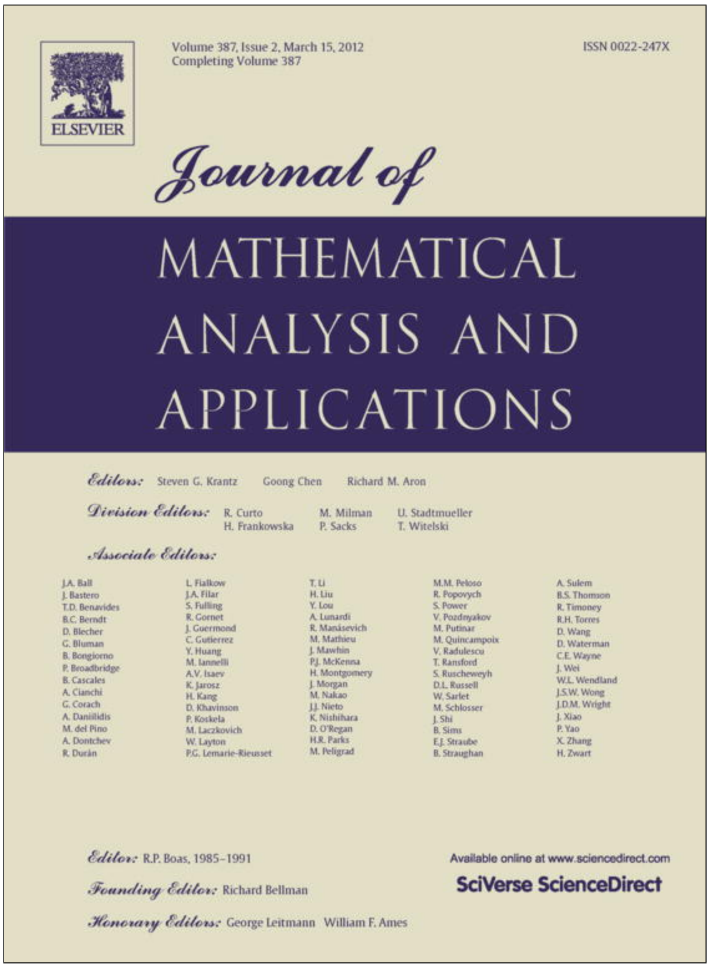
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On eigenvalues of a system of elliptic equations and of the biharmonic operator

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ABSTRACT

Let Ω be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$. We study eigenvalues of an eigenvalue problem of a system of elliptic equations:

$$\begin{cases} \Delta \mathbf{u} + \alpha \operatorname{grad}(\operatorname{div} \mathbf{u}) = -\sigma \mathbf{u}, & \text{in } \Omega, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}. \end{cases}$$

Estimates for eigenvalues of the above eigenvalue problem are obtained. Furthermore, we obtain an upper bound on the $(k+1)$th eigenvalue σ_{k+1}. We also obtain sharp lower bound for the first eigenvalue of two kinds of eigenvalue problems of the biharmonic operator on compact manifolds with boundary and positive Ricci curvature.

1. Introduction

Let Ω be a bounded domain with smooth boundary in an n-dimensional Euclidean space $\mathbb{R}^n$. Consider an eigenvalue problem of a system of n elliptic equations:

$$\begin{cases} \Delta \mathbf{u} + \alpha \operatorname{grad}(\operatorname{div} \mathbf{u}) = -\sigma \mathbf{u}, & \text{in } \Omega, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}, \end{cases} \tag{1.1}$$

where Δ is the Laplacian in $\mathbb{R}^n$, $\mathbf{u} = (u_1, u_2, \dots, u_n)$ is a vector-valued function from Ω to $\mathbb{R}^n$, α is a non-negative constant, $\operatorname{div} \mathbf{u}$ denotes the divergence of $\mathbf{u}$ and $\operatorname{grad} f$ is the gradient of a function f. Let

$$0 < \sigma_1 \leqslant \sigma_2 \leqslant \cdots \leqslant \sigma_k \leqslant \cdots \to \infty$$

be the eigenvalues of the problem (1.1). Here each eigenvalue is repeated according to its multiplicity. When $n = 3$, the problem (1.1) describes the behavior of the elastic vibration [32].

