Calc. Var. (2009) 36:507–523
DOI 10.1007/s00526-009-0240-7

Calculus of Variations

Estimates for eigenvalues of the poly-Laplacian with any order in a unit sphere

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Received: 28 April 2008 / Accepted: 30 March 2009 / Published online: 21 April 2009

Abstract In this paper we study eigenvalues of the poly-Laplacian with any order on a domain in an n-dimensional unit sphere and obtain estimates for eigenvalues. In particular, the optimal result of Cheng and Yang (Math Ann 331:445–460, 2005) is included in our ones. In order to prove our results, we introduce $2(l+1)$ functions a_i and b_i, for $i = 0, 1, \dots, l$ and two operators μ and η. First of all, we study properties of functions a_i and b_i and the operators μ and η. By making use of these properties and introducing k free constants, we obtain estimates for eigenvalues.

Mathematics Subject Classification (2000) 35P15 · 58G25

1 Introduction

Let Ω be a bounded domain with a smooth boundary $\partial\Omega$ in an n-dimensional complete Riemannian manifold M. Assume that λ_i is the ith eigenvalue of the Dirichlet eigenvalue problem of the poly-Laplacian with order l:

$$\begin{cases} (-\Delta)^l u = \lambda u & \text{in } \Omega \\ u = \dfrac{\partial u}{\partial \nu} = \cdots = \dfrac{\partial^{l-1} u}{\partial \nu^{l-1}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ is the Laplacian in M and ν denotes the outward unit normal. It is well known that the spectrum of this eigenvalue problem is real and discrete.

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \longrightarrow \infty.$$

Research partially supported by a Grant-in-Aid for Scientific Research from the JSPS.

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When $l = 1$, the above problem is called a fixed membrane problem and it is called a clamped plate problem when $l = 2$.

When $M = \mathbf{R}^n$ and $l = 1$, Payne et al. [14] proved

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{kn} \sum_{i=1}^{k} \lambda_i, \quad k = 1, 2, \ldots.$$

Further, Hile and Protter [11] generalized the above result to

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{kn}{4}, \quad k = 1, 2, \ldots.$$

In 1991, Yang [18] (cf. [7]) proved the following sharp result:

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\lambda_i, \quad k = 1, 2, \ldots. \tag{1.2}$$

According to the inequality, we can derive

$$\lambda_{k+1} \leq \frac{1}{k} \left(1 + \frac{4}{n}\right) \sum_{i=1}^{k} \lambda_i, \quad k = 1, 2, \ldots. \tag{1.3}$$

When $l = 2$, one usually uses Γ_i to denote the ith eigenvalue of (1.1). In this case, Payne et al. [14] proved

$$\Gamma_{k+1} - \Gamma_k \leq \frac{8(n+2)}{n^2 k} \sum_{i=1}^{k} \Gamma_i, \quad k = 1, 2, \ldots.$$

Recently, Cheng and Yang [6] have proved

$$\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i) \leq \left(\frac{8(n+2)}{n^2}\right)^{\frac{1}{2}} \sum_{i=1}^{k} (\Gamma_i (\Gamma_{k+1} - \Gamma_i))^{\frac{1}{2}},$$

which is analogous to the inequality (1.2) of Yang.

For any integer l, Chen and Qian [4] and Hook [12] (cf. [13]), independently, proved

$$\frac{n^2 k^2}{4l(n+2l-2)} \leq \sum_{i=1}^{k} \frac{\lambda_i^{\frac{1}{l}}}{\lambda_{k+1} - \lambda_i} \sum_{i=1}^{k} \lambda_i^{\frac{l-1}{l}}, \quad k = 1, 2, \ldots. \tag{1.4}$$

By making use of the method of Cheng and Yang [6], Wu and Cao [17] have generalized the inequality (1.4) to

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \leq \frac{\{4l(n+2l-2)\}^{\frac{1}{2}}}{n} \left\{ \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\frac{1}{2}} \lambda_i^{\frac{l-1}{l}} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\frac{1}{2}} \lambda_i^{\frac{1}{l}} \right\}^{\frac{1}{2}}. \tag{1.5}$$

We should notice that when $l = 1$, the inequality (1.5) becomes

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \leq \frac{2}{\sqrt{n}} \left\{ \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\frac{1}{2}} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\frac{1}{2}} \lambda_i \right\}^{\frac{1}{2}}.$$

From this inequality, one can only infer the inequality (1.3). One cannot obtain the sharp inequality (1.2) of Yang. In [8], we have proved

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4l(n+2l-2)}{n^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i, \tag{1.6}$$

which includes the sharp inequality (1.2) of Yang.

When M is an n-dimensional unit sphere $S^n(1)$ and $l=1$, Cheng and Yang [5] proved

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i+\frac{n^2}{4}\right), \tag{1.7}$$

which is optimal since when Ω tends to $S^n(1)$, the above inequality becomes equality for any k. Recently, Chen and Cheng [3] and El Soufi et al. [9] have generalized this result to any complete Riemannian manifold, independently. When $l=2$, Wang and Xia [16] have obtained

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{1}{n}\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2(n^2+(2n+4)\lambda_i^{\frac{1}{2}})\right\}^{\frac{1}{2}} \times\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(n^2+4\lambda_i^{\frac{1}{2}})\right\}^{\frac{1}{2}}.$$

From the above inequality, we can infer

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4}{n^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left((2n+4)\lambda_i^{\frac{1}{2}}+n^2\right)\left(\lambda_i^{\frac{1}{2}}+\frac{n^2}{4}\right). \tag{1.8}$$

In this paper, we study eigenvalues of the eigenvalue problem (1.1) for any integer l when M is $S^n(1)$. We prove the following:

Theorem *Let Ω be a domain in an n-dimensional unit sphere $S^n(1)$, $n\ge 2$. Assume that λ_i, $i=1,2,\ldots$, is the ith eigenvalue of the eigenvalue problem* (1.1). *Then, we have*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4}{n^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left\{(\lambda_i^{\frac{1}{l}}+n)^l-\lambda_i + [2^{l+2}-4(l+1)]\lambda_i^{\frac{1}{l}}(\lambda_i^{\frac{1}{l}}+n)^{l-2}\right\}\left(\lambda_i^{\frac{1}{l}}+\frac{n^2}{4}\right). \tag{1.9}$$

Remark 1 For $l=1$, our inequality (1.9) becomes the optimal inequality (1.7) of Cheng and Yang [5]:

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i+\frac{n^2}{4}\right).$$

When $l=2$, our result becomes the inequality (1.8):

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4}{n^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left((2n+4)\lambda_i^{\frac{1}{2}}+n^2\right)\left(\lambda_i^{\frac{1}{2}}+\frac{n^2}{4}\right).$$

Since the inequality (1.9) is a quadratic inequality of λ_{k+1}, it is not difficult to derive an upper bound for λ_{k+1} according to the first k eigenvalues.

