International Journal of Mathematics

A CLASSIFICATION OF HYPERSURFACES WITH PARALLEL PARA-BLASCHKE TENSOR IN S^{m+1}

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In this paper, we classify all immersed hypersurfaces in the unit sphere S^{m+1} with parallel para-Blaschke tensor.

Keywords: Möbius form, Möbius metric, Blaschke tensor, Möbius second fundamental form, para-Blaschke tensor .

2001 *Mathematics Subject Classification*: Primary 53A30; Secondary 53B25

1. Introduction

Let $S^n(r)$ be an n-dimensional standard sphere of radius r, $S^n = S^n(1)$, $\mathbb{R}^n$ be an n-dimensional Euclidean space, and $H^n(c)$ be an n-dimensional hyperbolic space of constant curvature $c < 0$ defined by

$$H^n(c) = \{y = (y_0, y_1) \in \mathbb{R}^{n+1}_1;\ \langle y, y\rangle_1 = \frac{1}{c},\ y_0 > 0\},$$

where for any integer $N \geq 2$, $\mathbb{R}^N_1 \equiv \mathbb{R}_1 \times \mathbb{R}^{N-1}$ is the N-dimensional Lorentzian space with the standard Lorentzian inner product $\langle \cdot, \cdot\rangle_1$ given by

$$\langle y, y'\rangle_1 = -y_0 y'_0 + y_1 \cdot y'_1, \quad y = (y_0, y_1),\, y' = (y'_0, y'_1) \in \mathbb{R}^N_1,$$

where the dot "·" denotes the standard Euclidean inner product on $\mathbb{R}^{N-1}$. In the sequel, we write $H^n = H^n(-1)$.

Let S^n_+ be the hemisphere in S^n whose first coordinate is positive. Then, there are two conformal diffeomorphisms $\sigma : \mathbb{R}^n \to S^n \backslash \{(-1, 0)\}$ and $\tau : H^n \to S^n_+$

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defined by

$$\begin{aligned}\sigma(u) &= \left(\frac{1-|u|^2}{1+|u|^2}, \frac{2u}{1+|u|^2}\right), \quad u \in \mathbb{R}^n, \\ \tau(y) &= \left(\frac{1}{y_0}, \frac{y_1}{y_0}\right), \quad y = (y_0, y_1) \in H^n \subset \mathbb{R}^{n+1}_1.\end{aligned} \tag{1.1}$$

Let $x : M^m \to S^{m+p}$ be an immersed submanifold in S^{m+p} without umbilical points. Wang [27] introduced four basic Möbius invariants of x: the Möbius metric g, the Möbius form Φ, the Blaschke tensor A and the Möbius second fundamental form B, which will be given in section 2. These invariants is closely related to the research of Willmore hypersurfaces and of conformal differential geometry (see [1], [2] and [15]). In recent years, Möbius geometry on submanifolds in S^{m+p} has been studied by many authors and many interesting results are obtained (cf. [5], [6], [7], [9], [10], [11], [12], [13], [14], [16], [17], [20], [21], [22], and so on). A hypersurface without umbilical points is called Möbius isoparametric if the Möbius form vanishes and Möbius principal curvatures are constant. In [18], Li, Liu, Wang and Zhao has classified Möbius isoparametric hypersurfaces with two distinct Möbius principal curvatures. Furthermore, Hu and Li [8] have classified all immersed hypersurfaces in S^{m+1} with parallel Möbius second fundamental forms.

Recently, the second author and Zhang [24] have studied hypersurfaces without umbilical points in S^{m+1} with parallel Blaschke tensor and have given a classification of this kind of hypersurfaces. A hypersurface without umbilical points is called Blaschke isoparametric if eigenvalues of the Blaschke tensor are constant and the Möbius form vanishes. When $m \leq 4$ (see [25] and [26]), they have given a complete classification for Blaschke isoparametric hypersurfaces and for any dimension m, if the distinct Blaschke eigenvalues is two, then, they have also classified it in [25].

On the other hand, Li and Wang [19] gave a Möbius characterization of hypersurfaces without umbilical points in real space forms, and with constant mean curvature and constant scalar curvature under the following assumptions:

(1) the Möbius form vanishes identically;
(2) there are two functions λ, μ such that

$$A + \lambda B = -\mu g.$$

In this case, functions λ, μ are necessarily constant (for further developments, see [23]).

Define $D^\lambda = A + \lambda B$ for some real number λ. D^λ is called a para-Blaschke tensor. Zhong and Sun [28] have classified hypersurfaces without umbilical points if the Möbius form vanishes and the para-Blaschke tensor has exactly two distinct eigenvalues.

In this paper, we study the general case. We will give a complete classification of hypersurfaces without umbilical points if the para-Blaschke tensor is parallel.

Theorem 1.1. *Let* $x : M^m \to S^{m+1}$ $(m \geq 2)$ *be an immersed hypersurface without* *umbilical points. Suppose that, for some constant λ, the para-Blaschke tensor $D^\lambda = A + \lambda B$ of x is parallel.*

(1) *If the Möbius form vanishes identically, then we have*

  (i) *when the para-Blaschke tensor D^λ has only one distinct eigenvalue, M is locally Möbius equivalent to*

   (a) *an immersed hypersurface in S^{m+1} with constant scalar curvature and constant mean curvature, or*

   (b) *the image under σ of an immersed hypersurface in $\mathbb{R}^{m+1}$ with constant scalar curvature and constant mean curvature, or*

   (c) *the image under τ of an immersed hypersurface in H^{m+1} with constant scalar curvature and constant mean curvature;*

  (ii) *when the Möbius second fundamental form B is parallel, M is locally Möbius equivalent to*

   (d) *a standard torus $S^K(r) \times S^{m-K}(\sqrt{1-r^2})$ in S^{m+1} for some $r > 0$ and positive integer K, or*

   (e) *the image under σ of a standard cylinder $S^K(r) \times \mathbb{R}^{m-K}$ in $\mathbb{R}^{m+1}$ for some $r > 0$ and positive integer K, or*

   (f) *the image under τ of a standard cylinder $S^K(r) \times H^{m-K}(-\frac{1}{1+r^2})$ in H^{m+1} for some $r > 0$ and positive integer K; or*

   (g) *$CSS(p, q, r)$ for some constants p, q, r (see Example 3.1);*

  (iii) *when the para-Blaschke tensor D^λ has at least two distinct constant eigenvalues and the Möbius second fundamental form B is non-parallel, M is locally Möbius equivalent to*

   (h) *one of the immersed hypersurfaces as indicated in Example 3.2, or*

   (i) *one of the immersed hypersurfaces as indicated in Example 3.3.*

(2) *If the Möbius form does not vanish identically, then x has exactly two distinct principal curvatures with one of which being simple; Furthermore, $x(M^m)$ is foliated by a family of $m-1$-dimensional totally umbilical submanifolds of S^{m+1}.*

2. Preliminaries

Let $x : M^m \to S^{m+p}$ be an immersed submanifold without umbilical points, $n = m + p$. Denote by h the second fundamental form of x with components h^α_{ij} and $H = \frac{1}{m}\mathrm{tr}h$ the mean curvature vector field. Define

$$\rho = \left(\frac{m}{m-1}\left(|h|^2 - m|H|^2\right)\right)^{\frac{1}{2}}, \quad Y = \rho(1, x), \tag{2.1}$$

then $Y : M^m \to \mathbb{R}^{n+2}_1$ is an immersion of M^m into the Lorentzian space $\mathbb{R}^{n+2}_1$ and is called the Möbius position vector of x. The function ρ given by (2.1) is called the Möbius factor for the immersion x. Define

$$C^{n+1}_+ = \left\{Y = (Y_0, Y) \in \mathbb{R}_1 \times \mathbb{R}^{n+1};\ \langle Y, Y\rangle_1 = 0,\ Y_0 > 0\right\}.$$