Interesting estimates for the eigenvalues of (1.1) have been done during the past years. In 1985, Levine and Protter [28] proved

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$$\sum_{i=1}^{k}\sigma_i \geqslant \frac{4\pi^2 n}{n+2}\frac{k^{1+2/n}}{(V\omega_{n-1})^{2/n}}, \quad \text{for } k=1,2,\ldots, \tag{1.2}$$

where ω_{n-1} is the volume of the $(n-1)$-dimensional unit sphere. Furthermore, Hook [25] has studied universal inequalities for eigenvalues of (1.1) and proved

$$\sum_{i=1}^{k}\frac{\sigma_i}{\sigma_{k+1}-\sigma_i} \geqslant \frac{n^2 k}{4(n+\alpha)}, \quad \text{for } k=1,2,\ldots. \tag{1.3}$$

The method given by Hook to prove the above inequality is abstract. Levitin and Parnovski [29] have obtained

$$\sigma_{k+1}-\sigma_k \leqslant \frac{\max\{4+\alpha^2; (n+2)\alpha+8\}}{n+\alpha}\frac{1}{k}\sum_{i=1}^{k}\sigma_i, \quad \text{for } k=1,2,\ldots. \tag{1.4}$$

Recently, by making use of a direct and explicit method, Cheng and Yang [14] have proved the following universal inequality of Yang type:

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i) \leqslant \frac{2\sqrt{n+\alpha}}{n}\left\{\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{\frac{1}{2}}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{\frac{1}{2}}\sigma_i\right\}^{\frac{1}{2}}. \tag{1.5}$$

In this paper, we strengthen the above Cheng–Yang's inequality.

Theorem 1.1. *Let Ω be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$. Eigenvalues of the eigenvalue problem* (1.1) *satisfy*

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \leqslant \min\left\{\frac{4(n+\alpha)}{n^2}, \frac{A(n,\alpha)}{n+\alpha}\right\}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\sigma_i, \tag{1.6}$$

where $A(n,\alpha)$ is defined by

$$A(n,\alpha) = \begin{cases} 4+\alpha^2, & \text{if } \alpha \geqslant \frac{n+2+\sqrt{(n+2)^2+16}}{2}, \\ \frac{8+(n+2)\alpha}{1+L}, & \text{if } 0 \leqslant \alpha < \frac{n+2+\sqrt{(n+2)^2+16}}{2}, \end{cases}$$

with $L = \frac{\{4+(n+2)\alpha-\alpha^2\}n^2}{4(n+\alpha)^2} > 0$, σ_i denotes the ith eigenvalue of (1.1)*.*

Corollary 1.2. *Under the same assumptions as in Theorem* 1.1*, we have*

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \leqslant \frac{4(n+\alpha)}{n^2}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\sigma_i. \tag{1.7}$$

We will show in the next section that (1.7) implies (1.5).

Remark 1.1. For $\alpha=0$, our result becomes the sharper inequality of Yang [37] (cf. [13]). Universal inequalities of Payne–Pólya–Weinberger–Yang type for eigenvalues of elliptic operators on Riemannian manifolds have been studied recently by many mathematicians. One can find various interesting results in this direction, e.g., in [1–3,5–13,15,17–31,34–37], etc.

The inequality (1.6) is a quadratic inequality of σ_{k+1}. By solving it, one can get an explicit upper bound on σ_{k+1} in terms of $\sigma_1,\ldots,\sigma_k$.

Corollary 1.3. *From Theorem* 1.1*, it is not hard to obtain the following simple inequality*

$$\sigma_{k+1} \leqslant \left(1+\min\left\{\frac{4(n+\alpha)}{n^2}, \frac{A(n,\alpha)}{n+\alpha}\right\}\right)\frac{1}{k}\sum_{i=1}^{k}\sigma_i$$

and the gap of any consecutive eigenvalues

$$\sigma_{k+1}-\sigma_k \leqslant \min\left\{\frac{4(n+\alpha)}{n^2}, \frac{A(n,\alpha)}{n+\alpha}\right\}\frac{1}{k}\sum_{i=1}^{k}\sigma_i.$$

For lower order eigenvalues of the eigenvalue problem (1.1), Yang and the second author [14] proved the following

$$\sigma_2 + \sigma_3 + \cdots + \sigma_{n+1} \leqslant n\sigma_1 + 4(1+\alpha)\sigma_1. \tag{1.8}$$

Combining Theorem 1.1 and (1.8), we can derive an upper bound for eigenvalue σ_{k+1}.

Corollary 1.4. *Under the same assumptions as in Theorem* 1.1, *we have*

$$\sigma_{k+1} \leqslant \left(1 + \frac{a(n)(n+\alpha)}{n^2}\right) k^{\frac{2(n+\alpha)}{n^2}} \sigma_1,$$

where $a(n) \leqslant 4$ can be explicitly given.