2 Proof of Theorem

Assume that u_i is an eigenfunction corresponding to the ith eigenvalue λ_i such that $\{u_i\}_{i=1}^{\infty}$ becomes an orthonormal basis of $L^2(\Omega)$. Thus, u_i satisfies

$$\begin{cases} (-\Delta)^l u_i = \lambda_i u_i, & \text{in } \Omega \\ u_i = \dfrac{\partial u_i}{\partial \nu} = \cdots = \dfrac{\partial^{l-1} u_i}{\partial \nu^{l-1}} = 0, & \text{on } \partial\Omega \\ \displaystyle\int_{\Omega} u_i u_j = \delta_{ij}. & \end{cases} \tag{2.1}$$

Let $x_1, x_2, \ldots, x_{n+1}$ be the standard coordinate functions of the $n+1$-dimensional Euclidean space $\mathbf{R}^{n+1}$, then

$$S^n(1) = \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} | \sum_{\alpha=1}^{n+1} x_\alpha^2 = 1 \right\}.$$

It is well known that, for any $i, j = 1, 2, \ldots, n$,

$$\nabla_i \nabla_j x_\alpha = -g_{ij} x_\alpha, \quad \alpha = 1, \ldots, n+1, \tag{2.2}$$

where ∇_i and g_{ij} denote the covariant differentiation and components of metric tensor of the unit sphere $S^n(1)$, respectively.

In order to prove our theorem, the following Propositions 2.1 and 2.2 play an essential role.

Proposition 2.1

$$\begin{aligned} (-\Delta)^l (u_i x_\alpha) &= (-\Delta)^{l-1}(a_1 x_\alpha) + (-\Delta)^{l-1}(\nabla b_1 \cdot \nabla x_\alpha) \\ &= (-\Delta)^{l-2}(a_2 x_\alpha) + (-\Delta)^{l-2}(\nabla b_2 \cdot \nabla x_\alpha) \\ &= \cdots \\ &= a_l x_\alpha + \nabla b_l \cdot \nabla x_\alpha, \end{aligned}$$

where a_k and b_k are functions of $u_i, (-\Delta)u_i, \ldots, (-\Delta)^k u_i$ and satisfy

$$a_0 = u_i, \quad b_0 = 0, \quad a_1 = \mu u_i, \quad b_1 = -2u_i.$$

and

$$\begin{aligned} a_{k+1} &= (-\Delta)a_k + n a_k - 2(-\Delta)b_k = \mu a_k - 2(-\Delta)b_k \\ b_{k+1} &= -2a_k + (-\Delta)b_k - (n-2)b_k = -2a_k + \eta b_k. \end{aligned}$$

Here μ and η are defined by $\mu = (-\Delta) + n$, $\eta = (-\Delta) - (n-2)$, respectively.

Proof Since the Ricci curvature of the unit sphere $S^n(1)$ is $n-1$, from the Ricci identity, we have, for any functions f and g,

$$(-\Delta)(\nabla f) \cdot \nabla g = \nabla\{(-\Delta) f\} \cdot \nabla g - (n-1)\nabla f \cdot \nabla g.$$

For any $k \le l$, we infer from (2.2)

$$\begin{aligned}(-\Delta)^{l-k}(a_k x_\alpha) &= (-\Delta)^{l-k-1}\{(-\Delta)a_k x_\alpha - 2\nabla a_k \cdot \nabla x_\alpha + a_k(-\Delta)x_\alpha\}\\ &= (-\Delta)^{l-k-1}\{((-\Delta)a_k + na_k)x_\alpha\} + (-\Delta)^{l-k-1}\{\nabla(-2a_k)\cdot\nabla x_\alpha\}\end{aligned}$$

and

$$\begin{aligned}&(-\Delta)^{l-k}(\nabla b_k \cdot \nabla x_\alpha)\\ &= (-\Delta)^{l-k-1}\left\{(-\Delta)(\nabla b_k)\cdot\nabla x_\alpha - 2\sum_{i,j=1}^{n}\nabla_i\nabla_j b_k \nabla_i\nabla_j x_\alpha + \nabla b_k \cdot (-\Delta)\nabla x_\alpha\right\}\\ &= (-\Delta)^{l-k-1}\{\nabla((-\Delta)b_k)\cdot\nabla x_\alpha - 2(-\Delta)b_k x_\alpha + \nabla b_k\cdot\nabla((-\Delta)x_\alpha)\\ &\quad - 2(n-1)\nabla b_k\cdot\nabla x_\alpha\}\\ &= (-\Delta)^{l-k-1}\{-2(-\Delta)b_k x_\alpha\} + (-\Delta)^{l-k-1}\{\nabla[(-\Delta)b_k - (n-2)b_k]\cdot\nabla x_\alpha\}.\end{aligned}$$

According to the above two equalities, we have

$$\begin{aligned}a_{k+1} &= (-\Delta)a_k + na_k - 2(-\Delta)b_k = \mu a_k - 2(-\Delta)b_k\\ b_{k+1} &= -2a_k + (-\Delta)b_k - (n-2)b_k = -2a_k + \eta b_k\end{aligned}$$

with

$$a_0 = u_i, \quad b_0 = 0, \quad a_1 = \mu u_i, \quad b_1 = -2u_i.$$

□

Lemma 2.1 *a_k and b_k in the Proposition 2.1 satisfy*

$$b_k = -2\sum_{j=0}^{k-1}\eta^{k-j-1}a_j, \tag{2.3}$$

$$a_k = \mu^k a_0 + 4(-\Delta)\sum_{j=0}^{k-2}\frac{\mu^{k-j-1}-\eta^{k-j-1}}{\mu-\eta}a_j, \tag{2.4}$$

for $k = 2, 3, \ldots, l$.

