ESTIMATES FOR EIGENVALUES OF A CLAMPED PLATE PROBLEM ON RIEMANNIAN MANIFOLDS*

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Abstract. In this paper we study eigenvalues of a clamped plate problem on a bounded domain in an n-dimensional complete Riemannian manifold. By making use of Nash's theorem and introducing k free constants, we derive a universal bound for eigenvalues, which solves a problem proposed by Wang and Xia [16].

1. introduction

Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. The following is called a Dirichlet eigenvalue problem of Laplacian:

$$\begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ is the Laplacian on M. This eigenvalue problem has a real and discrete spectrum:

$$0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_k \le \cdots \longrightarrow \infty,$$

where each eigenvalue is repeated according to its multiplicity.

When M is a Euclidean space $\mathbf{R}^n$, namely, when Ω is a bounded domain in $\mathbf{R}^n$, Payne, Pólya and Weinberger [15] proved

$$\lambda_{k+1} - \lambda_k \le \frac{4}{kn} \sum_{i=1}^{k} \lambda_i. \tag{1.2}$$

Hile and Protter [11] generalized the above result to

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \ge \frac{kn}{4}. \tag{1.3}$$

In 1991, a much sharper inequality was obtained by Yang [17] (cf. [7]):

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \le \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\lambda_i, \tag{1.4}$$

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which is called Yang's first inequality (see [1] and [2]). According to the inequality, one can infer

$$\lambda_{k+1} \leq \frac{1}{k}(1+\frac{4}{n})\sum_{i=1}^{k}\lambda_i, \tag{1.5}$$

which is called Yang's second inequality.

For the Dirichlet eigenvalue problem on a complete Riemannian manifold M, Chen and Cheng [3] and El Soufi, Harrell and Ilias [9] have proved, independently,

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i+\frac{n^2}{4}H_0^2), \tag{1.6}$$

where H_0^2 is an nonnegative constant which only depends on M and Ω. When M is the unit sphere, the above inequality is best possible, which has been obtained in [5]. In particular, when M is an n-dimensional hypersurface in $\mathbf{R}^{n+1}$, Harrell [10] has also proved the above inequality.

On the other hand, we consider an eigenvalue problem of the biharmonic operator Δ^2 on a bounded domain in an n-dimensional complete Riemannian manifold M, which is also called a clamped plate problem:

$$\begin{cases} \Delta^2 u = \Gamma u & \text{in } \Omega \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where Δ^2 denotes the biharmonic operator on M, and ν is the outward unit normal of $\partial\Omega$.

When Ω is a bounded domain in $\mathbf{R}^n$, Payne, Pólya and Weinberger [15] proved

$$\Gamma_{k+1}-\Gamma_k \leq \frac{8(n+2)}{n^2 k}\sum_{i=1}^{k}\Gamma_i. \tag{1.8}$$

Chen and Qian [4] and Hook [12], independently, extended the above inequality to

$$\frac{n^2k^2}{8(n+2)} \leq \sum_{i=1}^{k}\frac{\Gamma_i^{\frac{1}{2}}}{\Gamma_{k+1}-\Gamma_i}\sum_{i=1}^{k}\Gamma_i^{\frac{1}{2}} \quad (\text{ cf. } [13]). \tag{1.9}$$

Recently, answering a question of Ashbaugh [1], Cheng and Yang [6] have proved the following remarkable estimate:

$$\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i) \leq (\frac{8(n+2)}{n^2})^{\frac{1}{2}}\sum_{i=1}^{k}(\Gamma_i(\Gamma_{k+1}-\Gamma_i))^{\frac{1}{2}}, \tag{1.10}$$

which is analogous to Yang's first inequality.

In 2007, Wang and Xia ([16] p. 336) have proposed that for what kind of M, there exists a universal bound on the $(k+1)^{\text{th}}$ eigenvalue in terms of the first k eigenvalues of (1.7). When M is either a complete minimal submanifold in a Euclidean space or the unit sphere, Wang and Xia [16] have solved this problem. Namely, they have proved the following: when M is an n-dimensional minimal submanifold in a Euclidean space,

$$\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le \frac{8(n+2)}{n^2}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\Gamma_i \tag{1.11}$$

and when M is an n-dimensional unit sphere,

$$\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le \frac{1}{n^2}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)(n^2+(2n+4)\Gamma_i^{\frac{1}{2}})(n^2+4\Gamma_i^{\frac{1}{2}}). \tag{1.12}$$

have been proved.