Proof. From (1.7) and (1.8), our result is proved by applying the recursion formula of Cheng and Yang [13] to our case. We need to notice that the recursion formula of Cheng and Yang [13] does hold for any positive real number n. In our case, it is $\frac{n^2}{n+\alpha}$. □

The classical Lichnerowicz–Obata theorem states that if M is an n-dimensional complete connected Riemannian manifold with Ricci curvature bounded below by $(n-1)$ then the first non-zero eigenvalue of the Laplacian of M is bigger than or equal to n with equality holding if and only if M is isometric to a unit n-sphere (cf. [4]). In 1977, Reilly obtained a similar result for the first Dirichlet eigenvalue of the Laplacian of compact manifold with boundary. Reilly's theorem can be stated as follows. Let M be an $n(\geqslant 2)$-dimensional compact connected Riemannian manifold with boundary ∂M. Assume that the Ricci curvature of M is bounded below by $(n-1)$. If the mean curvature of ∂M is non-negative, then the first Dirichlet eigenvalue λ_1 of the Laplacian of M satisfies $\lambda_1 \geqslant n$ with equality holding if and only M is isometric to an n-dimensional Euclidean unit semi-sphere (cf. [33]). A similar estimate for the first non-zero Neumann eigenvalue of the Laplacian of the same manifolds has been obtained in [16] and [38] independently.

The second part of this paper is to estimate lower bounds for the first eigenvalue of four kinds of eigenvalue problems of the biharmonic operator on compact manifolds with boundary and positive Ricci curvature. The first two results in this direction concern the clamped and the buckling problem.

Theorem 1.5. *Let $(M, \langle,\rangle)$ be an $n(\geqslant 2)$-dimensional compact connected Riemannian manifold with boundary ∂M and denote by ν the outward unit normal vector field of ∂M. Assume that the Ricci curvature of M is bounded below by $(n-1)$. Let λ_1 be the first eigenvalue with Dirichlet boundary condition of the Laplacian of M and let Γ_1 be the first eigenvalue of the clamped plate problem on M*:

$$\begin{cases} \Delta^2 u = \Gamma u & \text{in } M, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M. \end{cases} \tag{1.9}$$

Then we have $\Gamma_1 > n\lambda_1$.

Theorem 1.6. *Assume M satisfies the conditions in Theorem* 1.5 *and let Λ_1 be the first eigenvalue of the following buckling problem*:

$$\begin{cases} \Delta^2 u = -\Lambda \Delta u & \text{in } M, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M. \end{cases} \tag{1.10}$$

Then $\Lambda_1 > n$.

We then consider two different eigenvalue problems of the biharmonic operator and obtain sharp lower bound for the first eigenvalues of them.

Theorem 1.7. *Let $(M, \langle,\rangle)$ be an $n(\geqslant 2)$-dimensional compact connected Riemannian manifold with boundary ∂M. Assume that the Ricci curvature of M is bounded below by $(n-1)$ and that the mean curvature of ∂M is non-negative. Let λ_1 be the first eigenvalue with Dirichlet boundary condition of the Laplacian of M and let p_1 be the first eigenvalue of the following problem*:

$$\begin{cases} \Delta^2 u = pu & \text{in } M, \\ u = \frac{\partial^2 u}{\partial \nu^2} = 0 & \text{on } \partial M. \end{cases} \tag{1.11}$$

Then $p_1 \geqslant n\lambda_1$ with equality holding if and only if M is isometric to an n-dimensional Euclidean unit semi-sphere.

Theorem 1.8. *Assume M satisfies the conditions in Theorem* 1.7 *and let q_1 be the first eigenvalue of the following problem*:

$$\begin{cases} \Delta^2 u = -q\Delta u & \text{in } M, \\ u = \frac{\partial^2 u}{\partial \nu^2} = 0 & \text{on } \partial M. \end{cases} \tag{1.12}$$

Then $q_1 \geqslant n$ with equality holding if and only if M is isometric to an n-dimensional Euclidean unit semi-sphere.

2. Proof of Theorem 1.1

In this section, we shall prove the inequality (1.6) and show that this inequality implies Cheng–Yang's inequality (1.5). Firstly, we give some general estimates for eigenvalues of the problem (1.1).