If $O(n+1,1)$ is the Lorentzian group of all elements in $GL(n+2;\mathbb{R})$ preserving the standard Lorentzian inner product $\langle\cdot,\cdot\rangle_1$ on $\mathbb{R}^{n+2}_1$, then there exists a subgroup $O^+(n+1,1)$ of $O(n+1,1)$ given by

$$O^+(n+1,1) = \left\{T \in O(n+1,1);\ T(C^{n+1}_+) \subset C^{n+1}_+\right\}. \tag{2.2}$$

In [27], Wang proved the following theorem:

Theorem 2.1. *Two submanifolds $x, \tilde{x} : M^m \to S^{m+p}$ with Möbius position vectors $Y, \tilde{Y}$ respectively are Möbius equivalent if and only if there is a $T \in O^+(n+1,1)$ such that $\tilde{Y} = T(Y)$.*

By Theorem 2.1, the induced metric $g = \langle dY, dY\rangle_1 = \rho^2 dx\cdot dx$ on M^m from the Lorentzian inner product $\langle\cdot,\cdot\rangle_1$ is a Möbius invariant and is called the Möbius metric of x. Let Δ denote the Laplacian with respect to the Möbius metric g. Defining $N : M \to \mathbb{R}^{n+1}_1$ by

$$N = -\frac{1}{m}\Delta Y - \frac{1}{2m^2}\langle\Delta Y, \Delta Y\rangle_1 Y, \tag{2.3}$$

we can infer

$$\begin{aligned}&\langle\Delta Y, Y\rangle_1 = -m, \quad \langle\Delta Y, dY\rangle_1 = 0, \quad \langle\Delta Y, \Delta Y\rangle_1 = 1 + m^2\kappa,\\ &\langle Y, Y\rangle_1 = \langle N, N\rangle_1 = 0, \quad \langle Y, N\rangle_1 = 1,\end{aligned} \tag{2.4}$$

where $m(m-1)\kappa$ denotes the scalar curvature M with respect to the Möbius metric g.

Let $V \to M$ be the vector subbundle of the trivial Lorentzian bundle $M \times \mathbb{R}^{n+2}_1$ defined as the orthogonal complement of $\mathbb{R}Y \oplus \mathbb{R}N \oplus Y_*(TM)$ with respect to the Lorentzian product $\langle\cdot,\cdot\rangle_1$. One calls V Möbius normal bundle of the immersion x. Clearly we have the following vector bundle decomposition:

$$M \times \mathbb{R}^{n+2}_1 = \mathbb{R}Y \oplus \mathbb{R}N \oplus Y_*(TM) \oplus V. \tag{2.5}$$

Let $T^\perp M$ be the normal bundle of the immersion $x : M \to S^n$, then the mean curvature vector field H of x defines a bundle isomorphism $f : T^\perp M \to V$ by

$$f(e) = (H\cdot e, (H\cdot e)x + e), \quad \forall e \in T^\perp M. \tag{2.6}$$

It is easily seen that f preserves the inner products and connections on $T^\perp M$ and V. We make use of the following conventions on the ranges of indices throughout this article:

$$1 \le i, j, k, \cdots \le m, \quad m+1 \le \alpha, \beta, \gamma, \cdots \le n.$$

For any local orthonormal frame field $\{e_i\}$ with respect to the induced metric $dx\cdot dx$ with its dual frame field $\{\theta^i\}$ and any orthonormal normal frame field $\{e_\alpha\}$ of x, setting

$$E_i = \rho^{-1}e_i, \quad \omega^i = \rho\theta^i, \quad E_\alpha = f(e_\alpha), \tag{2.7}$$

$\{E_i\}$ is a local orthonormal frame field with respect to the Möbius metric g, $\{\omega^i\}$ is the dual frame field of $\{E_i\}$, and $\{E_\alpha\}$ is a local orthonormal frame field of the Möbius normal bundle $V \to M$.

$$\Phi = \sum \Phi_i^\alpha \omega^i E_\alpha, \quad A = \sum A_{ij}\omega^i\omega^j, \quad B = \sum B_{ij}^\alpha \omega^i\omega^j E_\alpha \tag{2.8}$$

are called Möbius form, Blaschke tensor and Möbius second fundamental form, respectively, where

$$\begin{aligned} \Phi_i^\alpha =& -\rho^{-2}\left(H_{,i}^\alpha + \sum (h_{ij}^\alpha - H^\alpha\delta_{ij})e_j(\log\rho)\right), \\ A_{ij} =& -\rho^{-2}\left(\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - \sum H^\alpha h_{ij}^\alpha\right) \\ &-\frac{1}{2}\rho^{-2}\left(|d\log\rho|^2 - 1 + |H|^2\right)\delta_{ij} \\ B_{ij}^\alpha =& \rho^{-1}\left(h_{ij}^\alpha - H^\alpha\delta_{ij}\right), \end{aligned} \tag{2.9}$$

in which the subscript "$,i$" and Hess_{ij} denote, respectively, components of covariant derivatives and Hessian with respect to the induced metric $dx \cdot dx$.

Denote by R_{ijkl} and R_{ij}, respectively, components of the Riemannian curvature tensor and the Ricci tensor with respect to the Möbius metric g, then we have

$$\begin{aligned} &\mathrm{tr}A = \frac{1}{2m}(1 + m^2\kappa), \\ &\mathrm{tr}B = \sum B_{ii}^\alpha E_\alpha = 0, \quad |B|^2 = \sum (B_{ij}^\alpha)^2 = \frac{m-1}{m}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} R_{ijkl} &= \sum (B_{il}^\alpha B_{jk}^\alpha - B_{ik}^\alpha B_{jl}^\alpha) + A_{il}\delta_{jk} - A_{ik}\delta_{jl} + A_{jk}\delta_{il} - A_{jl}\delta_{ik} \\ R_{ij} &= -\sum B_{ik}^\alpha B_{kj}^\alpha + \delta_{ij}\mathrm{tr}A + (m-2)A_{ij}. \end{aligned} \tag{2.11}$$

Let Φ_{ij}^α, A_{ijk}, B_{ijk}^α denote components of the covariant derivatives of Φ, A, B, respectively. We can infer

$$\begin{aligned} \Phi_{ij}^\alpha - \Phi_{ji}^\alpha =& \sum (B_{ik}^\alpha A_{kj} - B_{kj}^\alpha A_{ki}), \\ A_{ijk} - A_{ikj} =& \sum (B_{ik}^\alpha \Phi_j^\alpha - B_{ij}^\alpha \Phi_k^\alpha), \\ B_{ijk}^\alpha - B_{ikj}^\alpha =& \delta_{ij}\Phi_k^\alpha - \delta_{ik}\Phi_j^\alpha. \end{aligned} \tag{2.12}$$

By (2.10) and (2.12), one infers

$$(m-1)\Phi_i^\alpha = -\sum B_{ijj}^\alpha. \tag{2.13}$$

According to (2.10), (2.11) and (2.13), if $m \geq 3$, the Möbius form Φ and the Blaschke tensor A are determined by the Möbius metric g and Möbius second fundamental form B. The following theorem can be found in [27]:

Theorem 2.2. *Two immersed hypersurfaces* $x, : M^m \to S^{m+1}$ *and* $\tilde{x} : \tilde{M}^m \to S^{m+1}$, $m \geq 3$, *are Möbius equivalent if and only if there exists a diffeomorphism* $\varphi : M^m \to \tilde{M}^m$ *which preserves the Möbius metric and the Möbius second fundamental forms.*

For a hypersurface without umbilical points $x : M^m \to S^{m+1}$, one calls $D^\lambda = A + \lambda B$ a *para-Blaschke tensor* of x with parameter λ, where λ is a real number. From (2.12), we have then

$$D^\lambda_{ijk} - D^\lambda_{ikj} = (B_{ik} - \lambda\delta_{ik})\Phi_j - (B_{ij} - \lambda\delta_{ij})\Phi_k, \tag{2.14}$$

where D^λ_{ijk} are components of the covariant derivatives of D^λ.

3. Examples

In this section, we would like to present several immersed hypersurfaces in S^{m+1} and show that their para-Blaschke tensors are parallel.

