k - A_k^3 B_k C_k + A_k C_k \sum_{i=1}^{k-1} A_i^2 B_i + A_k^2 B_k \sum_{i=1}^{k-1} A_i C_i \\
&- A_k B_k C_k \sum_{i=1}^{k-1} A_i^2 - A_k^2 \sum_{i=1}^{k-1} A_i B_i C_i \\
&\leq A_k C_k \sum_{i=1}^{k-1} A_i^2 (B_i - B_k) - A_k^2 \sum_{i=1}^{k-1} A_i C_i (B_i - B_k) \\
&= A_k \sum_{i=1}^{k-1} A_i (A_i C_k - A_k C_i)(B_i - B_k) \leq 0.
\end{aligned}$$

Thus, the inequality (2.1) is proved. □

Proof of Theorem 1. Assume that u_i is an eigenfunction corresponding to the i^{th} eigenvalue λ_i such that $\{u_i\}_{i-1}^{\infty}$ becomes an orthonormal basis of $L^2(\Omega)$. Thus, u_i satisfies

$$\begin{cases} (-\Delta)^l u_i = \lambda_i u_i, & \text{in } \Omega \\ u_i = \dfrac{\partial u_i}{\partial \nu} = \cdots = \dfrac{\partial^{l-1} u_i}{\partial \nu^{l-1}} = 0, & \text{on } \partial\Omega \\ \displaystyle\int_\Omega u_i u_j = \delta_{ij}. \end{cases} \tag{2.2}$$

Let $y^1, y^2, \cdots, y^n$ be the standard coordinate functions of the n-dimensional Euclidean space $\mathbf{R}^n$. We define φ_i^α and r_{ij}^α, for $i, j = 1, 2, \cdots, k$, by

$$\varphi_i^\alpha = u_i y^\alpha - \sum_{j=1}^k r_{ij}^\alpha u_j, \tag{2.3}$$

$$r_{ij}^\alpha = \int_\Omega u_i u_j y^\alpha. \tag{2.4}$$

Then, we have

$$\int_\Omega \varphi_i^\alpha u_j = 0, \quad \text{for any } \ i, j = 1, 2, \cdots, k. \tag{2.5}$$

It follows from the Rayleigh-Ritz inequality that

$$\lambda_{k+1} \le \frac{\displaystyle\int_\Omega \varphi_i^\alpha (-\Delta)^l \varphi_i^\alpha}{\displaystyle\int_\Omega (\varphi_i^\alpha)^2}. \tag{2.6}$$

By induction, we can prove

$$(-\Delta)^l (u_i y^\alpha) = y^\alpha (-\Delta)^l u_i - 2l \nabla (-\Delta)^{l-1} u_i \cdot \nabla y^\alpha, \tag{2.7}$$

where $\nabla u_i \cdot \nabla y^\alpha = \langle \nabla u_i, \nabla y^\alpha \rangle$, $\langle\ ,\ \rangle$ is an inner product on $\mathbf{R}^n$. Since

$$\begin{aligned} \int_\Omega \varphi_i^\alpha (-\Delta)^l \varphi_i^\alpha &= \int_\Omega \varphi_i^\alpha (-\Delta)^l (u_i y^\alpha) \\ &= \int_\Omega \varphi_i^\alpha \Big\{ y^\alpha (-\Delta)^l u_i - 2l \nabla (-\Delta)^{l-1} u_i \cdot \nabla y^\alpha \Big\} \\ &= \lambda_i \int_\Omega (\varphi_i^\alpha)^2 - 2l \int_\Omega \varphi_i^\alpha \nabla (-\Delta)^{l-1} u_i \cdot \nabla y^\alpha \end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega u_i y^\alpha \nabla(-\Delta)^{l-1}u_i\cdot\nabla y^\alpha = \int_\Omega (-\Delta)^{l-1}(u_i y^\alpha)\nabla u_i\cdot\nabla y^\alpha\\
&= \int_\Omega \Big\{ y^\alpha(-\Delta)^{l-1}u_i - 2(l-1)\nabla(-\Delta)^{l-2}u_i\cdot\nabla y^\alpha\Big\}\nabla u_i\cdot\nabla y^\alpha\\
&= -\int_\Omega u_i y^\alpha\nabla(-\Delta)^{l-1}u_i\cdot\nabla y^\alpha - \int_\Omega u_i(-\Delta)^{l-1}u_i\\
&+ 2(l-1)\int_\Omega (-\Delta)^{l-2}u_i\nabla y^\alpha\cdot\nabla(\nabla u_i\cdot\nabla y^\alpha),
\end{aligned}$$