Here we should notice $\mu - \eta = 2(n-1)$, which is scalar. But, for convenience, we still write it in $\mu - \eta$.

Proof For $k = 2$, from the Proposition 2.1, we have

$$b_2 = -2a_1 + (-\Delta b_1) - (n-2)b_1 = -2a_1 - 2\eta a_0.$$

Suppose that equality (2.3) holds for $k-1$, that is,

$$b_{k-1} = -2\sum_{j=0}^{k-2}\eta^{k-j-2}a_j, \quad k = 2, 3, \ldots, l.$$

Then, for integer k, from the Proposition 2.1, we derive

$$b_k = -2a_{k-1} + \eta b_{k-1} = -2a_{k-1} - 2\sum_{j=0}^{k-2}\eta^{k-j-1}a_j = -2\sum_{j=0}^{k-1}\eta^{k-j-1}a_j.$$

Therefore, (2.3) is proved.

For $k=2$, (2.4) is obvious. We assume that (2.4) holds for $k-1$. Then, from the Proposition 2.1 and (2.3), we have

$$\begin{aligned} a_k &= \mu a_{k-1} + 4(-\Delta)\sum_{j=0}^{k-2}\eta^{k-j-2}a_j \\ &= \mu\left\{\mu^{k-1}a_0 + 4(-\Delta)\sum_{j=0}^{k-3}\frac{\mu^{k-j-2}-\eta^{k-j-2}}{\mu-\eta}a_j\right\} + 4(-\Delta)\sum_{j=0}^{k-2}\eta^{k-j-2}a_j \\ &= \mu^k a_0 + 4(-\Delta)\sum_{j=0}^{k-2}\frac{\mu^{k-j-1}-\eta^{k-j-1}}{\mu-\eta}a_j. \end{aligned}$$

Thus, it completes the proof of (2.4). □

Proposition 2.2 *Defining p_i by*

$$p_i = (-\Delta)^l(u_i x_\alpha) - x_\alpha(-\Delta)^l u_i,$$

we have

$$\sum_{\alpha=1}^{n+1}\int_\Omega u_i x_\alpha p_i \le \left(\lambda_i^{\frac{1}{l}}+n\right)^l - \lambda_i + [2^{l+2}-4(l+1)]\lambda_i^{\frac{1}{l}}\left(\lambda_i^{\frac{1}{l}}+n\right)^{l-2}.$$

In order to prove this proposition, we need to prepare several lemmas.

Define a set $V_k(t)$ of polynomials in t by

$$V_k(t) := \left\{ v_k(t) = v_k t^k + v_{k-1}t^{k-1} + \cdots + v_1 t + v_0 \,\middle|\, v_i \ge 0, \quad i = 0, 1, \ldots k \right\}.$$

Defining

$$\nabla^k = \begin{cases} \Delta^{\frac{k}{2}}, & k \text{ is even} \\ \nabla(\Delta^{\frac{k-1}{2}}), & k \text{ is odd}, \end{cases} \tag{2.5}$$

The following inequality can be proved by the same method as in [4], in which Chen and Qian proved it for a bounded domain in the Euclidean space $\mathbf{R}^n$. For any integer k satisfying $1 \le k \le l$,

$$\int_\Omega |\nabla^k u_i|^2 \le \lambda_i^{\frac{k}{l}}. \tag{2.6}$$

Lemma 2.2 *For any $k \le l$ and any polynomial $v_{l-k}(t) \in V_{l-k}(t)$, we have*

$$\int_\Omega u_i v_{l-k}((-\Delta))a_k \ge 0. \tag{2.7}$$

Proof For $k = 0$, we know

$$\int_\Omega u_i v_l((-\Delta))a_0 = \int_\Omega u_i v_l((-\Delta))u_i = \int_\Omega v_l u_i(-\Delta)^l u_i + \int_\Omega v_{l-1}u_i(-\Delta)^{l-1}u_i + \cdots$$
$$+ \int_\Omega v_1 u_i(-\Delta)u_i + \int_\Omega v_0 u_i^2 = v_l \int_\Omega |\nabla^l u_i|^2 + v_{l-1}\int_\Omega |\nabla^{l-1}u_i|^2 + \cdots$$
$$+ v_1 \int_\Omega |\nabla u_i|^2 + v_0 \geq 0.$$

Suppose that the inequality (2.7) holds for $0, 1, \ldots, k-1$. Then, for the integer k, we can derive, from the Lemma 2.1,

$$\int_\Omega u_i v_{l-k}((-\Delta))a_k = \int_\Omega u_i v_{l-k}((-\Delta))\mu^k a_0 + 4\int_\Omega u_i v_{l-k}((-\Delta))(-\Delta)$$
$$\times \sum_{j=0}^{k-2} \frac{\mu^{k-j-1} - \eta^{k-j-1}}{\mu - \eta} a_j = \int_\Omega u_i v_l((-\Delta))a_0$$
$$+ 4\sum_{j=0}^{k-2}\int_\Omega u_i v_{l-k}((-\Delta))(-\Delta) \sum_{r=0}^{k-j-1} \binom{k-j-1}{k-j-1-r}(-\Delta)^r$$
$$\times \{n^{k-j-1-r} - (-n+2)^{k-j-1-r}\}a_j$$
$$= \int_\Omega u_i v_l((-\Delta))a_0 + 4\sum_{j=0}^{k-2}\int_\Omega u_i v_{l-k}((-\Delta))(-\Delta)v_{k-j-1}((-\Delta))a_j$$
$$= \int_\Omega u_i v_l((-\Delta))a_0 + 4\sum_{j=0}^{k-2}\int_\Omega u_i v_{l-j}((-\Delta))a_j \geq 0.$$