Example 3.1([8]). Let $\mathbb{R}^+$ be the half line of positive real numbers. For any two positive integers p, q satisfying $p + q < m$ and a real number $r \in (0, 1)$, we consider an imbedded hypersurface $u : S^p(r) \times S^q\left(\sqrt{1-r^2}\right) \times \mathbb{R}^+ \times \mathbb{R}^{m-p-q-1} \to \mathbb{R}^{m+1}$ defined by $u = (tu', tu'', u''')$, where

$$u' \in S^p(r) \subset \mathbb{R}^{p+1},\ u'' \in S^q\left(\sqrt{1-r^2}\right) \subset \mathbb{R}^{q+1},\ t \in \mathbb{R}^+,\ u''' \in \mathbb{R}^{m-p-q-1}.$$

Then $x = \sigma \circ u : S^p(r) \times S^q\left(\sqrt{1-r^2}\right) \times \mathbb{R}^+ \times \mathbb{R}^{m-p-q-1} \to S^{m+1}$ defines a hypersurface in S^{m+1} without umbilical points, which is denoted by $CSS(p, q, r)$. By a direct calculation, one derives that $CSS(p, q, r)$ has three distinct Möbius principal curvatures and that the Möbius second fundamental form and the Blaschke tensor of it are parallel. Therefore the para-Blaschke tensor D^λ for any λ is parallel.

The following two families of examples can be found in [24], [25] and [28].

Example 3.2. Let $\lambda \in \mathbb{R}$. For any integers m, K satisfying $m \geq 3$ and $2 \leq K \leq m - 1$, let $\tilde{y}_1 : M_1 \to S^{K+1}(r) \subset \mathbb{R}^{K+2}$ be an immersed hypersurface without umbilical points such that the scalar curvature S_1 and the mean curvature H_1 of it satisfy

$$S_1 = \frac{mK(K-1) - (m-1)r^2}{mr^2} + m(m-1)\lambda^2, \quad H_1 = -\frac{m}{K}\lambda. \tag{3.1}$$

Let

$$\tilde{y} = (\tilde{y}_0, \tilde{y}_2) : H^{m-K}\left(-\frac{1}{r^2}\right) \to \mathbb{R}^{m-K+1}_1 \tag{3.2}$$

be the canonical embedding and

$$\tilde{M}^m = M_1 \times H^{m-K}\left(-\frac{1}{r^2}\right), \quad \tilde{Y} = (\tilde{y}_0, \tilde{y}_1, \tilde{y}_2). \tag{3.3}$$

We have that $\tilde{Y} : \tilde{M}^m \to \mathbb{R}^{m+3}_1$ is an immersion satisfying $\langle\tilde{Y}, \tilde{Y}\rangle_1 = 0$ and inducing a Riemannian metric

$$g = \langle d\tilde{Y}, d\tilde{Y}\rangle_1 = -d\tilde{y}_0^2 + d\tilde{y}_1^2 + d\tilde{y}_2^2. \tag{3.4}$$

Obviously,

$$(\tilde{M}^m, g) = (M_1, d\tilde{y}_1^2) \times \left(H^{m-K}\left(-\frac{1}{r^2}\right), \langle d\tilde{y}, d\tilde{y}\rangle_1 \right) \tag{3.5}$$

is a Riemannian manifold. Define

$$\tilde{x}_1 = \frac{\tilde{y}_1}{\tilde{y}_0}, \quad \tilde{x}_2 = \frac{\tilde{y}_2}{\tilde{y}_0}, \quad \tilde{x} = (\tilde{x}_1, \tilde{x}_2), \tag{3.6}$$

then $|\tilde{x}|^2 = 1$. Hence, $\tilde{x} : M^m \to S^{m+1}$ defines an immersed hypersurface without umbilical points. It is easy to have

$$d\tilde{x} = -\frac{d\tilde{y}_0}{\tilde{y}_0^2}(\tilde{y}_1, \tilde{y}_2) + \frac{1}{\tilde{y}_0}(d\tilde{y}_1, d\tilde{y}_2). \tag{3.7}$$

Therefore the induced metric $\tilde{g} = d\tilde{x} \cdot d\tilde{x}$ is given by

$$\tilde{g} = \tilde{y}_0^{-2} g. \tag{3.8}$$

If $\tilde{n}_1$ be a unit normal vector field of $\tilde{y}_1$ in $S^{K+1}(r) \subset \mathbb{R}^{K+2}$, then $\tilde{n} = (\tilde{n}_1, 0) \in \mathbb{R}^{m+2}$ is a unit normal vector field of $\tilde{x}$. Consequently, by (3.6), the second fundamental form $\tilde{h}$ of $\tilde{x}$ is given by

$$\tilde{h} = -d\tilde{n} \cdot d\tilde{x} = -\tilde{y}_0^{-1}(d\tilde{n}_1 \cdot d\tilde{y}_1) = y_0^{-1} h, \tag{3.9}$$

where h is the second fundamental form of the immersion $\tilde{y}$. Let $\{E_i, 1 \le i \le K\}$ and $\{E_i, K+1 \le i \le m\})$ be a local orthonormal frame field on $(M_1, d\tilde{y}_1^2)$ and on $H^{m-K}(-\frac{1}{r^2})$, respectively. We know that $\{E_i, 1 \le i \le m\}$ is a local orthonormal frame field on $(\tilde{M}^m, g)$. Putting $e_i = \tilde{y}_0 E_i$, $i = 1, \cdots, m$, then $\{e_i, 1 \le i \le m\}$ is a local orthonormal frame field on $(\tilde{M}^m, \tilde{g})$. Thus

$$\begin{aligned} &\tilde{h}_{ij} = \tilde{y}_0 h_{ij}, \quad \text{for } \ 1 \le i, j \le K; \\ &\tilde{h}_{ij} = 0, \quad \text{for } \ i > K \text{ or } j > K. \end{aligned} \tag{3.10}$$

The mean curvature of $\tilde{x}$ is given by

$$\tilde{H} = \frac{K}{m}\tilde{y}_0 H_1 = -\tilde{y}_0 \lambda. \tag{3.11}$$

Therefore, by definition, the Möbius factor $\tilde{\rho}$ of $\tilde{x}$ is determined by

$$\tilde{\rho}^2 = \frac{m}{m-1}\left(\sum_{i,j} \tilde{h}_{ij}^2 - m|\tilde{H}|^2\right) = \frac{m}{m-1}\tilde{y}_0^2 \left(\sum_{i,j=1}^{K} h_{ij}^2 - m\lambda^2\right) = \tilde{y}_0^2.$$

Here we have used (3.1) and the Gauss equation of $\tilde{y}_1$. Hence, $\tilde{\rho} = \tilde{y}_0$ and $\tilde{Y}$ is the Möbius position of $\tilde{x}$. Therefore, the Möbius metric of $\tilde{x}$ is $\langle d\tilde{Y}, d\tilde{Y}\rangle_1 = g$ and the Möbius second fundamental form of $\tilde{x}$ is given by

$$\tilde{B} = \tilde{\rho}^{-1} \sum (\tilde{h}_{ij} - \tilde{H}\delta_{ij})\omega^i\omega^j = \sum_{i,j=1}^{K} (h_{ij} + \lambda\delta_{ij})\,\omega^i\omega^j + \sum_{i=K+1}^{m} \lambda(\omega^i)^2, \tag{3.12}$$

where $\{\omega^i\}$ is the local coframe field with respect to $\{E_i\}$ on M^m.