that is,

$$\begin{aligned}
&2\int_\Omega u_i y^\alpha\nabla(-\Delta)^{l-1}u_i\cdot\nabla y^\alpha\\
&= -\int_\Omega u_i(-\Delta)^{l-1}u_i + 2(l-1)\int_\Omega (-\Delta)^{l-2}u_i\nabla y^\alpha\cdot\nabla(\nabla u_i\cdot\nabla y^\alpha).
\end{aligned}\tag{2.8}$$

Thus, we infer

$$\begin{aligned}
\int_\Omega \varphi_i^\alpha(-\Delta)^l\varphi_i^\alpha &= \lambda_i\int_\Omega(\varphi_i^\alpha)^2 - 2l\int_\Omega u_i y^\alpha\nabla(-\Delta)^{l-1}u_i\cdot\nabla y^\alpha\\
&+ 2l\sum_{j=1}^k r_{ij}^\alpha\int_\Omega u_j\nabla(-\Delta)^{l-1}u_i\cdot\nabla y^\alpha\\
&= \lambda_i\int_\Omega(\varphi_i^\alpha)^2 + l\int_\Omega u_i(-\Delta)^{l-1}u_i\\
&- 2l(l-1)\int_\Omega(-\Delta)^{l-2}u_i\nabla y^\alpha\cdot\nabla(\nabla u_i\cdot\nabla y^\alpha) - 2l\sum_{j=1}^k r_{ij}^\alpha s_{ij}^\alpha,
\end{aligned}\tag{2.9}$$

where

$$s_{ij}^\alpha = \int_\Omega u_i\nabla(-\Delta)^{l-1}u_j\cdot\nabla y^\alpha = -\int_\Omega u_j\nabla(-\Delta)^{l-1}u_i\cdot\nabla y^\alpha = -s_{ji}^\alpha.$$

From (2.6) and (2.9), we derive

$$\begin{aligned}
&(\lambda_{k+1}-\lambda_i)\|\varphi_i\|^2\\
&\le l\Big\{\int_\Omega u_i(-\Delta)^{l-1}u_i - 2(l-1)\int_\Omega(-\Delta)^{l-2}u_i\nabla y^\alpha\cdot\nabla(\nabla u_i\cdot\nabla y^\alpha)\Big\}\\
&- 2l\sum_{j=1}^k r_{ij}^\alpha s_{ij}^\alpha,
\end{aligned}\tag{2.10}$$

where

$$\|\cdot\|^2 = \int|\cdot|^2.$$

On the other hand, from

$$\int_\Omega u_i y^\alpha \nabla u_i \cdot \nabla y^\alpha = -\frac{1}{2}, \tag{2.11}$$

we have

$$\begin{aligned}\int_\Omega \varphi_i^\alpha \nabla u_i \cdot \nabla y^\alpha &= \int_\Omega (u_i y^\alpha - \sum_{j=i}^k r_{ij}^\alpha u_j)\nabla u_i \cdot \nabla y^\alpha \\ &= -\frac{1}{2} - \sum_{j=1}^k r_{ij}^\alpha t_{ij}^\alpha,\end{aligned} \tag{2.12}$$

where

$$t_{ij}^\alpha = \int_\Omega u_j \nabla u_i \cdot \nabla y^\alpha = -\int_\Omega u_i \nabla u_j \cdot \nabla y^\alpha = -t_{ji}^\alpha. \tag{2.13}$$

Thus, for any constant $\delta > 0$, we have

$$\begin{aligned}1 + 2\sum_{j=1}^k r_{ij}^\alpha t_{ij}^\alpha &= -2\int_\Omega \varphi_i^\alpha \nabla u_i \cdot \nabla y^\alpha \\ &= -2\int_\Omega \varphi_i^\alpha (\nabla u_i \cdot \nabla y^\alpha - \sum_{j=1}^k t_{ij}^\alpha u_j) \\ &\le \delta\|\varphi_i^\alpha\|^2 + \frac{1}{\delta}\| - \nabla u_i \cdot \nabla y^\alpha + \sum_{j=1}^k t_{ij}^\alpha u_j\|^2 \\ &= \delta\|\varphi_i^\alpha\|^2 + \frac{1}{\delta}\|\nabla u_i \cdot \nabla y^\alpha\|^2 - \frac{1}{\delta}\sum_{j=1}^k (t_{ij}^\alpha)^2.\end{aligned} \tag{2.14}$$