When M is a hyperbolic space $H^n(-1)$, Cheng and Yang [8] have also solved this problem, that is, they have proved

$$\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le 24\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left\{\Gamma_i^{\frac{1}{2}}-\frac{(n-1)^2}{4}\right\}\left\{\Gamma_j^{\frac{1}{2}}-\frac{(n-1)^2}{6}\right\}. \tag{1.13}$$

In this paper, our purpose is to solve the problem proposed by Wang and Xia, completely. We derive that, for any complete Riemannian manifold M, there exists a universal bound on the $(k+1)^{\text{th}}$ eigenvalue in terms of the first k eigenvalues of (1.7). In order to prove our result, we making use of Nash's theorem [14] to construct trial functions and introduce k free constants to deal with the undesired terms.

Theorem. *Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Assume that Γ_i is the i^{th} eigenvalue of the clamped plate problem* (1.7)*. Then, there exists a constant H_0, which only depends on M and Ω such that*

$$\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le \frac{1}{n^2}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(n^2H_0^2+(2n+4)\Gamma_i^{\frac{1}{2}}\right)\left(n^2H_0^2+4\Gamma_i^{\frac{1}{2}}\right). \tag{1.14}$$

holds.

Remark 1. *For a complete minimal submanifold M in a Euclidean space, we can infer $H_0=0$. For an n-dimensional unit sphere $M=S^n(1)$, which can be considered as a hypersurface in $\mathbf{R}^{n+1}$ with the mean curvature $H=1$, we have $H_0=1$. Hence, the results of Wang and Xia* [16] *are simple consequences of our result.*

When M is the unit sphere $S^n(1)$ and Ω tends to $S^n(1)$, we know that Γ_1 tends to zero and Γ_i, for $i=2,\cdots,n+1$, tends to n^2. Therefore, for $k=1,2,\cdots,n$, our inequality (1.13) *becomes equality.*

Since our inequality (1.13) is a quadratic inequality of Γ_{k+1}, it is not difficult to derive an upper bound on Γ_{k+1} according to the first k eigenvalues and H_0^2.

Corollary 1. *Under the assumptions of the theorem, we have*

$$\Gamma_{k+1} \le A_k + \sqrt{A_k^2 - B_k}, \tag{1.15}$$

where

$$A_k = \frac{1}{k}\left\{\sum_{i=1}^{k}\Gamma_i + \frac{1}{2n^2}\sum_{i=1}^{k}\left(n^2H_0^2 + (2n+4)\Gamma_i^{\frac{1}{2}}\right)\left(n^2H_0^2 + 4\Gamma_i^{\frac{1}{2}}\right)\right\}$$

and

$$B_k = \frac{1}{k}\left\{\sum_{i=1}^{k}\Gamma_i^2 + \frac{1}{n^2}\sum_{i=1}^{k}\Gamma_i\left(n^2H_0^2 + (2n+4)\Gamma_i^{\frac{1}{2}}\right)\left(n^2H_0^2 + 4\Gamma_i^{\frac{1}{2}}\right)\right\}.$$

Since k is an any integer, we know that (1.13) also holds if we replace $k+1$ with k, that is, we have

$$\sum_{i=1}^{k-1}(\Gamma_k - \Gamma_i)^2 \le \frac{1}{n^2}\sum_{i=1}^{k-1}(\Gamma_k - \Gamma_i)\left(n^2H_0^2 + (2n+4)\Gamma_i^{\frac{1}{2}}\right)\left(n^2H_0^2 + 4\Gamma_i^{\frac{1}{2}}\right).$$

Therefore, we infer

$$\sum_{i=1}^{k}(\Gamma_k - \Gamma_i)^2 \le \frac{1}{n^2}\sum_{i=1}^{k}(\Gamma_k - \Gamma_i)\left(n^2H_0^2 + (2n+4)\Gamma_i^{\frac{1}{2}}\right)\left(n^2H_0^2 + 4\Gamma_i^{\frac{1}{2}}\right).$$

Namely, Γ_k also satisfies the same quadratic inequality. We derive

$$\Gamma_k \ge A_k - \sqrt{A_k^2 - B_k}.$$

Thus, we can obtain an estimate on $\Gamma_{k+1} - \Gamma_k$ as following:

Corollary 2. *Under the assumptions of the theorem, we have*

$$\Gamma_{k+1} - \Gamma_k \le 2\sqrt{A_k^2 - B_k}, \tag{1.16}$$

where A_k and B_k are given in the corollary 1.