Lemma 2.1. *Let Ω be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$. Let σ_i denote the ith eigenvalue of the eigenvalue problem* (1.1) *and $\mathbf{u}_i$ be the orthonormal vector-valued eigenfunction corresponding to σ_i. For any function $f \in C^2(\Omega) \cap C^1(\bar{\Omega})$, we have*

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\left\{\int_\Omega |\operatorname{grad} f|^2|\mathbf{u}_i|^2+\alpha\int_\Omega |\operatorname{grad} f\cdot\mathbf{u}_i|^2\right\}$$
$$\leqslant \sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\left\|2\operatorname{grad} f\cdot\operatorname{grad}(\mathbf{u}_i)+\Delta f\mathbf{u}_i+\alpha\left\{\operatorname{grad}(\operatorname{grad} f\cdot\mathbf{u}_i)+\operatorname{div}(\mathbf{u}_i)\operatorname{grad} f\right\}\right\|^2$$

and, for any positive constant B,

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\left\{(1-B)\int |\operatorname{grad} f|^2|\mathbf{u}_i|^2-B\alpha\int |\operatorname{grad} f\cdot\mathbf{u}_i|^2\right\}$$
$$\leqslant \frac{1}{B}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\left\|\operatorname{grad} f\cdot\operatorname{grad}(\mathbf{u}_i)+\frac{1}{2}\Delta f\mathbf{u}_i\right\|^2, \tag{2.1}$$

where $\operatorname{grad} f\cdot\operatorname{grad}(\mathbf{u}_i)$ *is defined by*

$$\operatorname{grad} f\cdot\operatorname{grad}(\mathbf{u}_i)=\left(\operatorname{grad} f\cdot\operatorname{grad}\left(u_i^1\right),\operatorname{grad} f\cdot\operatorname{grad}\left(u_i^2\right),\dots,\operatorname{grad} f\cdot\operatorname{grad}\left(u_i^n\right)\right).$$

Proof. Since $\mathbf{u}_i$ is the orthonormal vector-valued eigenfunction corresponding to the ith eigenvalue σ_i, $\mathbf{u}_i$ satisfies

$$\begin{cases}\Delta\mathbf{u}_i+\alpha\operatorname{grad}(\operatorname{div}(\mathbf{u}_i))=-\sigma_i\mathbf{u}_i, & \text{in } \Omega,\\ \mathbf{u}_i|_{\partial\Omega}=\mathbf{0}, & \\ \int_\Omega \mathbf{u}_i\cdot\mathbf{u}_j=\delta_{ij}, & \text{for any } i, j.\end{cases} \tag{2.2}$$

Defining vector-valued functions $\mathbf{v}_i$ by

$$\mathbf{v}_i=f\mathbf{u}_i-\sum_{j=1}^{k}a_{ij}\mathbf{u}_j, \tag{2.3}$$

where $a_{ij}=\int_\Omega f\mathbf{u}_i\cdot\mathbf{u}_j=a_{ji}$, we have

$$\mathbf{v}_i|_{\partial\Omega}=\mathbf{0},\qquad \int_\Omega \mathbf{u}_j\cdot\mathbf{v}_i=0,\quad \text{for any } i, j=1,\dots,k. \tag{2.4}$$

It then follows from the Rayleigh–Ritz inequality (cf. [26]) that

$$\sigma_{k+1}\leqslant\frac{\int_\Omega\{-\Delta\mathbf{v}_i\cdot\mathbf{v}_i+\alpha(\operatorname{div}(\mathbf{v}_i))^2\}}{\int_\Omega|\mathbf{v}_i|^2}. \tag{2.5}$$

From the definition of $\mathbf{v}_i$, we derive

$$\begin{aligned}\Delta\mathbf{v}_i&=\Delta(f\mathbf{u}_i)-\sum_{j=1}^{k}a_{ij}\Delta\mathbf{u}_j\\&=f\Delta\mathbf{u}_i+2\operatorname{grad} f\cdot\operatorname{grad}(\mathbf{u}_i)+\Delta f\mathbf{u}_i-\sum_{j=1}^{k}a_{ij}\Delta\mathbf{u}_j\\&=f\left(-\sigma_i\mathbf{u}_i-\alpha\operatorname{grad}\left(\operatorname{div}(\mathbf{u}_i)\right)\right)+2\operatorname{grad} f\cdot\operatorname{grad}(\mathbf{u}_i)+\Delta f\mathbf{u}_i-\sum_{j=1}^{k}a_{ij}\left(-\sigma_j\mathbf{u}_j-\alpha\operatorname{grad}\left(\operatorname{div}(\mathbf{u}_j)\right)\right)\\&=-\sigma_i f\mathbf{u}_i+\sum_{j=1}^{k}a_{ij}\sigma_j\mathbf{u}_j+2\operatorname{grad} f\cdot\operatorname{grad}(\mathbf{u}_i)+\Delta f\mathbf{u}_i-\alpha f\operatorname{grad}\left(\operatorname{div}(\mathbf{u}_i)\right)+\alpha\sum_{j=1}^{k}a_{ij}\operatorname{grad}\left(\operatorname{div}(\mathbf{u}_j)\right).\end{aligned}$$