Multiplying (2.14) by $(\lambda_{k+1} - \lambda_i)^2$, we infer

$$\begin{aligned}&(\lambda_{k+1} - \lambda_i)^2 + 2(\lambda_{k+1} - \lambda_i)^2 \sum_{j=1}^k r_{ij}^\alpha t_{ij}^\alpha \\ &\le \delta(\lambda_{k+1} - \lambda_i)^2\|\varphi_i^\alpha\|^2 + \frac{1}{\delta}(\lambda_{k+1} - \lambda_i)^2\|\nabla u_i \cdot \nabla y^\alpha\|^2 \\ &- \frac{1}{\delta}(\lambda_{k+1} - \lambda_i)^2 \sum_{j=1}^k (t_{ij}^\alpha)^2.\end{aligned} \tag{2.15}$$

Putting $\delta = (\lambda_{k+1} - \lambda_i)\delta_1$, where δ_1 is a positive constant, and taking sum on i from 1 to k, we have

$$
\begin{aligned}
&\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 + 2\sum_{i,j=1}^{k}(\lambda_{k+1} - \lambda_i)^2 r_{ij}^{\alpha} t_{ij}^{\alpha} \\
&\leq \delta_1 \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^3 \|\varphi_i^{\alpha}\|^2 + \frac{1}{\delta_1}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)\|\nabla u_i \cdot \nabla y^{\alpha}\|^2 \\
&- \frac{1}{\delta_1}\sum_{i,j=1}^{k}(\lambda_{k+1} - \lambda_i)(t_{ij}^{\alpha})^2.
\end{aligned}
\tag{2.16}
$$

From (2.10), we have

$$
\begin{aligned}
&\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 + 2\sum_{i,j=1}^{k}(\lambda_{k+1} - \lambda_i)^2 r_{ij}^{\alpha} t_{ij}^{\alpha} \\
&\leq \delta_1 \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 l \Big\{ \int u_i(-\Delta)^{l-1}u_i \\
&- 2(l-1)\int (-\Delta)^{l-2}u_i \nabla y^{\alpha} \cdot \nabla(\nabla u_i \cdot \nabla y^{\alpha}) - 2\sum_{j=1}^{k} r_{ij}^{\alpha} s_{ij}^{\alpha} \Big\} \\
&+ \frac{1}{\delta_1}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)\|\nabla u_i \cdot \nabla y^{\alpha}\|^2 - \frac{1}{\delta_1}\sum_{i,j=1}^{k}(\lambda_{k+1} - \lambda_i)(t_{ij}^{\alpha})^2 \\
&= \delta_1 \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 l \Big\{ \int u_i(-\Delta)^{l-1}u_i \\
&- 2(l-1)\int (-\Delta)^{l-2}u_i \nabla y^{\alpha} \cdot \nabla(\nabla u_i \cdot \nabla y^{\alpha}) \Big\} \\
&+ \frac{1}{\delta_1}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)\|\nabla u_i \cdot \nabla y^{\alpha}\|^2 - 2l\delta_1 \sum_{i,j=1}^{k}(\lambda_{k+1} - \lambda_i)^2 r_{ij}^{\alpha} s_{ij}^{\alpha} \\
&- \frac{1}{\delta_1}\sum_{i,j=1}^{k}(\lambda_{k+1} - \lambda_i)(t_{ij}^{\alpha})^2.
\end{aligned}
\tag{2.17}
$$

Since $r_{ij}^{\alpha} = r_{ji}^{\alpha}$, $t_{ij}^{\alpha} = -t_{ji}^{\alpha}$ and $(\lambda_i - \lambda_j) r_{ij}^{\alpha} = -2l s_{ij}^{\alpha}$, we have

$$
2\sum_{i,j=1}^{k}(\lambda_{k+1} - \lambda_i)^2 r_{ij}^{\alpha} t_{ij}^{\alpha} = -2\sum_{i,j=1}^{k}(\lambda_{k+1} - \lambda_i)(\lambda_i - \lambda_j) r_{ij}^{\alpha} t_{ij}^{\alpha}
\tag{2.18}
$$

and

$$\begin{aligned}
-2l\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)^2 r_{ij}^{\alpha}s_{ij}^{\alpha} &= \sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)^2(\lambda_i-\lambda_j)(r_{ij}^{\alpha})^2 \\
&= -\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2(r_{ij}^{\alpha})^2.
\end{aligned} \tag{2.19}$$