2. Proof of Theorem

In order to prove our theorem, the following Nash's theorem plays an important role.

Nash's theorem ([14]). *Each complete Riemannian manifold M can be isometrically immersed into a Euclidean space $\mathbf{R}^N$.*

Let M be an n-dimensional isometrically immersed submanifold in $\mathbf{R}^N$. For an arbitrary point $p \in M$, let $(x^1, \cdots, x^n)$ be an arbitrary coordinate system in a neighberfood U of $p \in M$. Let y be the position vector of $p \in M$ which is defined by

$$y = (y^1(x^1, \cdots, x^n), \cdots, y^N(x^1, \cdots, x^n)).$$

Since M is isometrically immersed in $\mathbf{R}^N$,

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left\langle\sum_{\alpha=1}^{N}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha}, \sum_{\beta=1}^{N}\frac{\partial y^\beta}{\partial x^j}\frac{\partial}{\partial y^\beta}\right\rangle = \sum_{\alpha=1}^{N}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\alpha}{\partial x^j}, \tag{2.1}$$

where g denotes the induced metric of M from $\mathbf{R}^N$ and $\langle\ ,\ \rangle$ is the standard inner product in $\mathbf{R}^N$. The following lemma of [3] is necessary for proving our theorem. For reader's convenience, we will give its proof.

Lemma (Chen and Cheng [3]). For any function $u \in C^\infty(M)$,

$$\sum_{\alpha=1}^{N}\Big(g(\nabla y^\alpha, \nabla u)\Big)^2 = |\nabla u|^2, \tag{2.2}$$

$$\sum_{\alpha=1}^{N} g(\nabla y^\alpha, \nabla y^\alpha) = \sum_{\alpha=1}^{N} |\nabla y^\alpha|^2 = n, \tag{2.3}$$

$$\sum_{\alpha=1}^{N} (\Delta y^\alpha)^2 = n^2|H|^2, \tag{2.4}$$

$$\sum_{\alpha=1}^{N} \Delta y^\alpha \nabla y^\alpha = 0, \tag{2.5}$$

where ∇ denotes the gradient operator on M, and $|H|$ is the mean curvature of M.

Proof. For any point p, we define $\overline{y} = (\overline{y}^1, \cdots, \overline{y}^N)$ by $y - y(p) = \overline{y}A$ such that $(\frac{\partial}{\partial \overline{y}^1})_p, \cdots, (\frac{\partial}{\partial \overline{y}^n})_p$ span T_pM and $g(\frac{\partial}{\partial \overline{y}^i}, \frac{\partial}{\partial \overline{y}^j}) = \delta_{ij}$, where $A = (a^\alpha_\beta) \in O(N)$ is an orthogonal matrix. For any function $u \in C^\infty(M)$, at p,

$$\begin{aligned}
\sum_{\alpha=1}^{N}\Big(g(\nabla y^\alpha, \nabla u)\Big)^2 &= \sum_{\alpha=1}^{N}\Big[g(\nabla(y^\alpha(p) + \sum_{\beta=1} a^\alpha_\beta \overline{y}^\beta), \nabla u)\Big]^2 \\
&= \sum_{\alpha=1}^{N}\Big[g(\nabla \sum_{\beta=1}^{N} a^\alpha_\beta \overline{y}^\beta, \nabla u)\Big]^2 \\
&= \sum_{\alpha=1}^{N}\Big(\sum_{\beta=1}^{N}\sum_{i=1}^{n} a^\alpha_\beta \frac{\partial \overline{y}^\beta}{\partial \overline{y}^i}\frac{\partial u}{\partial \overline{y}^i}\Big)^2 \\
&= \sum_{i=1}^{n}\sum_{\alpha=1}^{N} \frac{\partial \overline{y}^\alpha}{\partial \overline{y}^i}\frac{\partial \overline{y}^\alpha}{\partial \overline{y}^i}\frac{\partial u}{\partial \overline{y}^i}\frac{\partial u}{\partial \overline{y}^i} \\
&= |\nabla u|^2.
\end{aligned} \tag{2.6}$$

This completes the proof of (2.2).