Here and in the sequel, without loss of the generality, we use the same sign to denote a different polynomial in $V_k(t)$. Therefore, for any $k \leq l$, we have

$$\int_\Omega u_i v_{l-k}((-\Delta))a_k \geq 0. \tag{2.8}$$

□

Lemma 2.3 *For $r < l - k$, we have*

$$\int_\Omega u_i v_{l-k-r}((-\Delta))\mu^r a_k \geq \int_\Omega u_i v_{l-k-r}((-\Delta))\eta^r a_k. \tag{2.9}$$

Proof Since the Lemma 2.2,

$$\begin{aligned}
&\int_\Omega u_i v_{l-k-r}((-\Delta))(\mu^r-\eta^r)a_k \\
&\quad = \int_\Omega u_i v_{l-k-r}((-\Delta))\sum_{j=0}^{r}\binom{r}{r-j}(-\Delta)^{r-j}\{n^j-(-n+2)^j\}a_k \\
&\quad = \sum_{j=0}^{r}\binom{r}{r-j}\{n^j-(-n+2)^j\}\int_\Omega u_i v_{l-k-r}((-\Delta))(-\Delta)^{r-j}a_k \geq 0. \qquad (2.10)
\end{aligned}$$

□

Lemma 2.4 *For any $k \leq l$, we have*

$$\int_\Omega u_i v_{l-k}((-\Delta))a_k \geq \int_\Omega u_i v_{l-k}((-\Delta))\mu a_{k-1}. \qquad (2.11)$$

Proof For $k=1$, since $a_1=\mu a_0$, (2.11) is obvious. Suppose that inequality (2.11) holds for $0,1,\ldots,k-1$. Then, for integer k, from the Proposition 2.1 and the Lemma 2.1, we infer

$$\begin{aligned}
&\int_\Omega u_i v_{l-k}((-\Delta))(a_k-\mu a_{k-1}) \\
&\quad = 4\int_\Omega u_i v_{l-k}((-\Delta))(-\Delta)\{a_{k-2}+\eta a_{k-3}+\cdots+\eta^{k-2}a_0\} \\
&\quad = 4\int_\Omega u_i v_{l-k+1}((-\Delta))\{a_{k-2}+\eta a_{k-3}+\cdots+\eta^{k-2}a_0\} \\
&\quad \geq 4\int_\Omega u_i v_{l-k+1}((-\Delta))\{(\mu+\eta)a_{k-3}+\eta^2 a_{k-4}+\cdots+\eta^{k-2}a_0\}.
\end{aligned}$$

Since $v_{l-k+1}((-\Delta))(\mu+\eta) \in V_{l-k+2}((-\Delta))$, we derive

$$\int_\Omega u_i v_{l-k+1}((-\Delta))(\mu+\eta)a_{k-3} \geq \int_\Omega u_i v_{l-k+1}((-\Delta))(\mu+\eta)\mu a_{k-4}.$$

Hence, we have

$$\begin{aligned}
&4\int_\Omega u_i v_{l-k+1}((-\Delta))\{(\mu+\eta)a_{k-3}+\eta^2 a_{k-4}+\cdots+\eta^{k-2}a_0\} \\
&\quad \geq 4\int_\Omega u_i v_{l-k+1}((-\Delta))\{(\mu^2+\mu\eta+\eta^2)a_{k-4}+\cdots+\eta^{k-2}a_0\} \\
&\quad \geq \cdots \geq 4\int_\Omega u_i v_{l-k+1}((-\Delta))(\mu^{k-2}+\mu^{k-3}\eta+\cdots+\mu\eta^{k-3}+\eta^{k-2})a_0 \\
&\quad = 4\int_\Omega u_i v_{l-1}((-\Delta))a_0 \geq 0.
\end{aligned}$$

Therefore,

$$\int_{\Omega} u_i v_{l-k}((-\Delta))a_k \geq \int_{\Omega} u_i v_{l-k}((-\Delta))\mu a_{k-1}.$$

□

Lemma 2.5 *For any $k \leq l$, we have*

$$\int_{\Omega} u_i v_{l-k}((-\Delta))a_k \leq 2\int_{\Omega} u_i v_{l-k}((-\Delta))\mu a_{k-1}. \tag{2.12}$$

Proof For $k = 1$, we know

$$\begin{aligned}\int_{\Omega} u_i v_{l-1}((-\Delta))(a_1 - 2\mu a_0) &= -\int_{\Omega} u_i v_{l-1}((-\Delta))\mu a_0 \\ &= -\int_{\Omega} u_i v_l((-\Delta))a_0 \leq 0.\end{aligned}$$

Suppose that the inequality (2.12) holds for $0, 1, \ldots, k-1$. Then, for the integer k, we obtain, from the Proposition 2.1 and the Lemma 2.1,

$$\begin{aligned}&\int_{\Omega} u_i v_{l-k}((-\Delta))(a_k - 2\mu a_{k-1}) \\ &= \int_{\Omega} u_i v_{l-k}((-\Delta))\left(-\mu a_{k-1} + 4(-\Delta)\{a_{k-2} + \eta a_{k-3} + \cdots + \eta^{k-2}a_0\}\right) \\ &= \int_{\Omega} u_i v_{l-k}((-\Delta))\left(-\mu^2 a_{k-2} - 4(-\Delta)\{\mu a_{k-3} + \mu\eta a_{k-4} + \cdots + \mu\eta^{k-3}a_0\}\right. \\ &\quad \left. + 4(-\Delta)\{a_{k-2} + \eta a_{k-3} + \cdots + \eta^{k-2}a_0\}\right) \\ &= \int_{\Omega} u_i v_{l-k}((-\Delta))\left((-\mu^2 + 4(-\Delta))a_{k-2}\right. \\ &\quad \left. + 4(-\Delta)\{-(\mu-\eta)a_{k-3} - (\mu-\eta)\eta a_{k-4} - \cdots - (\mu-\eta)\eta^{k-3}a_0\}\right)\end{aligned}$$