It is obvious that

$$\begin{aligned}
&2\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)r_{ij}^{\alpha}t_{ij}^{\alpha} \\
&\leq \delta_1\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2(r_{ij}^{\alpha})^2+\frac{1}{\delta_1}\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)(t_{ij}^{\alpha})^2.
\end{aligned} \tag{2.20}$$

We infer, from (2.17) to (2.20)

$$\begin{aligned}
&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \\
&\leq \delta_1\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 l\Big\{\int_{\Omega}u_i(-\Delta)^{l-1}u_i \\
&-2(l-1)\int_{\Omega}(-\Delta)^{l-2}u_i\nabla y^{\alpha}\cdot\nabla(\nabla u_i\cdot\nabla y^{\alpha})\Big\} \\
&+\frac{1}{\delta_1}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\|\nabla u_i\cdot\nabla y^{\alpha}\|^2.
\end{aligned} \tag{2.21}$$

Defining

$$\nabla^k=\begin{cases}\Delta^{\frac{k}{2}}, & k \text{ is even}\\ \nabla(\Delta^{\frac{k-1}{2}}), & k \text{ is odd,}\end{cases}$$

by the result of [5]

$$\int_{\Omega}|\nabla^k u_i|^2\leq\lambda_i^{\frac{k}{l}}, \tag{2.22}$$

we have

$$\int_{\Omega}u_i(-\Delta)^{l-1}u_i=\int_{\Omega}|\nabla^{l-1}u_i|^2\leq\lambda_i^{\frac{l-1}{l}}, \tag{2.23}$$

$$\begin{aligned}&-\sum_{\alpha=1}^{n}\int_\Omega(-\Delta)^{l-2}u_i\nabla y^\alpha\cdot\nabla(\nabla u_i\cdot\nabla y^\alpha)=-\int_\Omega(-\Delta)^{l-2}u_i\Delta u_i\\&=\int_\Omega(-\Delta)^{l-2}u_i(-\Delta)u_i=\int_\Omega|\nabla^{l-1}u_i|^2\leq\lambda_i^{\frac{l-1}{l}}\end{aligned}\tag{2.24}$$

and

$$\sum_{\alpha=1}^{n}\|\nabla u_i\cdot\nabla y^\alpha\|^2=\int_\Omega|\nabla u_i|^2\leq\lambda_i^{\frac{1}{l}}.\tag{2.25}$$

Summing on α from 1 to n for (2.21) and making use of (2.23), (2.24) and (2.25) we have

$$\begin{aligned}&n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\\&\leq\delta_1 l(n+2l-2)\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\lambda_i^{\frac{l-1}{l}}+\frac{1}{\delta_1}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i^{\frac{1}{l}}.\end{aligned}\tag{2.26}$$

Letting

$$\delta_1=\left\{\frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i^{\frac{1}{l}}}{l(n+2l-2)\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\lambda_i^{\frac{l-1}{l}}}\right\}^{\frac{1}{2}},$$

we have

$$\left(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\right)^2\leq\frac{4l(n+2l-2)}{n^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\lambda_i^{\frac{l-1}{l}}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i^{\frac{1}{l}}.\tag{2.27}$$

By the lemma 1 with $A_i=\lambda_{k+1}-\lambda_i$, $B_i=\lambda_i^{\frac{l-1}{l}}$ and $C_i=\lambda_i^{\frac{1}{l}}$, we obtain

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\lambda_i^{\frac{l-1}{l}}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i^{\frac{1}{l}}\\&\leq\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i,\end{aligned}\tag{2.28}$$

namely,

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\leq\frac{4l(n+2l-2)}{n^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i.\tag{2.29}$$

It completes the proof of the theorem 1.