On the other hand, by (3.4) and the Gauss equations of $\tilde{y}_1, \tilde{y}$, the Ricci tensor with respect to g is given by

$$\begin{aligned} R_{ij} =& \frac{K-1}{r^2}\delta_{ij} - m\lambda h_{ij} - \sum_{k=1}^{K} h_{ik}h_{kj}, \quad \text{if } 1 \le i,j \le K; \\ R_{ij} =& -\frac{m-K-1}{r^2}\delta_{ij}, \quad \text{if } K+1 \le i,j \le m; \\ R_{ij} =& 0, \quad \text{if } 1 \le i \le K,\ K+1 \le j \le m, \text{ or } K+1 \le i \le m,\ 1 \le j \le K. \end{aligned} \tag{3.13}$$

Hence, we derive the scalar curvature $m(m-1)\kappa$ with respect to g

$$\kappa = \frac{m(K(K-1)-(m-K)(m-K-1))-(m-1)r^2}{m^2(m-1)r^2} + \lambda^2. \tag{3.14}$$

Thus

$$\frac{1}{2m}(1+m^2\kappa) = \frac{K(K-1)-(m-K)(m-K-1)}{2(m-1)r^2} + \frac{1}{2}m\lambda^2. \tag{3.15}$$

From (2.11), (3.12)∼(3.15) and $m \ge 3$, we infer that the Blaschke tensor of $\tilde{x}$ is given by $A = \sum A_{ij}\omega^i\omega^j$ with

$$\begin{aligned} A_{ij} =& \left(\frac{1}{2r^2} - \frac{1}{2}\lambda^2\right)\delta_{ij} - \lambda h_{ij}, \quad \text{if } 1 \le i,j \le K; \\ A_{ij} =& -\left(\frac{1}{2r^2} + \frac{1}{2}\lambda^2\right)\delta_{ij}, \quad \text{if } K+1 \le i,j \le m; \\ A_{ij} =& 0, \quad \text{if } 1 \le i \le K,\ K+1 \le j \le m, \text{ or } K+1 \le i \le m,\ 1 \le j \le K. \end{aligned} \tag{3.16}$$

Therefore, the para-Blaschke tensor $D^\lambda = A + \lambda B = \sum D^\lambda_{ij}\omega^i\omega^j$ satisfies

$$\begin{aligned} D^\lambda_{ij} =& A_{ij} + \lambda B_{ij} = \left(\frac{1}{2r^2} + \frac{1}{2}\lambda^2\right)\delta_{ij}, \quad \text{for } 1 \le i,j \le K; \\ D^\lambda_{ij} =& A_{ij} + \lambda B_{ij} = \left(-\frac{1}{2r^2} + \frac{1}{2}\lambda^2\right)\delta_{ij}, \quad \text{for } K+1 \le i,j \le m; \\ D^\lambda_{ij} =& 0, \quad \text{for } 1 \le i \le K,\ K+1 \le j \le m, \text{ or } K+1 \le i \le m,\ 1 \le j \le K. \end{aligned} \tag{3.17}$$

Thus, we know that D^λ is parallel.

Example 3.3. For $\lambda \in \mathbb{R}$ and integers m, K satisfying $m \ge 3$ and $2 \le K \le m-1$, let $\tilde{y} = (\tilde{y}_0, \tilde{y}_1) : M_1 \to H^{K+1}\left(-\frac{1}{r^2}\right) \subset \mathbb{R}^{K+2}_1$ be an immersed hypersurface without umbilical points so that its scalar curvature S_1 and mean curvature H_1 are given by

$$S_1 = -\frac{mK(K-1)+(m-1)r^2}{mr^2} + m(m-1)\lambda^2, \quad H_1 = -\frac{m}{K}\lambda. \tag{3.18}$$

Assume that

$$\tilde{y}_2 : S^{m-K}(r) \to \mathbb{R}^{m-K+1} \tag{3.19}$$

is the canonical embedding. Putting

$$\tilde{M}^m = M_1 \times S^{m-K}(r), \quad \tilde{Y} = (\tilde{y}_0, \tilde{y}_1, \tilde{y}_2), \tag{3.20}$$

$\tilde{Y} : M^m \to \mathbb{R}^{m+3}_1$ is an immersion satisfying $\langle \tilde{Y}, \tilde{Y} \rangle_1 = 0$ and the induced metric

$$g = \langle d\tilde{Y}, d\tilde{Y} \rangle_1 = -d\tilde{y}_0^2 + d\tilde{y}_1^2 + d\tilde{y}_2^2$$

is a Riemannian metric. Defining

$$\tilde{x}_1 = \frac{\tilde{y}_1}{\tilde{y}_0}, \quad \tilde{x}_2 = \frac{\tilde{y}_2}{\tilde{y}_0}, \quad \tilde{x} = (\tilde{x}_1, \tilde{x}_2), \tag{3.21}$$

$|\tilde{x}|^2 = 1$, $\tilde{x} : \tilde{M}^m \to S^{m+1}$ determines an immersed hypersurface without umbilical points and $\tilde{Y}$ is its Möbius position vector. By the same assertions as in the example 3.2, we have

$$d\tilde{x} = -\frac{d\tilde{y}_0}{\tilde{y}_0^2}(\tilde{y}_1, \tilde{y}_2) + \frac{1}{\tilde{y}_0}(d\tilde{y}_1, d\tilde{y}_2), \tag{3.22}$$

and the induced metric $\tilde{g} = d\tilde{x} \cdot d\tilde{x}$ is given by

$$\tilde{g} = \tilde{y}_0^{-2}(-d\tilde{y}_0^2 + d\tilde{y}_1^2 + d\tilde{y}_2^2) = \tilde{y}_0^{-2} g. \tag{3.23}$$

If $(\tilde{n}_0, \tilde{n}_1)$ is the unit normal vector field of $\tilde{y}$ in $H^{K+1}\left(-\frac{1}{r^2}\right) \subset \mathbb{R}^{m+2}_1$, then it is easy to verify that

$$\tilde{n} = (\tilde{n}_1, 0) - \tilde{n}_0 \tilde{x} \in \mathbb{R}^{m+2}$$

is a unit normal vector field of $\tilde{x}$. Consequently, by (3.22)

$$\begin{aligned} d\tilde{n} \cdot d\tilde{x} =& (d\tilde{n}_1, 0) \cdot d\tilde{x} - \tilde{n}_0 d\tilde{x} \cdot d\tilde{x} \\ =& -(\tilde{y}_0^{-2} d\tilde{y}_0) d\tilde{n}_1 \cdot \tilde{y}_1 + \tilde{y}_0^{-1} d\tilde{n}_1 \cdot d\tilde{y}_1 - \tilde{n}_0 \tilde{y}_0^{-2}(-d\tilde{y}_0^2 + d\tilde{y}_1^2 + d\tilde{y}_2^2) \\ =& \tilde{y}_0^{-1}(-d\tilde{n}_0 d\tilde{y}_0 + d\tilde{n}_1 \cdot d\tilde{y}_1) - \tilde{n}_0 \tilde{y}_0^{-2} g, \end{aligned} \tag{3.24}$$

where, in the third equality, we have used

$$-\tilde{n}_0 d\tilde{y}_0 + \tilde{n}_1 \cdot d\tilde{y}_1 = 0. \tag{3.25}$$

Thus, the second fundamental form $\tilde{h}$ of the immersion $\tilde{x}$ is related to the second fundamental form h of the immersion $\tilde{y}$ and the metric g in the following way:

$$\tilde{h} = -d\tilde{n} \cdot d\tilde{x} = -\tilde{y}_0^{-1} \langle d(\tilde{n}_0, \tilde{n}_1), d\tilde{y} \rangle_1 + \tilde{n}_0 \tilde{y}_0^{-2} g = y_0^{-1} h + \tilde{n}_0 \tilde{y}_0^{-2} g. \tag{3.26}$$

Let $\{E_i, 1 \le i \le K\}$ and $\{E_i, K+1 \le i \le m\}$ be a local orthonormal frame field on $(M_1, d\tilde{y}^2)$ and on $S^{m-K}(r))$, respectively. $\{E_i, 1 \le i \le m\}$ becomes a local orthonormal frame field on (M^m, g). Putting $e_i = \tilde{y}_0 E_i$, $i = 1, \cdots, m$, $\{e_i, 1 \le i \le m\}$ is a local orthonormal frame field on $(M^m, \tilde{g})$ and we have

$$\begin{aligned} \tilde{h}_{ij} =& \tilde{h}(e_i, e_j) = \tilde{y}_0^2 \tilde{h}(E_i, E_j) = \tilde{y}_0 h(E_i, E_j) + \tilde{n}_0 g(E_i, E_j) \\ =& \tilde{y}_0 h_{ij} + \tilde{n}_0 \delta_{ij}, \quad \text{when } 1 \le i, j \le K; \\ \tilde{h}_{ij} =& \tilde{n}_0 \delta_{ij}, \quad \text{when } i > K \text{ or } j > K. \end{aligned} \tag{3.27}$$