From $\mu - \eta = 2(n-1)$ and the Lemma 2.4, we have

$$\int_{\Omega} -u_i v_{l-k}((-\Delta))(-\Delta)(\mu-\eta)a_{k-3} \leq \int_{\Omega} -u_i v_{l-k}((-\Delta))(-\Delta)(\mu-\eta)\mu a_{k-4}.$$

Thus,

$$\begin{aligned}&\int_{\Omega} u_i v_{l-k}((-\Delta))\left((-\mu^2 + 4(-\Delta))a_{k-2}\right. \\ &\qquad \left. + 4(-\Delta)\{-(\mu-\eta)a_{k-3} - (\mu-\eta)\eta a_{k-4} - \cdots - (\mu-\eta)\eta^{k-3}a_0\}\right)\end{aligned}$$

$$
\begin{aligned}
&\leq \int_\Omega u_i v_{l-k}((-\Delta)) \Big((-\mu^2 + 4(-\Delta))a_{k-2} \\
&\quad +4(-\Delta)\{-(\mu-\eta)(\mu+\eta)a_{k-4} - \cdots - (\mu-\eta)\eta^{k-3}a_0\}\Big).
\end{aligned}
$$

Since $v_{l-k}((-\Delta))(-\Delta)(\mu-\eta)(\mu+\eta) \in V_{l-k+2}((-\Delta))$, according to the Lemma 2.4, we infer

$$
\begin{aligned}
&\int_\Omega -u_i v_{l-k}((-\Delta))(-\Delta)(\mu-\eta)(\mu+\eta)a_{k-4} \\
&\quad \leq \int_\Omega -u_i v_{l-k}((-\Delta))(-\Delta)(\mu-\eta)(\mu+\eta)\mu a_{k-5}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_\Omega u_i v_{l-k}((-\Delta)) \Big((-\mu^2 + 4(-\Delta))a_{k-2} \\
&\quad +4(-\Delta)\{-(\mu-\eta)(\mu+\eta)a_{k-4} - \cdots - (\mu-\eta)\eta^{k-3}a_0\}\Big) \\
&\leq \int_\Omega u_i v_{l-k}((-\Delta)) \Big((-\mu^2 + 4(-\Delta))a_{k-2} \\
&\quad +4(-\Delta)\{-(\mu-\eta)(\mu^2+\mu\eta+\eta^2)a_{k-5} - \cdots - (\mu-\eta)\eta^{k-3}a_0\}\Big) \\
&\leq \cdots \leq \int_\Omega u_i v_{l-k}((-\Delta)) \Big\{(-\mu^2 + 4(-\Delta))a_{k-2} \\
&\quad -4(-\Delta)(\mu-\eta)(\mu^{k-3}+\mu^{k-4}\eta+\cdots+\mu\eta^{k-4}+\eta^{k-3})a_0\Big\} \\
&\leq 0.
\end{aligned}
$$

Here we have used $v_{l-k}((-\Delta))(\mu^2 - 4(-\Delta)) \in V_{l-k+2}((-\Delta))$ with $n \geq 2$. Therefore,

$$
\int_\Omega u_i v_{l-k}((-\Delta))a_k \leq 2\int_\Omega u_i v_{l-k}((-\Delta))\mu a_{k-1}.
$$

It finishes the proof of the Lemma 2.5. □

Proof of Proposition 2.2 Since $\sum_{\alpha=1}^{n+1} x_\alpha^2 = 1$, from the Proposition 2.1, we have

$$
\begin{aligned}
\sum_{\alpha=1}^{n+1}\int_\Omega u_i x_\alpha p_i &= \sum_{\alpha=1}^{n+1}\left\{\int_\Omega (-\Delta)^l(u_i x_\alpha)u_i x_\alpha - \int_\Omega u_i(-\Delta)^l u_i (x_\alpha)^2\right\} \\
&= \sum_{\alpha=1}^{n+1}\left\{\int_\Omega \{a_l x_\alpha + \nabla b_l \cdot \nabla x_\alpha\}u_i x_\alpha - \int_\Omega u_i(-\Delta)^l u_i (x_\alpha)^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \int\limits_{\Omega} u_i a_l - \int\limits_{\Omega} u_i(-\Delta)^l u_i \\
&= \int\limits_{\Omega} u_i a_l - \lambda_i .
\end{aligned}
\tag{2.13}
$$

Since $a_{k+1} = \mu a_k - 2(-\Delta)b_k$ and $b_k = -2\sum_{j=0}^{k-1}\eta^{k-j-1}a_j$, we infer, from the Lemmas 2.3 and 2.5, we have

$$
\begin{aligned}
\int\limits_{\Omega} a_l u_i &= \int\limits_{\Omega} u_i \left\{\mu a_{l-1} - 2(-\Delta)b_{l-1}\right\} \\
&= \int\limits_{\Omega} u_i \left\{\mu a_{l-1} + 4(-\Delta)\sum_{j=0}^{l-2}\eta^{l-j-2}a_j\right\} \\
&\le \int\limits_{\Omega} u_i \left\{\mu a_{l-1} + 4(-\Delta)\sum_{j=0}^{l-2}\mu^{l-j-2}a_j\right\} \\
&\qquad \text{(from the Lemma 2.3)} \\
&= \int\limits_{\Omega} u_i \left\{\mu^2 a_{l-2} + 4\mu(-\Delta)\sum_{j=0}^{l-3}\eta^{l-j-3}a_j + 4(-\Delta)\sum_{j=0}^{l-2}\mu^{l-j-2}a_j\right\} \\
&\le \int\limits_{\Omega} u_i \left\{\mu^2 a_{l-2} + 4(-\Delta)\sum_{j=0}^{l-3}\mu^{l-j-2}a_j + 4(-\Delta)\sum_{j=0}^{l-2}\mu^{l-j-2}a_j\right\} \\
&\le \int\limits_{\Omega} u_i\mu^l a_0 + 4\int\limits_{\Omega} u_i(-\Delta)\left\{a_{l-2} + 2\mu a_{l-3} + \cdots + (l-1)\mu^{l-2}a_0\right\} \\
&\le \int\limits_{\Omega} u_i\mu^l a_0 \\
&\quad + 4\int\limits_{\Omega} u_i(-\Delta)\left\{2\mu a_{l-3} + 4\mu^2 a_{l-4} + \cdots + 2(l-2)\mu^{l-2}a_0 + (l-1)\mu^{l-2}a_0\right\} \\
&\qquad \text{(from the Lemma 2.5)} \\
&= \int\limits_{\Omega} u_i\mu^l a_0 + 8\int\limits_{\Omega} u_i(-\Delta)\mu\left\{a_{l-3} + 2\mu a_{l-4} + \cdots + (l-3)\mu^{l-4}a_1\right\} \\
&\quad + \int\limits_{\Omega} 8(l-2)u_i(-\Delta)\mu^{l-2}a_0 + \int\limits_{\Omega} 4(l-1)u_i(-\Delta)\mu^{l-2}a_0 \\
&\le \int\limits_{\Omega} u_i\mu^l a_0 + \left(\sum_{k=1}^{l-1}2^{l-k+1}k\right)\int\limits_{\Omega} u_i(-\Delta)\mu^{l-2}a_0 \\
&= \int\limits_{\Omega} u_i\mu^l a_0 + \left[2^{l+2} - 4(l+1)\right]\int\limits_{\Omega} u_i(-\Delta)\mu^{l-2}a_0
\end{aligned}
$$