□

3. Proof of Theorem 2

In this section, we will prove the theorem 2. At first, we will construct the test function by using the method of Sun, Cheng and Yang [12]. Let u_i be an eigenfunction corresponding to the i^{th} eigenvalue λ_i such that $\{u_i\}_{i=1}^{\infty}$ becomes an orthonormal basis of $L^2(\Omega)$. We define an $n \times n$-matrix B as following:

$$B := (b_{\alpha\beta})$$

where $b_{\alpha\beta} := \displaystyle\int_\Omega y^\alpha u_1 u_{\beta+1}$ and (y^α) is a position vector of $\mathbf{R}^n$. From the orthogonalization of Gram-Schmidt, there exists an upper triangle matrix $R = (R_{\alpha\beta})$ and an orthogonal matrix $Q = (q_{\alpha\beta})$ such that $R = QB$. Thus,

$$R_{\alpha\beta} = \sum_{\gamma=1}^{n} q_{\alpha\gamma} b_{\gamma\beta} = \int_\Omega \sum_{\gamma=1}^{n} q_{\alpha\gamma} y^\gamma u_1 u_{\beta+1} = 0, \quad 1 \leq \beta < \alpha \leq n.$$

By defining $g^\alpha := \displaystyle\sum_{\gamma=1}^{n} q_{\alpha\gamma} y^\gamma$, we have

$$\int_\Omega g^\alpha u_1 u_{\beta+1} = \int_\Omega \sum_{\gamma=1}^{n} q_{\alpha\gamma} y^\gamma u_1 u_{\beta+1} = 0, \quad 1 \leq \beta < \alpha \leq n.$$

Putting

$$\psi_\alpha := (g^\alpha - a^\alpha) u_1 \quad \text{and} \quad a^\alpha := \int_\Omega g^\alpha u_1^2,$$

we obtain

$$\int_\Omega \psi_\alpha u_{\beta+1} = 0, \quad 0 \leq \beta < \alpha \leq n.$$

Next, we will prove the following lemma 2.

Lemma 2. *Let u_i be an eigenfunction corresponding to the i^{th} eigenvalue λ_i of the eigenvalue problem* (1.1). *Then, for $k = 1, 2, \cdots, l-1$, we have*

$$\left(\int_\Omega |\nabla^{k-1} u_{i,\alpha}|^2\right)^{\frac{1}{k}} \leq \left(\int_\Omega |\nabla^{k} u_{i,\alpha}|^2\right)^{\frac{1}{k+1}},$$

where $u_{i,\alpha} = \langle \nabla u_i, \nabla g^\alpha \rangle$.

Proof. When $k = 1$, from Stokes' theorem,

$$\int_\Omega (u_{i,\alpha})^2 = -\int_\Omega u_i \langle \nabla u_{i,\alpha}, \nabla g^\alpha \rangle \leq \left(\int_\Omega \langle \nabla u_{i,\alpha}, \nabla g^\alpha \rangle^2\right)^{\frac{1}{2}} \leq \left(\int_\Omega |\nabla u_{i,\alpha}|^2\right)^{\frac{1}{2}},$$

where we have used $< \nabla g^\alpha, \nabla g^\alpha >= 1$. Assume that the lemma 2 holds for $k-1$, that is,

$$\left(\int_\Omega |\nabla^{k-2} u_{i,\alpha}|^2\right)^{\frac{1}{k-1}} \leq \left(\int_\Omega |\nabla^{k-1} u_{i,\alpha}|^2\right)^{\frac{1}{k}}.$$

Then, for k,

$$\begin{aligned}\int_\Omega |\nabla^{k-1}u_{i,\alpha}|^2 &= -\int_\Omega \nabla^{k-2}u_{i,\alpha}\nabla^k u_{i,\alpha}\\ &\le \Big(\int_\Omega |\nabla^{k-2}u_{i,\alpha}|^2\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla^{k}u_{i,\alpha}|^2\Big)^{\frac{1}{2}}\\ &\le \Big(\int_\Omega |\nabla^{k-1}u_{i,\alpha}|^2\Big)^{\frac{k-1}{2k}}\Big(\int_\Omega |\nabla^{k}u_{i,\alpha}|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

Therefore,

$$\Big(\int_\Omega |\nabla^{k-1}u_{i,\alpha}|^2\Big)^{\frac{k+1}{2k}} \le \Big(\int_\Omega |\nabla^{k}u_{i,\alpha}|^2\Big)^{\frac{1}{2}},$$

that is,

$$\Big(\int_\Omega |\nabla^{k-1}u_{i,\alpha}|^2\Big)^{\frac{1}{k}} \le \Big(\int_\Omega |\nabla^{k}u_{i,\alpha}|^2\Big)^{\frac{1}{k+1}}.$$