The mean curvature of the immersion $\tilde{x}$ is given by

$$\tilde{H} = \frac{K}{m}\tilde{y}_0 H_1 + \tilde{n}_0 = \tilde{n}_0 - \tilde{y}_0\lambda. \tag{3.28}$$

Therefore, by definition, the Möbius factor $\tilde{\rho}$ of $\tilde{x}$ is determined by

$$\tilde{\rho}^2 = \frac{m}{m-1}\left(\sum_{i,j}\tilde{h}_{ij}^2 - m|\tilde{H}|^2\right) = \frac{m}{m-1}\tilde{y}_0^2\left(\sum_{i,j}h_{ij}^2 - m\lambda^2\right) = \tilde{y}_0^2.$$

Here we have used (3.18) and the Gauss equation of the immersion $\tilde{y}$. Hence $\tilde{\rho} = \tilde{y}_0$ and $\tilde{Y}$ is the Möbius position of $\tilde{x}$. Consequently, the Möbius metric of $\tilde{x}$ is $\langle d\tilde{Y}, d\tilde{Y}\rangle_1 = g$ and the Möbius second fundamental form of $\tilde{x}$ is

$$\tilde{B} = \tilde{\rho}^{-1}(\tilde{h}_{ij} - \tilde{H}\delta_{ij})\omega^i\omega^j = \sum_{i,j=1}^{K}(h_{ij} + \lambda\delta_{ij})\omega^i\omega^j + \sum_{i=K+1}^{m}\lambda(\omega^i)^2, \tag{3.29}$$

where $\{\omega^i\}$ is the local coframe field with respect to $\{E_i\}$ on M^m.

On the other hand, by (3.26) and the Gauss' equations of $\tilde{y}_1, \tilde{y}$, the Ricci tensor with respect to g follows:

$$\begin{aligned}
R_{ij} =& -\frac{K-1}{r^2}\delta_{ij} - m\lambda h_{ij} - \sum_{k=1}^{K}h_{ik}h_{kj}, \quad \text{if } 1 \le i,j \le K;\\
R_{ij} =& \frac{m-K-1}{r^2}\delta_{ij}, \quad \text{if } K+1 \le i,j \le m;\\
R_{ij} =& 0, \quad \text{if } 1 \le i \le K,\ K+1 \le j \le m, \text{ or } K+1 \le i \le m,\ 1 \le j \le K,
\end{aligned} \tag{3.30}$$

and the scalar curvature $m(m-1)\kappa$ with respect to this metric satisfies

$$\kappa = \frac{m((m-K)(m-K-1) - K(K-1)) - (m-1)r^2}{m^2(m-1)r^2} + \lambda^2. \tag{3.31}$$

Thus

$$\frac{1}{2m}(1 + m^2\kappa) = \frac{(m-K)(m-K-1) - K(K-1)}{2(m-1)r^2} + \frac{1}{2}m\lambda^2. \tag{3.32}$$

From (2.11), (3.29) $\sim$ (3.32) and $m \ge 3$, we infer

$$\begin{aligned}
A_{ij} =& -\left(\frac{1}{2r^2} + \frac{1}{2}\lambda^2\right)\delta_{ij} - \lambda h_{ij}, \quad \text{if } 1 \le i,j \le K;\\
A_{ij} =& \left(\frac{1}{2r^2} - \frac{1}{2}\lambda^2\right)\delta_{ij}, \quad \text{if } K+1 \le i,j \le m;\\
A_{ij} =& 0, \quad \text{if } 1 \le i \le K,\ K+1 \le j \le m, \text{ or } K+1 \le i \le m,\ 1 \le j \le K.
\end{aligned} \tag{3.33}$$

Thus, the para-Blaschke tensor $D^\lambda = A + \lambda B = \sum D_{ij}^\lambda\omega^i\omega^j$ is given by

$$\begin{aligned}
D_{ij}^\lambda =& A_{ij} + \lambda B_{ij} = \left(-\frac{1}{2r^2} + \frac{1}{2}\lambda^2\right)\delta_{ij}, \quad \text{for } 1 \le i,j \le K;\\
D_{ij}^\lambda =& A_{ij} + \lambda B_{ij} = \left(\frac{1}{2r^2} + \frac{1}{2}\lambda^2\right)\delta_{ij}, \quad \text{for } K+1 \le i,j \le m;\\
D_{ij}^\lambda =& 0, \quad \text{for } 1 \le i \le K,\ K+1 \le j \le m, \text{ or } K+1 \le i \le m,\ 1 \le j \le K.
\end{aligned} \tag{3.34}$$

Hence, we derive that D^λ is parallel.

The main theorem in [28] can be restated as follows:

Theorem 3.1. *Let $x : M^m \to S^{m+1}$ be a hypersurface without umbilical points and with vanishing Möbius form Φ. If there exists a $\lambda \in \mathbb{R}$ such that the para-Blaschke tensor D^λ has only two distinct constant eigenvalues, then x is locally Möbius equivalent to*

(1) *a standard torus $S^K(r) \times S^{m-K}(\sqrt{1-r^2})$ in S^{m+1} for some $r > 0$ and positive integer K; or*
(2) *the image under σ of a standard cylinder $S^K(r) \times \mathbb{R}^{m-K}$ in $\mathbb{R}^{m+1}$ for some $r > 0$ and positive integer K; or*
(3) *the image under τ of a standard cylinder $S^K(r) \times H^{m-K}(-\frac{1}{1+r^2})$ in $H^{m+1}(-1)$ for some $r > 0$ and positive integer K; or*
(4) *one of the immersed hypersurfaces as indicated in Example* 3.2*; or*
(5) *one of the immersed hypersurfaces as indicated in Example* 3.3.

4. Proof of Theorem 1.1

Let $x : M^m \to S^{m+1}$ be an immersed hypersurface without umbilical points. First of all, we restate a theorem given by Li and Wang [19],

Theorem 4.1. *For an immersed hypersurface $x : M^m \to S^{m+1}$ without umbilical points and with vanishing Möbius form Φ, if the para-Blaschke tensor D^λ satisfies $D^\lambda = fg$ for some function f on M, then f is constant and x is Möbius equivalent to one of the following:*

(1) *an immersed hypersurface $\tilde{x} : M^m \to S^{m+1}$ with constant scalar curvature and constant mean curvature;*
(2) *the image under σ of an immersed hypersurface in $\mathbb{R}^{m+1}$ with constant scalar curvature and constant mean curvature;*
(3) *the image under τ of an immersed hypersurface in $\mathbb{H}^{m+1}$ with constant scalar curvature and constant mean curvature.*

In order to prove Theorem 1.1, we prepare several lemmas.

Lemma 4.1. *If D^λ is parallel, then eigenvalues of D^λ are constant on M^m.*

Proof. Since D^λ is symmetric, there exists a local orthonormal frame field $\{E_i\}$ such that, at each point

$$D^\lambda_{ij} = D^\lambda_i \delta_{ij}. \tag{4.1}$$

Because D^λ is parallel, we have

$$0 = \sum D^\lambda_{ijk}\omega^k = dD^\lambda_{ij} - D^\lambda_{kj}\omega^k_i - D^\lambda_{ik}\omega^k_j, \tag{4.2}$$

where $\{\omega^i\}$ is the local coframe field dual to $\{E_i\}$ and ω_j^i are the Levi-Civita connection form with respect to the metric g. Then it follows easily that

$$dD_{ij}^{\lambda} - (D_i^{\lambda} - D_j^{\lambda})\omega_j^i = 0, \tag{4.3}$$

which implies that

$$dD_{ii}^{\lambda} = 0.$$

Hence, D_{ii}^{λ} is constant and eigenvalues of D^{λ} are constant on M^m. Thus, we also have

$$\omega_j^i = 0 \quad \text{if} \quad D_i^{\lambda} \neq D_j^{\lambda} \tag{4.4}$$

since D^{λ} is parallel and $dD_{ij}^{\lambda} = 0$. □

At first, we consider the case that $\Phi \equiv 0$.