By definition,

$$\sum_{\alpha=1}^{N} g(\nabla y^\alpha, \nabla y^\alpha) = \sum_{\alpha=1}^{N}\sum_{i,j}^{n} \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\alpha}{\partial x^j} g^{ij} = \sum_{i,j}^{n} g_{ij}g^{ij} = n. \tag{2.7}$$

Since y is the position vector of M, we have

$$\Delta y = nH. \tag{2.8}$$

Thus we can derive

$$\sum_{\alpha=1}^{N}(\Delta y^\alpha)^2 = n^2|H|^2. \tag{2.9}$$

Since ∇y is tangent to M, we have

$$\sum_{\alpha=1}^{N}\Delta y^\alpha \nabla_i y^\alpha = 0. \tag{2.10}$$

Therefore,

$$\sum_{\alpha=1}^{N}\Delta y^\alpha \nabla y^\alpha = 0. \tag{2.11}$$

$\square$

Proof of Theorem. Since M is a complete Riemannian manifold, Nash's theorem implies that there exists an isometric immersion from M into a Euclidean space $\mathbf{R}^N$. Thus, M can be considered as an n-dimensional complete isometrically immersed submanifold in $\mathbf{R}^N$. We denote, by $y = (y^\alpha)$, the position vector of M in $\mathbf{R}^N$. Let u_i be an eigenfunction corresponding to the eigenvalue Γ_i such that

$$\begin{cases} \Delta^2 u_i = \Gamma_i u_i & \text{in } \Omega \\ u_i = \dfrac{\partial u_i}{\partial \nu} = 0 & \text{on } \partial\Omega \\ \displaystyle\int_\Omega u_i u_j = \delta_{ij} \ (i, j = 1, 2 \cdots). \end{cases} \tag{2.12}$$

For $i = 1, \cdots, k$, and $\alpha = 1, \cdots, N$, we define

$$\phi_i^\alpha := y^\alpha u_i - \sum_{j=1}^{k} r_{ij}^\alpha u_j, \tag{2.13}$$

where $r_{ij}^\alpha = \displaystyle\int_\Omega y^\alpha u_i u_j$. By a simple calculation, we obtain

$$\int_\Omega u_j \phi_i^\alpha = 0, \quad i, j = 1, \cdots, k. \tag{2.14}$$

From the Rayleigh-Ritz inequality, we have

$$\Gamma_{k+1} \le \frac{\displaystyle\int_\Omega \phi_i^\alpha \Delta^2 \phi_i^\alpha}{\displaystyle\int_\Omega (\phi_i^\alpha)^2}, \quad 1 \le i \le k. \tag{2.15}$$

Since

$$
\begin{aligned}
\int_\Omega \phi_i^\alpha \Delta^2 \phi_i^\alpha &= \int_\Omega \phi_i^\alpha \Delta^2 (y^\alpha u_i - \sum_{j=1}^k r_{ij}^\alpha u_j) \\
&= \int_\Omega \phi_i^\alpha \Big\{ \Delta^2 y^\alpha \cdot u_i + 2\nabla(\Delta y^\alpha)\cdot \nabla u_i + 2\Delta y^\alpha \Delta u_i \\
&\quad + 2\Delta(\nabla y^\alpha \cdot \nabla u_i) + 2\nabla y^\alpha \cdot \nabla(\Delta u_i) + \Gamma_i y^\alpha u_i \Big\},
\end{aligned} \tag{2.16}
$$

we infer from (2.14), (2.15) and (2.16)

$$
(\Gamma_{k+1} - \Gamma_i)||\phi_i^\alpha||^2 \leq \int_\Omega p_i^\alpha \phi_i^\alpha = \omega_i^\alpha - \sum_{j=1}^k r_{ij}^\alpha s_{ij}^\alpha, \tag{2.17}
$$

where

$$
p_i^\alpha = \Delta^2 y^\alpha \cdot u_i + 2\nabla(\Delta y^\alpha)\cdot \nabla u_i + 2\Delta y^\alpha \Delta u_i + 2\Delta(\nabla y^\alpha \cdot \nabla u_i) + 2\nabla y^\alpha \cdot \nabla(\Delta u_i),
$$

$$
s_{ij}^\alpha = \int_\Omega p_i^\alpha u_j, \qquad \omega_i^\alpha = \int_\Omega p_i^\alpha y^\alpha u_i.
$$