□

Proof of Theorem 2. By making use of the Rayleigh-Ritz inequality, we have

$$\lambda_{\alpha+1} \le \frac{\displaystyle\int_\Omega \psi_\alpha(-\Delta)^l\psi_\alpha}{\displaystyle\int_\Omega \psi_\alpha^2}, \quad 1\le\alpha\le n. \tag{3.1}$$

By induction, we can derive

$$(-\Delta)^l(g^\alpha u_1) = (-\Delta)^l u_1\cdot g^\alpha - 2l(-\Delta)^{l-1}u_{1,\alpha},$$

where $u_{1,\alpha} = \langle\nabla u_1, \nabla g^\alpha\rangle$. Hence, it follows that

$$\begin{aligned}&(-\Delta)^l\psi_\alpha = (-\Delta)^l u_1\cdot g^\alpha - 2l(-\Delta)^{l-1}u_{1,\alpha} - a^\alpha(-\Delta)^l u_1\\ &= -2l(-\Delta)^{l-1}u_{1,\alpha} + \lambda_1\psi_\alpha.\end{aligned}$$

Therefore, we have, from (3.1),

$$\begin{aligned}&(\lambda_{\alpha+1}-\lambda_1)||\psi_\alpha||^2 \le -2l\int_\Omega(-\Delta)^{l-1}u_{1,\alpha}\cdot\psi_\alpha\\ &= -2l\Big\{\int_\Omega g^\alpha u_1(-\Delta)^{l-1}u_{1,\alpha} - a^\alpha\int_\Omega u_1(-\Delta)^{l-1}u_{1,\alpha}\Big\}.\end{aligned}$$

By Stokes' Theorem,

$$\int_\Omega u_1(-\Delta)^{l-1}u_{1,\alpha} = 0$$

and

$$\begin{aligned}&\int_\Omega g^\alpha u_1(-\Delta)^{l-1}u_{1,\alpha}=\int_\Omega u_{1,\alpha}(-\Delta)^{l-1}(g^\alpha u_1)\\&=\int_\Omega u_{1,\alpha}\{\ (-\Delta)^{l-1}u_1\cdot g^\alpha-2(l-1)(-\Delta)^{l-2}u_{1,\alpha}\}\\&=-\int_\Omega\Big\{u_1\{\ (-\Delta)^{l-1}u_{1,\alpha}\cdot g^\alpha+(-\Delta)^{l-1}u_1\}\ +2(l-1)u_{1,\alpha}(-\Delta)^{l-2}u_{1,\alpha}\Big\},\end{aligned}$$

that is,

$$2\int_\Omega g^\alpha u_1(-\Delta)^{l-1}u_{1,\alpha}=-\int_\Omega\Big\{u_1(-\Delta)^{l-1}u_1+2(l-1)u_{1,\alpha}(-\Delta)^{l-2}u_{1,\alpha}\Big\}.$$

Hence, we derive

$$(\lambda_{\alpha+1}-\lambda_1)||\psi_\alpha||^2\le l\int_\Omega u_1(-\Delta)^{l-1}u_1+2(l-1)\int_\Omega u_{1,\alpha}(-\Delta)^{l-2}u_{1,\alpha}. \tag{3.2}$$

Since

$$\nabla^k=\begin{cases}\Delta^{\frac{k}{2}}, & k\text{ is even}\\ \nabla(\Delta^{\frac{k-1}{2}}), & k\text{ is odd,}\end{cases}$$

then, we have

$$\lambda_1=\int_\Omega u_1(-\Delta)^l u_1=\int_\Omega|\nabla^l u_1|^2.$$

Therefore, (3.2) can be expressed as

$$(\lambda_{\alpha+1}-\lambda_1)||\psi_\alpha||^2\le l\int_\Omega|\nabla^{l-1}u_1|^2+2(l-1)\int_\Omega|\nabla^{l-2}u_{1,\alpha}|^2. \tag{3.3}$$