Lemma 4.2. *If $\Phi \equiv 0$, then $B_{ij} = 0$ whenever eigenvalues $D_i^{\lambda} \neq D_j^{\lambda}$ of D^{λ}.*

Proof. By (2.12), we derive

$$\sum B_{ik}D_{kj}^{\lambda} - D_{ik}^{\lambda}B_{kj} = \Phi_{ij} - \Phi_{ji} = 0.$$

From (4.1), we have $B_{ij}(D_j^{\lambda} - D_i^{\lambda}) = 0$. We infer $B_{ij} = 0$ whenever eigenvalues $D_i^{\lambda} \neq D_j^{\lambda}$. □

By Lemma 4.2, at any point p, we can choose an orthonormal basis such that both D^{λ} and B are diagonalized simultaneously.

Let t be the number of distinct eigenvalues of D^{λ}, and $d_1, \cdots, d_t$ be the distinct eigenvalues of D^{λ}. Let $\{E_i\}$ be an orthonormal frame field such that

$$(D_{ij}^{\lambda}) = \text{Diag}(\underbrace{d_1, \cdots, d_1}_{k_1}, \underbrace{d_2, \cdots, d_2}_{k_2}, \cdots, \underbrace{d_t, \cdots, d_t}_{k_t}), \tag{4.5}$$

namely,

$$D_1^{\lambda} = \cdots = D_{k_1}^{\lambda} = d_1, \cdots, D_{m-k_t+1}^{\lambda} = \cdots = D_m^{\lambda} = d_t. \tag{4.6}$$

Lemma 4.3. *Assume $\Phi \equiv 0$. If D^{λ} is parallel and $t \geq 3$, then,*

$$B_i = B_j \quad \textit{whenever} \quad D_i^{\lambda} = D_j^{\lambda}, \tag{4.7}$$

where B_i's are eigenvalues of (B_{ij}).

Proof. From Lemma 4.2, we can choose an appropriate orthonormal frame field $\{E_i\}$ such that (4.5) and $B_{ij} = B_i\delta_{ij}$ hold. By (4.4), for any i, j, $\omega_j^i = 0$ whenever $D_i^{\lambda} \neq D_j^{\lambda}$. Thus, $d\omega_j^i = 0$, which implies

$$0 = B_{ii}B_{jj} - B_{ij}^2 + (D_{ii}^{\lambda} - \lambda B_{ii}) - (D_{ij}^{\lambda} - \lambda B_{ij})\delta_{ij} + (D_{jj}^{\lambda} - \lambda B_{jj}) - (D_{ij}^{\lambda} - \lambda B_{ij})\delta_{ij}.$$

namely,

$$B_iB_j - \lambda(B_i + B_j) + D_i^\lambda + D_j^\lambda = 0. \tag{4.8}$$

If there exist i, j such that $D_i^\lambda = D_j^\lambda$ and $B_i \neq B_j$, then for all k satisfying $D_k^\lambda \neq D_i^\lambda$, we have

$$B_iB_k - \lambda(B_i + B_k) + D_i^\lambda + D_k^\lambda = 0, \quad B_jB_k - \lambda(B_j + B_k) + D_j^\lambda + D_k^\lambda = 0. \tag{4.9}$$

From (4.9), we have $(B_i - B_j)(B_k - \lambda) = 0$, which implies $B_k = \lambda$. Thus by (4.9), $D_k^\lambda - \lambda^2 = -D_i^\lambda = -D_j^\lambda$. This means that $t = 2$. The contradiction finishes the proof of Lemma 4.3. □

Corollary 4.1. *Under the assumptions of Lemma* 4.3*, there exists an orthonormal frame field* $\{E_i\}$ *such that*

$$D_{ij}^\lambda = D_i^\lambda \delta_{ij}, \quad B_{ij} = B_i\delta_{ij} \tag{4.10}$$

and

$$(B_{ij}) = Diag(\underbrace{b_1, \cdots, b_1}_{k_1}, \underbrace{b_2, \cdots, b_2}_{k_2}, \cdots, \underbrace{b_t, \cdots, b_t}_{k_t}), \tag{4.11}$$

that is,

$$B_1 = \cdots = B_{k_1} = b_1, \cdots, B_{m-k_t+1} = \cdots = B_m = b_t, \tag{4.12}$$

where $b_1, \cdots, b_t$ *are not necessarily different from each other.*

Lemma 4.4. *Under the assumptions, of Lemma* 4.3*, Möbius principal curvatures* $b_1, \cdots, b_t$ *of* x *are constant, namely,* x *is Möbius isoparametric.*

Proof. Without loss of generality, it suffices to show that b_1 is constant. According to the assumptions and Corollary 4.1, we can choose a frame field $\{E_i\}$ in a neighborhood of any point such that (4.5), (4.10) and (4.11) hold. Note that for $1 \leq i \leq k_1$ and $j > k_1$, by (4.4)

$$\sum B_{ijk}\omega^k = dB_{ij} - \sum B_{kj}\omega_i^k - \sum B_{ik}\omega_j^k = 0. \tag{4.13}$$

Therefore, $B_{ijk} = 0$. By the symmetry of B_{ijk}, we see that $B_{ijk} = 0$, in case that two of i, j, k are less than or equal to k_1 with the other larger than k_1, or one of i, j, k is less than or equal to k_1 with the other larger than k_1. Hence, for any i, j satisfying $1 \leq i, j \leq k_1$,

$$\sum_{k=1}^{k_1} B_{ijk}\omega^k = dB_{ij} - \sum B_{kj}\omega_i^k - \sum B_{ik}\omega_j^k = dB_i\delta_{ij} - B_j\omega_i^j - B_i\omega_j^i.$$

We infer

$$\sum_{k=1}^{k_1} B_{iik}\omega^k = db_1, \tag{4.14}$$

which yields

$$E_k(b_1) = 0, \quad k_1 + 1 \le k \le m. \tag{4.15}$$

Similarly,

$$E_i(B_j) = 0, \quad 1 \le i \le k_1, \ k_1 + 1 \le j \le m. \tag{4.16}$$

On the other hand, from (4.8) we have

$$b_1 B_j - \lambda(b_1 + B_j) + d_1 + D_j^\lambda = 0, \quad k_1 + 1 \le j \le m. \tag{4.17}$$

We derive, for $1 \le k \le k_1$,

$$E_k(b_1)(B_j - \lambda) = 0, \quad 1 \le k \le k_1, \ k_1 + 1 \le j \le m. \tag{4.18}$$

Define $U = \{q \in M^n; B_j(q) \ne \lambda$ for some $j > k_1\}$. For any point $p \in U$, we can find some $j > k_1$ such that $B_j \ne \lambda$ around p. Therefore by (4.18), $E_k(b_1) = 0$ for $1 \le k \le k_1$, which and (4.15) implies that b_1 is a constant. This proves that b_1 is constant on the closure $\overline{U}$ of U. On the other hand, for any $p \notin \overline{U}$, we have $B_{k_1+1} = \cdots = B_m = \lambda$ around p. By (4.12), B has exactly two distinct eigenvalues at p. If $M^m \backslash \overline{U}$ is an open set, from (2.10), we know that b_1 is a constant in $M^m \backslash \overline{U}$. Since M^m is connected, we have that b_1 is constant identically on M^m. □

Corollary 4.2. *Under the assumptions of the lemma 4.3, $t = 3$ and B is parallel.*

Proof. From (4.13) and (4.14), we infer that B is parallel.