From Stokes' theorem, we infer

$$
2\int_\Omega y^\alpha u_i \nabla(\Delta y^\alpha)\cdot\nabla u_i = \int_\Omega \Big\{ 2u_i \Delta y^\alpha \nabla u_i \cdot \nabla y^\alpha + u_i^2 (\Delta y^\alpha)^2 - y^\alpha u_i^2 \Delta^2 y^\alpha \Big\},
$$

$$
2\int_\Omega y^\alpha u_i \Delta(\nabla y^\alpha \cdot \nabla u_i) = \int_\Omega \Big\{ 2u_i \Delta y^\alpha \nabla y^\alpha \cdot \nabla u_i + 4(\nabla y^\alpha \cdot \nabla u_i)^2 + 2y^\alpha \Delta u_i \nabla y^\alpha \cdot \nabla u_i \Big\},
$$

$$
2\int_\Omega y^\alpha u_i \nabla y^\alpha \cdot \nabla(\Delta u_i) = -2\int_\Omega \Big(|\nabla y^\alpha|^2 u_i \Delta u_i + y^\alpha \Delta u_i \nabla y^\alpha \cdot \nabla u_i + y^\alpha \Delta y^\alpha u_i \Delta u_i \Big).
$$

Thus, we obtain

$$
\omega_i^\alpha = \int_\Omega \Big\{ (\Delta y^\alpha)^2 u_i^2 + 4(\nabla y^\alpha \cdot \nabla u_i)^2 - 2|\nabla y^\alpha|^2 u_i \Delta u_i + 4u_i \Delta y^\alpha \nabla y^\alpha \cdot \nabla u_i \Big\}. \tag{2.18}
$$

Since

$$
\begin{aligned}
&2\int_\Omega \Delta u_j \nabla y^\alpha \cdot \nabla u_i - \Delta u_i \nabla y^\alpha \cdot \nabla u_j \\
&= (\Gamma_j - \Gamma_i) r_{ij}^\alpha - \int_\Omega u_i \Delta u_j \Delta y^\alpha + \int_\Omega u_j \Delta u_i \Delta y^\alpha,
\end{aligned}
$$

we can infer

$$
s_{ij}^\alpha = (\Gamma_j - \Gamma_i) r_{ij}^\alpha. \tag{2.19}
$$

Then (2.17) can be written as

$$(\Gamma_{k+1}-\Gamma_i)||\phi_i^\alpha||^2 \le \omega_i^\alpha + \sum_{j=1}^{k}(\Gamma_i-\Gamma_j)(r_{ij}^\alpha)^2. \tag{2.20}$$

On the other hand, defining

$$v_i^\alpha = -2\int_\Omega y^\alpha u_i(\nabla y^\alpha \cdot \nabla u_i + \frac{u_i \Delta y^\alpha}{2})$$

and

$$t_{ij}^\alpha = \int_\Omega u_j(\nabla y^\alpha \cdot \nabla u_i + \frac{u_i\Delta y^\alpha}{2}) = -t_{ji}^\alpha,$$

then it follows that

$$\int_\Omega -2\phi_i^\alpha(\nabla y^\alpha\cdot\nabla u_i + \frac{u_i\Delta y^\alpha}{2}) = v_i^\alpha + 2\sum_{j=1}^{k} r_{ij}^\alpha t_{ij}^\alpha. \tag{2.21}$$