Now, we prove that either

$$\sum_{k=1}^{l-1}\lambda_1^{\frac{k-1}{l}}(\lambda_{\alpha+1}^{\frac{l-k}{l}}-\lambda_1^{\frac{l-k}{l}})\le 2l(l-1)\int_\Omega|\nabla^{l-2}u_{1,\alpha}|^2 \tag{3.4}$$

or

$$\lambda_{\alpha+1}^{\frac{1}{l}}-\lambda_1^{\frac{1}{l}}\le 4\int_\Omega(u_{1,\alpha})^2 \tag{3.5}$$

hold for any α, $1\le\alpha\le n$. Assume that there exists an α such that neither (3.4) nor (3.5) holds. From (2.22) and (3.3), we have

$$\begin{aligned}(\lambda_{\alpha+1}-\lambda_1)||\psi_\alpha||^2 &\le l\int_\Omega |\nabla^{l-1}u_1|^2 + 2l(l-1)\int_\Omega |\nabla^{l-2}u_{1,\alpha}|^2\\ &\le l\lambda_1^{\frac{l-1}{l}} + 2l(l-1)\int_\Omega |\nabla^{l-2}u_{1,\alpha}|^2\\ &< l\lambda_1^{\frac{l-1}{l}} + \sum_{k=1}^{l-1}\lambda_1^{\frac{k-1}{l}}(\lambda_{\alpha+1}^{\frac{l-k}{l}} - \lambda_1^{\frac{l-k}{l}})\\ &= \sum_{k=1}^{l}\lambda_1^{\frac{k-1}{l}}\lambda_{\alpha+1}^{\frac{l-k}{l}} = \frac{\lambda_{\alpha+1}-\lambda_1}{\lambda_{\alpha+1}^{\frac{1}{l}} - \lambda_1^{\frac{1}{l}}},\end{aligned}$$

thus,

$$(\lambda_{\alpha+1}^{\frac{1}{l}} - \lambda_1^{\frac{1}{l}})||\psi_\alpha||^2 < 1. \tag{3.6}$$

On the other hand, since

$$\int_\Omega g^\alpha u_1 u_{1,\alpha} = -\frac{1}{2},$$

we have

$$\int_\Omega \psi_\alpha\cdot(-2u_{1,\alpha}) = 1.$$

From the Cauchy-Schwarz inequality,

$$\begin{aligned}1 = \left(\int_\Omega \psi_\alpha\cdot(-2u_{1,\alpha})\right)^2 &\le ||\psi_\alpha||^2\cdot 4\int_\Omega (u_{1,\alpha})^2\\ &< (\lambda_{\alpha+1}^{\frac{1}{l}} - \lambda_1^{\frac{1}{l}})||\psi_\alpha||^2,\end{aligned} \tag{3.7}$$

which contradicts with (3.6). Therefore, either (3.4) or (3.5) holds for any α, $1 \le \alpha \le n$.

Next, we prove that

$$\lambda_{\alpha+1}^{\frac{1}{l}} - \lambda_1^{\frac{1}{l}} \le 2l(\int_\Omega |\nabla^{l-2}u_{1,\alpha}|^2)^{\frac{1}{l-1}} \tag{3.8}$$

hold for $1 \le \alpha \le n$. It suffices to show that both (3.4) implies (3.8) and (3.5) implies (3.8).

Assume that (3.4) holds. From the definition of $u_{1,\alpha}$ and (2.22), we have

$$\sum_{\alpha=1}^{n}\int_\Omega |\nabla^{k-1}u_{1,\alpha}|^2 = \int_\Omega |\nabla^k u_1|^2 \le \lambda_1^{\frac{k}{l}}, \quad k = 1,2,\cdots,l-1. \tag{3.9}$$

Then, we infer

$$\begin{aligned}
2l(l-1)\int_\Omega |\nabla^{l-2}u_{1,\alpha}|^2 &\geq \sum_{k=1}^{l-1}\lambda_1^{\frac{k-1}{l}}(\lambda_{\alpha+1}^{\frac{l-k}{l}}-\lambda_1^{\frac{l-k}{l}})\\
&\geq \sum_{k=1}^{l-1}\lambda_1^{\frac{k-1}{l}}(\lambda_1^{\frac{l-k-1}{l}}\lambda_{\alpha+1}^{\frac{1}{l}}-\lambda_1^{\frac{l-k}{l}})\\
&= (l-1)\lambda_1^{\frac{l-2}{l}}(\lambda_{\alpha+1}^{\frac{1}{l}}-\lambda_1^{\frac{1}{l}})\\
&\geq (l-1)(\int_\Omega |\nabla^{l-2}u_{1,\alpha}|^2)^{\frac{l-2}{l-1}}(\lambda_{\alpha+1}^{\frac{1}{l}}-\lambda_1^{\frac{1}{l}}).
\end{aligned}$$