Therefore, we have

$$\int_\Omega -\Delta \mathbf{v}_i \cdot \mathbf{v}_i = \sigma_i \|\mathbf{v}_i\|^2 - \int_\Omega \big(2\,\mathrm{grad}\, f \cdot \mathrm{grad}(\mathbf{u}_i) + \Delta f \mathbf{u}_i\big) \cdot \mathbf{v}_i + \alpha\left(\int_\Omega f\, \mathrm{grad}\big(\mathrm{div}(\mathbf{u}_i)\big) \cdot \mathbf{v}_i - \sum_{j=1}^k a_{ij} \int_\Omega \mathrm{grad}\big(\mathrm{div}(\mathbf{u}_j)\big) \cdot \mathbf{v}_i\right). \tag{2.6}$$

From Stokes' theorem, we infer

$$\begin{aligned}
&\int_\Omega f\, \mathrm{grad}\big(\mathrm{div}(\mathbf{u}_i)\big) \cdot \mathbf{v}_i - \sum_{j=1}^k a_{ij} \int_\Omega \mathrm{grad}\big(\mathrm{div}(\mathbf{u}_j)\big) \cdot \mathbf{v}_i \\
&\quad = -\int_\Omega \big(\mathrm{div}(\mathbf{v}_i)\big)^2 + \int_\Omega \big(\mathrm{div}(\mathbf{v}_i)\, \mathrm{grad}\, f \cdot \mathbf{u}_i - \mathrm{div}(\mathbf{u}_i)\, \mathrm{grad}\, f \cdot \mathbf{v}_i\big) \\
&\quad = -\int_\Omega \big(\mathrm{div}(\mathbf{v}_i)\big)^2 - \int_\Omega \big(\mathrm{grad}(\mathrm{grad}\, f \cdot \mathbf{u}_i) + \mathrm{div}(\mathbf{u}_i)\, \mathrm{grad}\, f\big) \cdot \mathbf{v}_i.
\end{aligned}$$

From (2.5) and (2.6), we have

$$(\sigma_{k+1} - \sigma_i)\|\mathbf{v}_i\|^2 \leqslant -\int_\Omega \big\{2\,\mathrm{grad}\, f \cdot \mathrm{grad}(\mathbf{u}_i) + \Delta f \mathbf{u}_i + \alpha\big(\mathrm{grad}(\mathrm{grad}\, f \cdot \mathbf{u}_i) + \mathrm{div}(\mathbf{u}_i)\, \mathrm{grad}\, f\big)\big\} \cdot \mathbf{v}_i. \tag{2.7}$$

Define

$$b_{ij} = \int_\Omega \left(\mathrm{grad}\, f \cdot \mathrm{grad}(\mathbf{u}_i) + \frac{1}{2}\Delta f \mathbf{u}_i\right) \cdot \mathbf{u}_j = -b_{ji}. \tag{2.8}$$

From (2.2), we derive

$$\begin{aligned}
b_{ij} &= \int_\Omega \left(\mathrm{grad}\, f \cdot \mathrm{grad}(\mathbf{u}_i) + \frac{1}{2}\Delta f \mathbf{u}_i\right) \cdot \mathbf{u}_j \\
&= \frac{1}{2}\int_\Omega \big(\Delta(f\mathbf{u}_i) - f\Delta\mathbf{u}_i\big)\mathbf{u}_j \\
&= \frac{1}{2}\int_\Omega f\mathbf{u}_i \Delta\mathbf{u}_j + (-\Delta\mathbf{u}_i) f\mathbf{u}_j \\
&= -\frac{1}{2}\int_\Omega f\mathbf{u}_i\big(\sigma_j\mathbf{u}_j + \alpha\, \mathrm{grad}(\mathrm{div}\, \mathbf{u}_j)\big) + \frac{1}{2}\int_\Omega f\mathbf{u}_j\big(\sigma_i\mathbf{u}_i + \alpha\, \mathrm{grad}(\mathrm{div}\, \mathbf{u}_i)\big) \\
&= \frac{1}{2}(\sigma_i - \sigma_j)a_{ij} + \frac{1}{2}\alpha \int_\Omega \big(\mathrm{grad}\, f \cdot \mathbf{u}_i\, \mathrm{div}(\mathbf{u}_j) - \mathrm{div}(\mathbf{u}_i)\, \mathrm{grad}\, f \cdot \mathbf{u}_j\big).
\end{aligned}$$