Multiplying (2.21) by $(\Gamma_{k+1}-\Gamma_i)^2$, we obtain, from (2.20),

$$\begin{aligned}
&(\Gamma_{k+1}-\Gamma_i)^2(v_i^\alpha + 2\sum_{j=1}^{k} r_{ij}^\alpha t_{ij}^\alpha)\\
&= (\Gamma_{k+1}-\Gamma_i)^2\int_\Omega -2\phi_i^\alpha\Big\{(\nabla y^\alpha\cdot\nabla u_i + \frac{u_i\Delta y^\alpha}{2}) - \sum_{j=1}^{k} t_{ij}^\alpha u_j\Big\}\\
&\le \delta_i(\Gamma_{k+1}-\Gamma_i)^3||\phi_i^\alpha||^2 + \frac{\Gamma_{k+1}-\Gamma_i}{\delta_i}||\nabla y^\alpha\cdot\nabla u_i + \frac{u_i\Delta y^\alpha}{2} - \sum_{j=1}^{k} t_{ij}^\alpha u_j||^2\\
&= \delta_i(\Gamma_{k+1}-\Gamma_i)^3||\phi_i^\alpha||^2 + \frac{\Gamma_{k+1}-\Gamma_i}{\delta_i}\Big\{||\nabla y^\alpha\cdot\nabla u_i + \frac{u_i\Delta y^\alpha}{2}||^2 - \sum_{j=1}^{k}(t_{ij}^\alpha)^2\Big\}\\
&\le \delta_i(\Gamma_{k+1}-\Gamma_i)^2\Big\{\omega_i^\alpha + \sum_{j=1}^{k}(\Gamma_i-\Gamma_j)(r_{ij}^\alpha)^2\Big\}\\
&+ \frac{\Gamma_{k+1}-\Gamma_i}{\delta_i}\Big\{||\nabla y^\alpha\cdot\nabla u_i + \frac{u_i\Delta y^\alpha}{2}||^2 - \sum_{j=1}^{k}(t_{ij}^\alpha)^2\Big\},
\end{aligned} \tag{2.22}$$

where δ_i is a positive constant. By the Stokes' theorem and the Schwarz's inequality, we have

$$\int_\Omega |\nabla u_i|^2 \le \Gamma_i^{\frac{1}{2}}.$$

From (2.18) and the lemma, we have

$$\begin{aligned}\sum_{\alpha=1}^{N}\omega_i^\alpha &= n^2\int_\Omega |H|^2u_i^2 + (2n+4)\int_\Omega |\nabla u_i|^2 \\ &\le n^2\sup_\Omega |H|^2 + (2n+4)\Gamma_i^{\frac{1}{2}}\end{aligned} \tag{2.23}$$

and

$$\sum_{\alpha=1}^{N}||\nabla y^\alpha\cdot\nabla u_i + \frac{u_i\Delta y^\alpha}{2}||^2 \le \frac{1}{4}n^2\sup_\Omega |H|^2 + \Gamma_i^{\frac{1}{2}}. \tag{2.24}$$

By a simple calculation, we derive

$$\sum_{\alpha=1}^{N} v_i^\alpha = \sum_{\alpha=1}^{N}\int_\Omega |\nabla y^\alpha|^2u_i^2 = n. \tag{2.25}$$

Summing on i from 1 to k for (2.22), we have

$$\begin{aligned}&\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2v_i^\alpha - 2\sum_{i,j}^{k}(\Gamma_{k+1}-\Gamma_i)(\Gamma_i-\Gamma_j)r_{ij}^\alpha t_{ij}^\alpha \\ &\le \sum_{i=1}^{k}\delta_i(\Gamma_{k+1}-\Gamma_i)^2\omega_i^\alpha + \sum_{i=1}^{k}\frac{1}{\delta_i}(\Gamma_{k+1}-\Gamma_i)||\nabla y^\alpha\cdot\nabla u_i + \frac{u_i\Delta y^\alpha}{2}||^2 \\ &+ \sum_{i,j}^{k}\delta_i(\Gamma_{k+1}-\Gamma_i)^2(\Gamma_i-\Gamma_j)(r_{ij}^\alpha)^2 - \sum_{i,j}^{k}\frac{1}{\delta_i}(\Gamma_{k+1}-\Gamma_i)(t_{ij}^\alpha)^2.\end{aligned} \tag{2.26}$$