That is

$$\lambda_{\alpha+1}^{\frac{1}{l}}-\lambda_1^{\frac{1}{l}} \leq 2l(\int_\Omega |\nabla^{l-2}u_{1,\alpha}|^2)^{\frac{1}{l-1}}.$$

If (3.5) holds, from the lemma 2 and $l\geq 2$, we have

$$\begin{aligned}
\lambda_{\alpha+1}^{\frac{1}{l}}-\lambda_1^{\frac{1}{l}} &\leq 2l\int_\Omega (u_{1,\alpha})^2\\
&\leq 2l(\int_\Omega |\nabla u_{1,\alpha}|^2)^{\frac{1}{2}}\\
&\leq \cdots\\
&\leq 2l(\int_\Omega |\nabla^{l-2}u_{1,\alpha}|^2)^{\frac{1}{l-1}}.
\end{aligned}$$

Thus, (3.8) holds. By taking sum on α from 1 to n, we can infer

$$\sum_{\alpha=1}^{n}(\lambda_{\alpha+1}^{\frac{1}{l}}-\lambda_1^{\frac{1}{l}})^{l-1} \leq (2l)^{l-1}\lambda_1^{\frac{l-1}{l}}.$$

This completes the proof of Theorem 2.

□

4. Proof of Theorem 3

In this section, we will prove the theorem 3.

Proof of Theorem 3. From (3.7),

$$1=\left(\int_\Omega \psi_\alpha\cdot(-2u_{1,\alpha})\right)^2 \leq ||\psi_\alpha||^2\cdot 4\int_\Omega (u_{1,\alpha})^2. \tag{4.1}$$

Multiplying (4.1) by $(\lambda_{\alpha+1}-\lambda_1)$, we have, from (3.3),

$$\begin{aligned}\lambda_{\alpha+1}-\lambda_1 &\leq (\lambda_{\alpha+1}-\lambda_1)||\psi_\alpha||^2\cdot 4\int_\Omega (u_{1,\alpha})^2\\ &\leq 4l\int_\Omega\{\ |\nabla^{l-1}u_1|^2+2(l-1)|\nabla^{l-2}u_{1,\alpha}|^2\}\ \int_\Omega (u_{1,\alpha})^2,\end{aligned}$$

namely,

$$\lambda_{\alpha+1}-\lambda_1\leq 4l\int_\Omega|\nabla^{l-1}u_1|^2\int_\Omega(u_{1,\alpha})^2+8l(l-1)\int_\Omega|\nabla^{l-2}u_{1,\alpha}|^2\int_\Omega(u_{1,\alpha})^2. \tag{4.2}$$

Clearly,

$$\int_\Omega (u_{1,\alpha})^2\geq 0, \tag{4.3}$$

$$\int_\Omega |\nabla^{l-2}u_{1,\alpha}|^2\geq 0. \tag{4.4}$$

Summing on α from 1 to n for (4.2), we derive from (3.9), (4.3), and (4.4),

$$\begin{aligned}&\sum_{\alpha=1}^n(\lambda_{\alpha+1}-\lambda_1)\\ &\leq 4l\int_\Omega|\nabla^{l-1}u_1|^2\sum_{\alpha=1}^n\int_\Omega(u_{1,\alpha})^2+8l(l-1)\sum_{\alpha=1}^n\int_\Omega|\nabla^{l-2}u_{1,\alpha}|^2\int_\Omega(u_{1,\alpha})^2\\ &\leq 4l\lambda_1^{\frac{l-1}{l}}\lambda_1^{\frac{1}{l}}+8l(l-1)(\sum_{\alpha=1}^n\int_\Omega|\nabla^{l-2}u_{1,\alpha}|^2)(\sum_{\alpha=1}^n\int_\Omega(u_{1,\alpha})^2)\\ &\leq 4l(2l-1)\lambda_1.\end{aligned}$$

This completes the proof of Theorem 3.

□

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