Hence, we have

$$2b_{ij} = (\sigma_i - \sigma_j)a_{ij} + \alpha \int_\Omega \big(\mathrm{grad}\, f \cdot \mathbf{u}_i\, \mathrm{div}(\mathbf{u}_j) - \mathrm{div}(\mathbf{u}_i)\, \mathrm{grad}\, f \cdot \mathbf{u}_j\big). \tag{2.9}$$

By a simple calculation, we have, from (2.3) and (2.8),

$$\int_\Omega \big(2\,\mathrm{grad}\, f \cdot \mathrm{grad}(\mathbf{u}_i) + \Delta f \mathbf{u}_i\big) \cdot \mathbf{v}_i = -\int_\Omega |\mathrm{grad}\, f|^2 |\mathbf{u}_i|^2 - 2\sum_{j=1}^k a_{ij} b_{ij}, \tag{2.10}$$

$$\int_\Omega \big(\mathrm{grad}(\mathrm{grad}\, f \cdot \mathbf{u}_i) + \mathrm{div}(\mathbf{u}_i)\, \mathrm{grad}\, f\big) \cdot \mathbf{v}_i = \sum_{j=1}^k a_{ij} \int_\Omega \big(\mathrm{grad}\, f \cdot \mathbf{u}_i\, \mathrm{div}(\mathbf{u}_j) - \mathrm{div}(\mathbf{u}_i)\, \mathrm{grad}\, f \cdot \mathbf{u}_j\big) - \int_\Omega |\mathrm{grad}\, f \cdot \mathbf{u}_i|^2. \tag{2.11}$$

Putting

$$w_i = -\int_\Omega \big\{2\,\mathrm{grad}\, f \cdot \mathrm{grad}(\mathbf{u}_i) + \Delta f\,\mathbf{u}_i + \alpha\big(\mathrm{grad}(\mathrm{grad}\, f \cdot \mathbf{u}_i) + \mathrm{div}(\mathbf{u}_i)\,\mathrm{grad}\, f\big)\big\} \cdot \mathbf{v}_i,$$

we derive from (2.9)–(2.11) that

$$w_i = \int_\Omega |\,\mathrm{grad}\, f|^2 |\mathbf{u}_i|^2 + \sum_{j=1}^{k} (\sigma_i - \sigma_j) a_{ij}^2 + \alpha \int_\Omega |\,\mathrm{grad}\, f \cdot \mathbf{u}_i|^2. \tag{2.12}$$

We infer, from (2.7) and (2.12),

$$(\sigma_{k+1} - \sigma_i)\|\mathbf{v}_i\|^2 \leqslant w_i. \tag{2.13}$$

On the other hand, from (2.4), (2.9) and the inequality of Cauchy–Schwarz, we have

$$\begin{aligned}
w_i^2 &= \left(-\int_\Omega \left\{2\,\mathrm{grad}\, f \cdot \mathrm{grad}(\mathbf{u}_i) + \Delta f\,\mathbf{u}_i + \alpha\big\{\mathrm{grad}(\mathrm{grad}\, f \cdot \mathbf{u}_i) + \mathrm{div}(\mathbf{u}_i)\,\mathrm{grad}\, f\big\} - \sum_{j=1}^{k} (\sigma_i - \sigma_j) a_{ij}\mathbf{u}_j\right\} \cdot \mathbf{v}_i\right)^2 \\
&\leqslant \|\mathbf{v}_i\|^2 \left\|2\,\mathrm{grad}\, f \cdot \mathrm{grad}(\mathbf{u}_i) + \Delta f\,\mathbf{u}_i + \alpha\big\{\mathrm{grad}(\mathrm{grad}\, f \cdot \mathbf{u}_i) + \mathrm{div}(\mathbf{u}_i)\,\mathrm{grad}\, f\big\} - \sum_{j=1}^{k} (\sigma_i - \sigma_j) a_{ij}\mathbf{u}_j\right\|^2 \\
&= \|\mathbf{v}_i\|^2 \left\{\big\|2\,\mathrm{grad}\, f \cdot \mathrm{grad}(\mathbf{u}_i) + \Delta f\,\mathbf{u}_i + \alpha\big\{\mathrm{grad}(\mathrm{grad}\, f \cdot \mathbf{u}_i) + \mathrm{div}(\mathbf{u}_i)\,\mathrm{grad}\, f\big\}\big\|^2 - \sum_{j=1}^{k} (\sigma_i - \sigma_j)^2 a_{ij}^2\right\}.
\end{aligned}$$