If $t > 3$, then there exist at least four indices i_1, i_2, i_3, i_4, such that $D_{i_1}^\lambda, D_{i_2}^\lambda, D_{i_3}^\lambda, D_{i_4}^\lambda$ are distinct each other. Then we have from (4.8) that

$$(B_{i_1} - \lambda)(B_{i_2} - \lambda) - \lambda^2 + D_{i_1}^\lambda + D_{i_2}^\lambda = 0, \quad (B_{i_3} - \lambda)(B_{i_4} - \lambda) - \lambda^2 + D_{i_3}^\lambda + D_{i_4}^\lambda = 0,$$
$$(B_{i_1} - \lambda)(B_{i_3} - \lambda) - \lambda^2 + D_{i_1}^\lambda + D_{i_3}^\lambda = 0, \quad (B_{i_2} - \lambda)(B_{i_4} - \lambda) - \lambda^2 + D_{i_2}^\lambda + D_{i_4}^\lambda = 0.$$

It follows that $(D_{i_1}^\lambda - D_{i_4}^\lambda)(D_{i_2}^\lambda - D_{i_3}^\lambda) = 0$. It is a contradiction. □

Lemma 4.5. *Assume $\Phi \equiv 0$. If D^λ is parallel and B is not parallel, then one of the following cases holds:*

(1) $t = 1$ and D^λ is proportional to the metric g;
(2) $t = 2$, $d_1 + d_2 = \lambda^2$ and $B_i = \lambda$ either for all $1 \le i \le k_1$, or for all $k_1 + 1 \le i \le m$.

Proof. From Corollary 4.2, it follows that $t \le 2$. It suffices to consider the case that $t = 2$. For any point $p \in M^m$, we can find an orthonormal frame field $\{E_i\}$ such that (4.1) holds around p and $B_{ij}(p) = B_i \delta_{ij}$. By (4.4)

$$\omega_j^i = 0, \quad 1 \le i \le k_1, \quad k_1 + 1 \le j \le m, \tag{4.19}$$

hold. By making use of the same assertion as (4.8), we have

$$B_i B_j - \lambda(B_i + B_j) + D_i^\lambda + D_j^\lambda = 0, \quad 1 \le i \le k_1, \quad k_1 + 1 \le j \le m. \tag{4.20}$$

If there exist i_0, j_0 with $1 \leq i_0 \leq k_1$, $k_1 + 1 \leq j_0 \leq m$ such that $B_{i_0} \neq \lambda$ and $B_{j_0} \neq \lambda$, there exist two Möbius principal curvatures different from λ. For any i, $1 \leq i \leq k_1$,

$$B_{i_0}B_{j_0} - \lambda(B_{i_0} + B_{j_0}) + D^\lambda_{i_0} + D^\lambda_{j_0} = 0, \quad B_iB_{j_0} - \lambda(B_i + B_{j_0}) + D^\lambda_i + D^\lambda_{j_0} = 0.$$

Thus, $(B_i - B_{i_0})(B_{j_0} - \lambda) = 0$. We obtain

$$B_i = B_{i_0}, \quad 1 \leq i \leq k_1. \tag{4.21}$$

Similarly,

$$B_j = B_{j_0}, \quad k_1 \leq j \leq m. \tag{4.22}$$

Therefore, there are exactly two distinct Möbius principal curvatures around point p. From (2.10) we know that Möbius principal curvatures B_i are constant. Thus, (4.11) and (4.19) hold. Hence, B is parallel from (4.13). It is impossible. Either $B_i = \lambda$ for i, $1 \leq i \leq k_1$, or $B_j = \lambda$ for j, $k_1 \leq j \leq m$. We have by (4.20), $d_1 + d_2 = \lambda^2$. □

Next, we consider the case that $\Phi \not\equiv 0$.

Since our classification theorem is a local one, we can assume from now on that $\Phi \neq 0$ everywhere on M^n without loss of generality.

Lemma 4.6. *If the Möbius form Φ of x does not vanish and D^λ is parallel, then x has exactly two distinct Möbius principal curvatures one of which is simple.*

Proof. For any given point $p \in M^m$, we can choose an orthonormal frame field $\{E_i\}$ around p with respect to the Möbius metric g, such that $B_{ij}(p) = B_i\delta_{ij}$. Since D^λ is parallel, we derive from (2.14)

$$(B_i - \lambda)(\delta_{ik}\Phi_j - \delta_{ij}\Phi_k) = 0. \tag{4.23}$$

If there exist $i_1 \neq i_2$ such that $B_{i_1} - \lambda \neq 0$, $B_{i_2} - \lambda \neq 0$, then, for any i, j, we have

$$\delta_{i_1j}\Phi_i(p) - \delta_{i_1i}\Phi_j(p) = 0, \quad \delta_{i_2j}\Phi_i(p) - \delta_{i_2i}\Phi_j(p) = 0. \tag{4.24}$$

It is not hard to derive $\Phi_i(p) = 0$. It is a contradiction, which finishes the proof of Lemma 4.6. □

Lemma 4.7. *If the Möbius form Φ of x does not vanish and D^λ is parallel, then x has exactly two distinct principal curvatures with one of which being simple. Furthermore $x(M^m)$ is foliated by a family of $m-1$-dimensional totally umbilical submanifolds of S^{m+1}.*

Proof. By Lemma 4.6, there are two different Möbius principal curvatures b_1 and b_2 with b_1 being simple. According to (2.10), $b_2 = \varepsilon\frac{1}{m}$, $b_1 = -\varepsilon\frac{m-1}{m}$, where $\varepsilon = \pm 1$. Furthermore, by the definition of the Möbius second fundamental form B, x has two distinct principal curvatures h_1 and h_2 with h_1 being simple. Therefore $x(M^m)$ is foliated by a family of its hypersurfaces tangent to the eigenvector space distribution corresponding to h_2.

For any $p \in M^m$, we take an orthonormal frame field $\{E_i\}$ such that $B_{ij} = B_i\delta_{ij}$, where

$$B_1 = b_1 = -\varepsilon\frac{m-1}{m}, \quad B_2 = \cdots = B_m = b_2 = \varepsilon\frac{1}{m}. \tag{4.25}$$

By (4.23), we know that the Möbius form $\Phi = \sum \Phi_i\omega^i$ satisfies

$$\Phi_1 \neq 0, \quad \Phi_j = 0, \quad j = 2, \cdots, m. \tag{4.26}$$

By a direct computation, we have

$$H = \frac{1}{m}(h_1 + (m-1)h_2), \quad |h|^2 - mH^2 = \frac{m-1}{m}(h_1 - h_2)^2, \quad \rho = |h_1 - h_2|. \tag{4.27}$$

We derive, from (2.9)

$$\begin{aligned}\Phi_i = -\rho^{-2}\bigg\{&e_i(h_1)\left(\frac{1}{m} + B_i\mathrm{Sgn}\,(h_1 - h_2)\right)\\ &+ e_i(h_2)\left(\frac{m-1}{m} - B_i\mathrm{Sgn}\,(h_1 - h_2)\right)\bigg\},\end{aligned} \tag{4.28}$$

where $e_i = \rho E_i$. Since $\varepsilon\frac{1}{m} = B_2 = \rho^{-1}(h_2 - H) = -\frac{1}{m}\mathrm{Sgn}\,(h_1 - h_2)$, we have $\mathrm{Sgn}\,(h_1 - h_2) = -\varepsilon$. We infer, from (4.25) and (4.28),

$$\Phi_1 = -\rho^{-2}e_1(h_1), \quad \Phi_j = -\rho^{-2}e_j(h_2), \quad j = 2, \cdots, m. \tag{4.29}$$

(4.26) and (4.29) yield $e_j(h_2) = 0$ which means that h_2 is constant along the leaves. On the other hand, from (4.13) and (4.25), we derive

$$B_{11i} = B_{abi} = 0, \quad B_{1ai} = B_{a1i} = -\varepsilon\Gamma^a_{1i}, \quad 1 \le i \le m, \quad 2 \le a, b \le m, \tag{4.30}$$

where Γ^k_{ij} is determined by $\omega^j_i = \sum \Gamma^j_{ik}\omega^k$.