$$
\begin{aligned}
&= \int_\Omega \sum_{k=0}^{l} \binom{l}{l-k} n^{l-k} u_i (-\Delta)^k a_0 \\
&\quad + [2^{l+2} - 4(l+1)] \int_\Omega \sum_{k=0}^{l-2} \binom{l-2}{l-k-2} n^{l-k-2} u_i (-\Delta)^{k+1} a_0 \\
&= \int_\Omega \sum_{k=0}^{l} \binom{l}{l-k} n^{l-k} u_i (-\Delta)^k u_i \\
&\quad + [2^{l+2} - 4(l+1)] \int_\Omega \sum_{k=0}^{l-2} \binom{l-2}{l-k-2} n^{l-k-2} u_i (-\Delta)^{k+1} u_i \\
&= \sum_{k=0}^{l} \binom{l}{l-k} n^{l-k} \int_\Omega |\nabla^k u_i|^2 \\
&\quad + (2^{l+2} - 4(l+1)) \sum_{k=0}^{l-2} \binom{l-2}{l-k-2} n^{l-k-2} \int_\Omega |\nabla^{k+1} u_i|^2 \\
&\le \sum_{k=0}^{l} \binom{l}{l-k} n^{l-k} \lambda_i^{\frac{k}{l}} + (2^{l+2} - 4(l+1)) \sum_{k=0}^{l-2} \binom{l-2}{l-k-2} n^{l-k-2} \lambda_i^{\frac{k+1}{l}} \\
&\qquad \text{(from (2.6))} \\
&= (\lambda_i^{\frac{1}{l}} + n)^l + [2^{l+2} - 4(l+1)] \lambda_i^{\frac{1}{l}} (\lambda_i^{\frac{1}{l}} + n)^{l-2}.
\end{aligned}
$$

In view of the formula (2.13), the Proposition 2.2 is proved.

From now we can prove our theorem by the Rayleigh–Ritz inequality. In order to prove our results, we need to introduce k free constants. For the Laplacian and the second-order Laplacian one free constant is introduced, but this is not sufficient in our case.

Proof We define ϕ_i^α and r_{ij} for $i, j = 1, 2, \ldots, k$, by

$$
\begin{aligned}
\phi_i^\alpha &= u_i x_\alpha - \sum_{j=1}^{k} r_{ij} u_j, \\
r_{ij} &= \int_\Omega u_i u_j x_\alpha.
\end{aligned}
\tag{2.14}
$$

Then, from (2.1) and (2.14), we have

$$
\begin{aligned}
&\phi_i^\alpha|_{\partial\Omega} = \frac{\partial \phi_i^\alpha}{\partial \nu}|_{\partial\Omega} = \cdots = \frac{\partial^{l-1} \phi_i^\alpha}{\partial \nu^{l-1}}|_{\partial\Omega} = 0, \\
&\int_\Omega u_j \phi_i^\alpha = 0.
\end{aligned}
$$

It follows from the Rayleigh–Ritz inequality that

$$\lambda_{k+1} \leq \frac{\int_\Omega \phi_i^\alpha (-\Delta)^l \phi_i^\alpha}{\int_\Omega (\phi_i^\alpha)^2}. \tag{2.15}$$

We have

$$\begin{aligned}
\int_\Omega \phi_i^\alpha (-\Delta)^l \phi_i^\alpha &= \int_\Omega \phi_i^\alpha (-\Delta)^l (u_i x_\alpha) \\
&= \int_\Omega \phi_i^\alpha \{((-\Delta)^l (u_i x_\alpha) - x_\alpha (-\Delta)^l u_i) + x_\alpha (-\Delta)^l u_i\} \\
&= \lambda_i \int_\Omega \phi_i^\alpha (u_i x_\alpha) + \int_\Omega \phi_i^\alpha p_i \\
&= \lambda_i \int_\Omega \phi_i^\alpha \left(u_i x_\alpha - \sum_{j=1}^k r_{ij} u_j \right) + \int_\Omega \phi_i^\alpha p_i \\
&= \lambda_i \|\phi_i^\alpha\|^2 + \int_\Omega \left(u_i x_\alpha - \sum_{j=1}^k r_{ij} u_j \right) p_i \\
&= \lambda_i \|\phi_i^\alpha\|^2 + \int_\Omega u_i x_\alpha p_i - \sum_{j=1}^k r_{ij} s_{ij},
\end{aligned}$$

where

$$\begin{aligned}
\|\phi_i^\alpha\|^2 &= \int_\Omega |\phi_i^\alpha|^2, \\
p_i &= (-\Delta)^l (u_i x_\alpha) - x_\alpha (-\Delta)^l u_i, \\
s_{ij} = \int_\Omega p_i u_j &= \int_\Omega \{(-\Delta)^l (u_i x_\alpha) - x_\alpha (-\Delta)^l u_i\} u_j \\
&= \int_\Omega \{u_i x_\alpha (-\Delta)^l u_j - u_j x_\alpha (-\Delta)^l u_i\} \\
&= (\lambda_j - \lambda_i) \int_\Omega x_\alpha u_i u_j \\
&= (\lambda_j - \lambda_i) r_{ij} \\
&= -s_{ji}.
\end{aligned} \tag{2.16}$$