Hence, we infer from (2.13)

$$\begin{aligned}
(\sigma_{k+1} - \sigma_i)^2 w_i^2 &\leqslant (\sigma_{k+1} - \sigma_i) w_i \Big\{\big\|2\,\mathrm{grad}\, f \cdot \mathrm{grad}(\mathbf{u}_i) + \Delta f\,\mathbf{u}_i \\
&\qquad + \alpha\big(\mathrm{grad}(\mathrm{grad}\, f \cdot \mathbf{u}_i) + \mathrm{div}(\mathbf{u}_i)\,\mathrm{grad}\, f\big)\big\|^2 - \sum_{j=1}^{k} (\sigma_i - \sigma_j)^2 a_{ij}^2\Big\}, \\
(\sigma_{k+1} - \sigma_i)^2 w_i &\leqslant (\sigma_{k+1} - \sigma_i) \Big\{\big\|2\,\mathrm{grad}\, f \cdot \mathrm{grad}(\mathbf{u}_i) + \Delta f\,\mathbf{u}_i \\
&\qquad + \alpha\big(\mathrm{grad}(\mathrm{grad}\, f \cdot \mathbf{u}_i) + \mathrm{div}(\mathbf{u}_i)\,\mathrm{grad}\, f\big)\big\|^2 - \sum_{j=1}^{k} (\sigma_i - \sigma_j)^2 a_{ij}^2\Big\}.
\end{aligned} \tag{2.14}$$

Taking sum on i from 1 to k for (2.14), we have, from (2.12) and $a_{ij} = a_{ji}$,

$$\begin{aligned}
&\sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)^2 \left\{\int_\Omega |\,\mathrm{grad}\, f|^2 |\mathbf{u}_i|^2 + \alpha \int_\Omega |\,\mathrm{grad}\, f \cdot \mathbf{u}_i|^2\right\} \\
&\quad\leqslant \sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i) \big\|2\,\mathrm{grad}\, f \cdot \mathrm{grad}(\mathbf{u}_i) + \Delta f\,\mathbf{u}_i + \alpha\big\{\mathrm{grad}(\mathrm{grad}\, f \cdot \mathbf{u}_i) + \mathrm{div}(\mathbf{u}_i)\,\mathrm{grad}\, f\big\}\big\|^2.
\end{aligned}$$

The first inequality of Lemma 2.1 is proved.

For any constant $B > 0$, we infer, from (2.4), (2.10) and (2.13),

$$\begin{aligned}
&(\sigma_{k+1} - \sigma_i)^2 \left(\int_\Omega |\,\mathrm{grad}\, f|^2 |\mathbf{u}_i|^2 + 2\sum_{j=1}^{k} a_{ij} b_{ij}\right) \\
&\quad= (\sigma_{k+1} - \sigma_i)^2 \left\{-2\int_\Omega \left(\mathrm{grad}\, f \cdot \mathrm{grad}(\mathbf{u}_i) + \frac{1}{2}\Delta f\,\mathbf{u}_i - \sum_{j=1}^{k} b_{ij}\mathbf{u}_j\right) \cdot \mathbf{v}_i\right\}
\end{aligned}$$

$$\leqslant (\sigma_{k+1}-\sigma_i)^3 B\|\mathbf{v}_i\|^2 + \frac{\sigma_{k+1}-\sigma_i}{B}\left(\left\|\operatorname{grad} f\cdot\operatorname{grad}(\mathbf{u}_i)+\frac{1}{2}\Delta f\mathbf{u}_i\right\|^2 - \sum_{j=1}^k b_{ij}^2\right)$$

$$\leqslant (\sigma_{k+1}-\sigma_i)^2 B\left(\int_\Omega |\operatorname{grad} f|^2|\mathbf{u}_i|^2 + \sum_{j=1}^k(\sigma_i-\sigma_j)a_{ij}^2 + \alpha\int_\Omega |\operatorname{grad} f\cdot\mathbf{u}_i|^2\right)$$
$$+\frac{\sigma_{k+1}-\sigma_i}{B}\left(\left\|\operatorname{grad} f\cdot\operatorname{grad}(\mathbf{u}_i)+\frac{1}{2}\Delta f\mathbf{u}_i\right\|^2 - \sum_{j=1}^k b_{ij}^2\right). \tag{2.15}$$