By (2.12), (4.26) and (4.30), we have

$$\Gamma^a_{1b} = -\varepsilon B_{1ab} = \varepsilon(B_{ab1} - B_{a1b}) = \varepsilon(\delta_{ab}\Phi_1 - \delta_{a1}\Phi_b),$$

that is

$$\Gamma^a_{1b} = \varepsilon\delta_{ab}\Phi_1, \quad 2 \le a, b \le m. \tag{4.31}$$

Denoting by $\overline{\nabla}$ the Levi-Civita connection with respect to the induced metric $\overline{g} = \rho^{-2}g$, the connection coefficients $\overline{\Gamma}^k_{ij}$ of $\overline{\nabla}$ are related with Γ^k_{ij} by

$$\Gamma^k_{ij} = \rho^{-1}\overline{\Gamma}^k_{ij} + \rho^{-2}\left(\delta_{jk}e_i(\rho) - \delta_{ij}e_k(\rho)\right).$$

It follows that

$$\overline{g}(\overline{\nabla}_{e_b}e_1, e_a) = \overline{\Gamma}^a_{1b} = \rho\Gamma^a_{1b} - \rho^{-1}e_1(\rho)\delta_{ab} = \delta_{ab}(\varepsilon\rho\Phi_1 - \rho^{-1}e_1(\rho)).$$

Since each leaf of the foliation has e_1 as it unit normal in $x(M^m)$, we see easily that all leaves of the foliation are totally umbilical with respect to their normals in $x(M^m)$, that is, those leaves are totally umbilical as $m-1$-submanifolds of S^{m+1}. □

Proof of Theorem 1.1. By Lemma 4.7, we need only to consider the case that the Möbius form $\Phi \equiv 0$. By making use of Theorem 4.1 and the classification of hypersurfaces with parallel Möbius second fundamental forms given by Hu, Li in [8], we need only to prove that if x does not have parallel Möbius second fundamental form and the number t of distinct eigenvalues of D^λ is larger than 1, then x must be Möbius equivalent to one of the immersed hypersurfaces given in Examples 3.2 and 3.3.

According to Lemma 4.1, we know that D^λ has constant eigenvalues. If $t \geq 3$, then the Möbius second fundamental form, from Corollary 4.2, is parallel. Hence, $t = 2$. Without loss of generality, we can assume, by Lemma 4.5, that

$$t = 2, \quad d_1 = d, \quad d_2 = \lambda^2 - d, \quad B_{K+1} = \cdots = B_m = \lambda, \tag{4.32}$$

where $K = k_1$. Since the Möbius second fundamental form B of x is not parallel, the number of distinct Möbius principal curvatures must be larger than 2 (see [18]). Then it follows easily that $m \geq 3$. Because D^λ is parallel, the tangent bundle TM^m has a decomposition $TM^m = V_1 \oplus V_2$, where V_1 and V_2 are eigenspaces of D^λ for eigenvalues $d_1 = d$ and $d_2 = \lambda^2 - d$, respectively. Let $\{E_i, 1 \leq i \leq K\}$, $\{E_j, K \leq j \leq m\}$ are orthonormal frame fields for subbundles V_1 and V_2, respectively. Then $\{E_i, 1 \leq i \leq m\}$ is an orthonormal frame field on M^m with respect to the Möbius metric g. (4.19) implies that both V_1 and V_2 are integrable, and thus Riemannian manifold (M^m, g) can be decomposed locally into a direct product of two Riemannian manifolds (M_1, g_1) and (M_2, g_2), that is,

$$(M, g) = (M_1, g_1) \times (M_2, g_2). \tag{4.33}$$

It follows from (2.11) and (4.6) that the Riemannian curvature tensors of (M_1, g_1) and (M_2, g_2) have respectively the following components:

$$\begin{aligned} R_{ijkl} =& (2d - \lambda^2)(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + (B_{il} - \lambda)(B_{jk} - \lambda) - (B_{ik} - \lambda)(B_{jl} - \lambda), \\ & \qquad 1 \leq i, j, k, l \leq K; \\ R_{ijkl} =& (\lambda^2 - 2d)(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}), \quad K + 1 \leq i, j, k, l \leq m; \end{aligned} \tag{4.34}$$

Thus (M_2, g_2) is of constant sectional curvature $\lambda^2 - 2d$. Since $d_1 \neq d_2$, (4.32) implies that $2d - \lambda^2 \neq 0$.

Next, we consider separately the following two subcases:

Subcase (1): $2d - \lambda^2 > 0$. In this case, then (M_2, g_2) is locally isometric to $H^{m-K}\left(-\frac{1}{r^2}\right)$ with $r = (2d - \lambda^2)^{-\frac{1}{2}}$. Let $\tilde{y} = (\tilde{y}_0, \tilde{y}_2) : H^{m-K}\left(-\frac{1}{r^2}\right) \to \mathbb{R}_1^{m-K+1}$ be the canonical embedding. Since $h = \sum_{k=1}^{K}(B_{ij} - \lambda\delta_{ij})\omega^i\omega^j$ is a Codazzi tensor on (M_1, g_1), it follows from (4.34) that there exists an isometric immersion

$$\tilde{y}_1 : (M_1, g_1) \to S^{K+1}(r) \subset \mathbb{R}^{K+2}, \quad 2 \leq K \leq m - 1, \tag{4.35}$$

such that h is its second fundamental form. Clearly, $\tilde{y}_1$ is without umbilical points. Furthermore, it has constant scalar curvature S_1 and constant mean curvature H_1

as follows:

$$S_1 = \frac{mK(K-1)-(m-1)r^2}{mr^2}, \quad H_1 = -\frac{m}{K}\lambda.$$

Note that M^m can be locally identified with $\tilde{M}^m = (M_1, g_1) \times H^{m-K}(-\frac{1}{r^2})$.

Define $\tilde{x}_1 = \frac{\tilde{y}_1}{\tilde{y}_0}$, $\tilde{x}_2 = \frac{\tilde{y}_2}{\tilde{y}_0}$ and $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$, then, by the discussion in Example 3.2, $\tilde{x} : \tilde{M}^m \to S^{m+1}$ gives an immersed hypersurface with the given metric g, B, respectively, as its Möbius metric and Möbius second fundamental form. Therefore, by Theorem 2.2, x is Möbius equivalent to $\tilde{x}$.

Subcase (2): $2d - \lambda^2 < 0$. In this case, (M_2, g_2) is locally isometric to $S^{m-K}(r)$ with $r = (\lambda^2 - 2d)^{-\frac{1}{2}}$. Let $\tilde{y}_2 : S^{m-K}(r) \to \mathbb{R}^{m-K+1}$ be the canonical embedding. Since $h = \sum_{k=1}^{K}(B_{ij} - \lambda\delta_{ij})\omega^i\omega^j$ is a Codazzi tensor on (M_1, g_1), it follows from (4.34) that there exists an isometric immersion

$$\tilde{y} = (\tilde{y}_0, \tilde{y}_1) : (M_1, g_1) \to H^{K+1}\left(-\frac{1}{r^2}\right) \subset \mathbb{R}^{K+2}_1, \quad 2 \le K \le m-1, \tag{4.36}$$

such that h is its second fundamental form. Clearly, $\tilde{y}$ is without umbilical points. Furthermore, it has constant scalar curvature S_1 and constant mean curvature H_1 as follows:

$$S_1 = -\frac{mK(K-1)-(m-1)r^2}{mr^2}, \quad H_1 = -\frac{m}{K}\lambda.$$

Note that M^m can be locally identified with $\tilde{M}^m = (M_1, g_1) \times H^{m-K}\left(-\frac{1}{r^2}\right)$.

Define $\tilde{x}_1 = \frac{\tilde{y}_1}{\tilde{y}_0}$, $\tilde{x}_2 = \frac{\tilde{y}_2}{\tilde{y}_0}$ and $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$, then, by the discussion in Example 3.3, $\tilde{x} : \tilde{M}^m \to S^{m+1}$ gives an immersed hypersurface with the given metric g, B, respectively, as its Möbius metric and Möbius second fundamental form. Therefore, by Theorem 2.2, x is Möbius equivalent to $\tilde{x}$.

Acknowledgments

The research of the first author is partially supported by a Grant-in-Aid for Scientific Research from the JSPS, while the research of the second and the third authors are partially supported by NNSF of China (No. 10671181). The second author would like to express his gratitude to Saga University for the hospitality during his visiting.

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