Thus, from (2.15), we obtain

$$(\lambda_{k+1} - \lambda_i) \|\phi_i^\alpha\|^2 \leq \int_\Omega \phi_i^\alpha p_i$$

$$= \int_\Omega u_i x_\alpha p_i - \sum_{j=1}^k r_{ij} s_{ij}$$
$$= \int_\Omega u_i x_\alpha p_i + \sum_{j=1}^k (\lambda_i - \lambda_j) r_{ij}^2. \tag{2.17}$$

On the other hand, putting

$$t_{ij} = \int_\Omega u_j \left(\nabla u_i \cdot \nabla x_\alpha + \frac{u_i \Delta x_\alpha}{2} \right),$$

t_{ij} satisfies

$$\begin{aligned} t_{ij} &= -\int_\Omega \nabla u_j \cdot \nabla x_\alpha u_i - \int_\Omega u_i u_j \Delta x_\alpha + \int_\Omega \frac{u_i u_j \Delta x_\alpha}{2} \\ &= -\int_\Omega u_i \left(\nabla u_j \cdot \nabla x_\alpha + \frac{u_j \Delta x_\alpha}{2} \right) \\ &= -t_{ji}. \end{aligned}$$

Thus, for any constant $\delta_i > 0$, we have

$$\begin{aligned} -2\int_\Omega \phi_i^\alpha \left(\nabla u_i \cdot \nabla x_\alpha + \frac{u_i \Delta x_\alpha}{2} \right) &= -2\int_\Omega \phi_i^\alpha \left\{ \left(\nabla u_i \cdot \nabla x_\alpha + \frac{u_i \Delta x_\alpha}{2} \right) - \sum_{j=1}^k t_{ij} u_j \right\} \\ &\le \delta_i \|\phi_i^\alpha\|^2 + \frac{1}{\delta_i} \left\| \nabla u_i \cdot \nabla x_\alpha + \frac{u_i \Delta x_\alpha}{2} - \sum_{j=1}^k t_{ij} u_j \right\|^2 \\ &\le \delta_i \|\phi_i^\alpha\|^2 + \frac{1}{\delta_i} \left(\left\| \nabla u_i \cdot \nabla x_\alpha + \frac{u_i \Delta x_\alpha}{2} \right\|^2 - \sum_{j=1}^k t_{ij}^2 \right). \end{aligned}$$

Putting $\delta_i = \delta_i'(\lambda_{k+1} - \lambda_i)$, where δ_i' is a positive constant, we infer from (2.17),

$$\begin{aligned} &-2(\lambda_{k+1} - \lambda_i)^2 \int_\Omega \phi_i^\alpha \left\{ \left(\nabla u_i \cdot \nabla x_\alpha + \frac{u_i \Delta x_\alpha}{2} \right) - \sum_{j=1}^k t_{ij} u_j \right\} \\ &\le \delta_i'(\lambda_{k+1} - \lambda_i)^3 \|\phi_i^\alpha\|^2 + \frac{\lambda_{k+1} - \lambda_i}{\delta_i'} \left(\left\| \nabla u_i \cdot \nabla x_\alpha + \frac{u_i \Delta x_\alpha}{2} \right\|^2 - \sum_{j=i}^k t_{ij}^2 \right) \\ &\le \delta_i'(\lambda_{k+1} - \lambda_i)^2 \left(\int_\Omega u_i x_\alpha p_i + \sum_{j=1}^k (\lambda_i - \lambda_j) r_{ij}^2 \right) \\ &\quad + \frac{\lambda_{k+1} - \lambda_i}{\delta_i'} \left(\left\| \nabla u_i \cdot \nabla x_\alpha + \frac{u_i \Delta x_\alpha}{2} \right\|^2 - \sum_{j=i}^k t_{ij}^2 \right) \end{aligned}$$

Taking the sum on i from 1 to k, we have

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_{\Omega}(-2u_ix_\alpha\nabla u_i\cdot\nabla x_\alpha-u_i^2x_\alpha\Delta x_\alpha)+2\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)^2r_{ij}t_{ij}\\&\quad\le\sum_{i=1}^{k}\delta_i'(\lambda_{k+1}-\lambda_i)^2\int_{\Omega}u_ix_\alpha p_i+\sum_{i=1}^{k}\frac{1}{\delta_i'}(\lambda_{k+1}-\lambda_i)\left\|\nabla u_i\cdot\nabla x_\alpha+\frac{u_i\Delta x_\alpha}{2}\right\|^2\\&\quad+\sum_{i,j=1}^{k}\delta_i'(\lambda_{k+1}-\lambda_i)^2(\lambda_i-\lambda_j)r_{ij}^2-\sum_{i,j=1}^{k}\frac{1}{\delta_i'}(\lambda_{k+1}-\lambda_i)t_{ij}^2.\end{aligned}\tag{2.18}$$

Since $r_{ij}=r_{ji}$, $t_{ij}=-t_{ji}$ and $(\lambda_i-\lambda_j)r_{ij}=-s_{ij}$, we have

$$2\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)^2r_{ij}t_{ij}=-2\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)r_{ij}t_{ij}.\tag{2.19}$$