Putting

$$\delta_i = \frac{\delta}{n^2\sup_\Omega |H|^2 + (2n+4)\Gamma_i^{\frac{1}{2}}}, \ \delta \text{ is a positive constant,}$$

then, we have

$$\begin{aligned}&-\sum_{i,j}^{k}\delta_i(\Gamma_{k+1}-\Gamma_i)^2(\Gamma_i-\Gamma_j)(r_{ij}^\alpha)^2 - \sum_{i,j}^{k}\delta_i(\Gamma_{k+1}-\Gamma_i)(\Gamma_i-\Gamma_j)^2(r_{ij}^\alpha)^2 \\ &= -\sum_{i,j}^{k}\delta_i(\Gamma_{k+1}-\Gamma_i)(\Gamma_{k+1}-\Gamma_j)(\Gamma_i-\Gamma_j)(r_{ij}^\alpha)^2 \\ &= -\frac{1}{2}\sum_{i,j}^{k}(\Gamma_{k+1}-\Gamma_i)(\Gamma_{k+1}-\Gamma_j)(\Gamma_i-\Gamma_j)(\delta_i-\delta_j)(r_{ij}^\alpha)^2 \\ &\ge 0.\end{aligned} \tag{2.27}$$

It is clear that

$$
\begin{aligned}
&-\sum_{i,j}^{k}\delta_i(\Gamma_{k+1}-\Gamma_i)(\Gamma_i-\Gamma_j)^2(r_{ij}^{\alpha})^2-\sum_{i,j}^{k}\frac{1}{\delta_i}(\Gamma_{k+1}-\Gamma_i)(t_{ij}^{\alpha})^2\\
&\qquad\le -2\sum_{i,j}^{k}(\Gamma_{k+1}-\Gamma_i)(\Gamma_i-\Gamma_j)r_{ij}^{\alpha}t_{ij}^{\alpha}.
\end{aligned}\tag{2.28}
$$

Therefore, it follows that

$$
\begin{aligned}
&\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2v_i^{\alpha}\\
&\le\sum_{i=1}^{k}\delta_i(\Gamma_{k+1}-\Gamma_i)^2\omega_i^{\alpha}+\sum_{i=1}^{k}\frac{1}{\delta_i}(\Gamma_{k+1}-\Gamma_i)||\nabla y^{\alpha}\cdot\nabla u_i+\frac{u_i\Delta y^{\alpha}}{2}||^2.
\end{aligned}\tag{2.29}
$$

Summing on α from 1 to N for (2.29), we infer, from(2.23), (2.24) and (2.25),

$$
\begin{aligned}
&n\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\\
&\le\sum_{i=1}^{k}\delta_i(\Gamma_{k+1}-\Gamma_i)^2\left(n^2\sup_{\Omega}|H|^2+(2n+4)\Gamma_i^{\frac{1}{2}}\right)\\
&+\sum_{i=1}^{k}\frac{1}{\delta_i}(\Gamma_{k+1}-\Gamma_i)\left(\frac{1}{4}n^2\sup_{\Omega}|H|^2+\Gamma_i^{\frac{1}{2}}\right)\\
&=\delta\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\\
&+\frac{1}{\delta}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(\frac{1}{4}n^2\sup_{\Omega}|H|^2+\Gamma_i^{\frac{1}{2}}\right)\left(n^2\sup_{\Omega}|H|^2+(2n+4)\Gamma_i^{\frac{1}{2}}\right).
\end{aligned}\tag{2.30}
$$

Putting

$$
\delta=\left[\frac{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(\frac{1}{4}n^2\sup_{\Omega}|H|^2+\Gamma_i^{\frac{1}{2}}\right)\left(n^2\sup_{\Omega}|H|^2+(2n+4)\Gamma_i^{\frac{1}{2}}\right)}{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2}\right]^{\frac{1}{2}},
$$

we obtain

$$
\begin{aligned}
&\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \\
&\leq \frac{1}{n^2}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(n^2\sup_{\Omega}|H|^2+(2n+4)\Gamma_i^{\frac{1}{2}}\right)\left(n^2\sup_{\Omega}|H|^2+4\Gamma_i^{\frac{1}{2}}\right).
\end{aligned}
\tag{2.31}
$$

Since the spectrum of the clamped plate problem is an invariant of isometries, we know that the above inequality holds for any isometric immersion from M into a Euclidean space.

Now we define ψ as

$$\psi := \{\phi;\phi \text{ is an isometric immersion from } M \text{ into a Euclidean space}\}.$$

Putting

$$H_0^2 := \inf_{\phi\in\psi}\sup_{\Omega}|H|^2,$$

We infer (1.13). This completes the proof of the theorem.

□

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