Defining

$$\delta_i'=\frac{\delta}{(\lambda_i^{\frac{1}{l}}+n)^l-\lambda_i+(2^{l+2}-4(l+1))\lambda_i^{\frac{1}{l}}(\lambda_i^{\frac{1}{l}}+n)^{l-2}},\tag{2.20}$$

where $\delta>0$ is any constant, we derive

$$\begin{aligned}&\sum_{i,j=1}^{k}\delta_i'(\lambda_{k+1}-\lambda_i)^2(\lambda_i-\lambda_j)r_{ij}^2+\sum_{i,j=1}^{k}\delta_i'(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2r_{ij}^2\\&\quad=\sum_{i,j=1}^{k}\delta_i'(\lambda_{k+1}-\lambda_i)(\lambda_{k+1}-\lambda_j)(\lambda_i-\lambda_j)r_{ij}^2\\&\quad=\frac{1}{2}\sum_{i,j=1}^{k}(\delta_i'-\delta_j')(\lambda_{k+1}-\lambda_i)(\lambda_{k+1}-\lambda_j)(\lambda_i-\lambda_j)r_{ij}^2\le 0\end{aligned}\tag{2.21}$$

since δ_i' is a monotone decreasing function of $\lambda_i^{\frac{1}{l}}$. It is obvious that

$$\begin{aligned}2\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)r_{ij}t_{ij}&\le\sum_{i,j=1}^{k}\delta_i'(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2r_{ij}^2\\&\quad+\sum_{i,j=1}^{k}\frac{1}{\delta_i'}(\lambda_{k+1}-\lambda_i)t_{ij}^2.\end{aligned}\tag{2.22}$$

We infer, from (2.18), (2.19), (2.21) and (2.22)

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_{\Omega}(-2u_ix_\alpha\nabla u_i\cdot\nabla x_\alpha-u_i^2x_\alpha\Delta x_\alpha)\\&\quad\le\sum_{i=1}^{k}\delta_i'(\lambda_{k+1}-\lambda_i)^2\int_{\Omega}u_ix_\alpha p_i+\sum_{i=1}^{k}\frac{1}{\delta_i'}(\lambda_{k+1}-\lambda_i)\left\|\nabla u_i\cdot\nabla x_\alpha+\frac{u_i\Delta x_\alpha}{2}\right\|^2.\end{aligned}\tag{2.23}$$

From (2.2) and $\sum_{\alpha=1}^{n+1} x_\alpha^2 = 1$, we have

$$\sum_{\alpha=1}^{n+1} \int_\Omega (-2u_i x_\alpha \nabla u_i \cdot \nabla x_\alpha - u_i^2 x_\alpha \Delta x_\alpha) = -\int_\Omega u_i \nabla u_i \nabla \left(\sum_{\alpha=1}^{n+1} x_\alpha^2 \right) + n \int_\Omega u_i^2 \sum_{\alpha=1}^{n+1} x_\alpha^2 = n \tag{2.24}$$

and

$$\begin{aligned} \sum_{\alpha=1}^{n+1} \left\| \nabla u_i \cdot \nabla x_\alpha + \frac{u_i \Delta x_\alpha}{2} \right\|^2 &= \sum_{\alpha=1}^{n+1} \left\| \nabla u_i \cdot \nabla x_\alpha - \frac{n}{2} u_i x_\alpha \right\|^2 \\ &= \int_\Omega \sum_{\alpha=1}^{n+1} (\nabla u_i \cdot \nabla x_\alpha)^2 - \frac{n}{2} \int_\Omega u_i \nabla u_i \nabla \left(\sum_{\alpha=1}^{n+1} x_\alpha^2 \right) \\ &\quad + \frac{n^2}{4} \int_\Omega u_i^2 \sum_{\alpha=1}^{n+1} x_\alpha^2 = \|\nabla u_i\|^2 + \frac{n^2}{4} \le \lambda_i^{\frac{1}{l}} + \frac{n^2}{4}. \end{aligned} \tag{2.25}$$

Taking the sum on α from 1 to $n+1$ for (2.23) and making use of (2.24) and (2.25), we obtain

$$n \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \le \sum_{i=1}^k \delta_i' (\lambda_{k+1} - \lambda_i)^2 \sum_{\alpha=1}^{n+1} \int_\Omega u_i x_\alpha p_i + \sum_{i=1}^k \frac{1}{\delta_i'} (\lambda_{k+1} - \lambda_j) \left(\lambda_i^{\frac{1}{l}} + \frac{n^2}{4} \right).$$

Therefore, the Proposition 2.2 implies

$$\begin{aligned} &n \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \\ &\le \sum_{i=1}^k \delta_i' (\lambda_{k+1} - \lambda_i)^2 \left\{ (\lambda_i^{\frac{1}{l}} + n)^l - \lambda_i + [2^{l+2} - 4(l+1)] \lambda_i^{\frac{1}{l}} (\lambda_i^{\frac{1}{l}} + n)^{l-2} \right\} \\ &\quad + \sum_{i=1}^k \frac{1}{\delta_i'} (\lambda_{k+1} - \lambda_i) \left(\lambda_i^{\frac{1}{l}} + \frac{n^2}{4} \right). \end{aligned}$$

According to (2.20), we infer

$$\begin{aligned} n \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 &\le \delta \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \\ &\quad + \frac{1}{\delta} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \left\{ (\lambda_i^{\frac{1}{l}} + n)^l - \lambda_i + [2^{l+2} - 4(l+1)] \lambda_i^{\frac{1}{l}} (\lambda_i^{\frac{1}{l}} + n)^{l-2} \right\} \left(\lambda_i^{\frac{1}{l}} + \frac{n^2}{4} \right). \end{aligned}$$

Putting

$$\delta = \left\{ \frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \left\{(\lambda_i^{\frac{1}{l}}+n)^l - \lambda_i + [2^{l+2}-4(l+1)]\lambda_i^{\frac{1}{l}}\left(\lambda_i^{\frac{1}{l}}+n\right)^{l-2}\right\}(\lambda_i^{\frac{1}{l}}+\frac{n^2}{4})}{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2} \right\}^{\frac{1}{2}},$$

we obtain

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4}{n^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left\{(\lambda_i^{\frac{1}{l}}+n)^l - \lambda_i + [2^{l+2}-4(l+1)]\lambda_i^{\frac{1}{l}}(\lambda_i^{\frac{1}{l}}+n)^{l-2}\right\}\left(\lambda_i^{\frac{1}{l}}+\frac{n^2}{4}\right).$$

It completes the proof of theorem. □

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