Taking sum on i from 1 to k for (2.15), we obtain

$$\sum_{i=1}^k(\sigma_{k+1}-\sigma_i)^2\left(\int_\Omega |\operatorname{grad} f|^2|\mathbf{u}_i|^2 + 2\sum_{j=1}^k a_{ij}b_{ij}\right)$$
$$\leqslant \sum_{i=1}^k(\sigma_{k+1}-\sigma_i)^2 B\left(\int_\Omega |\operatorname{grad} f|^2|\mathbf{u}_i|^2 + \sum_{j=1}^k(\sigma_i-\sigma_j)a_{ij}^2 + \alpha\int_\Omega |\operatorname{grad} f\cdot\mathbf{u}_i|^2\right)$$
$$+\sum_{i=1}^k\frac{\sigma_{k+1}-\sigma_i}{B}\left(\left\|\operatorname{grad} f\cdot\operatorname{grad}(\mathbf{u}_i)+\frac{1}{2}\Delta f\mathbf{u}_i\right\|^2 - \sum_{j=1}^k b_{ij}^2\right).$$

Since a_{ij} is symmetric and b_{ij} is anti-symmetric, we have

$$2\sum_{i,j=1}^k(\sigma_{k+1}-\sigma_i)^2 a_{ij}b_{ij} = -2\sum_{i,j=1}^k(\sigma_{k+1}-\sigma_i)(\sigma_i-\sigma_j)a_{ij}b_{ij},$$
$$\sum_{i,j=1}^k(\sigma_{k+1}-\sigma_i)^2(\sigma_i-\sigma_j)a_{ij}^2 = -\sum_{i,j=1}^k(\sigma_{k+1}-\sigma_i)(\sigma_i-\sigma_j)^2 a_{ij}^2.$$

Therefore, we infer

$$\sum_{i=1}^k(\sigma_{k+1}-\sigma_i)^2\int_\Omega |\operatorname{grad} f|^2|\mathbf{u}_i|^2 \leqslant \sum_{i=1}^k(\sigma_{k+1}-\sigma_i)^2 B\left(\int_\Omega |\operatorname{grad} f|^2|\mathbf{u}_i|^2 + \alpha\int_\Omega |\operatorname{grad} f\cdot\mathbf{u}_i|^2\right)$$
$$+\sum_{i=1}^k\frac{\sigma_{k+1}-\sigma_i}{B}\left(\left\|\operatorname{grad} f\cdot\operatorname{grad}(\mathbf{u}_i)+\frac{1}{2}\Delta f\mathbf{u}_i\right\|^2\right).$$

This finishes the proof of Lemma 2.1. □

Next, we shall give a proof of Theorem 1.1.

Proof of Theorem 1.1. For the standard Euclidean coordinate system $(x^1, x^2, \ldots, x^n)$ in $\mathbb{R}^n$, we have, for any $1 \leqslant \beta \leqslant n$,

$$\operatorname{grad}(x^\beta) = \mathbf{e}_\beta,$$

where $\mathbf{e}_1 = (1, 0, \ldots, 0), \mathbf{e}_2 = (0, 1, \ldots, 0), \mathbf{e}_n = (0, 0, \ldots, 1)$.

Taking $f = x^\beta$ in (2.1) and making sum on β from 1 to n for the resulted inequality, we obtain

$$\sum_{i=1}^k(\sigma_{k+1}-\sigma_i)^2\left\{(1-B)\int\sum_{\beta=1}^n\left|\operatorname{grad}(x^\beta)\right|^2|\mathbf{u}_i|^2 - B\alpha\int\sum_{\beta=1}^n\left|\operatorname{grad}(x^\beta)\cdot\mathbf{u}_i\right|^2\right\}$$
$$\leqslant \frac{1}{B}\sum_{i=1}^k(\sigma_{k+1}-\sigma_i)\sum_{\beta=1}^n\left\|\operatorname{grad}(x^\beta)\cdot\operatorname{grad}(\mathbf{u}_i)+\frac{1}{2}\Delta x^\beta\mathbf{u}_i\right\|^2.$$

A straightforward calculation yields

$$\sum_{\beta=1}^{n}\left|\operatorname{grad}\left(x^{\beta}\right)\right|^{2}=n,$$

$$\sum_{\beta=1}^{n}\left(\operatorname{grad}\left(x^{\beta}\right)\cdot\mathbf{u}_{i}\right)^{2}=|\mathbf{u}_{i}|^{2},$$

$$\sum_{\beta=1}^{n}\left|\operatorname{grad}\left(x^{\beta}\right)\cdot\operatorname{grad}(\mathbf{u}_{i})+\frac{1}{2}\Delta x^{\beta}\mathbf{u}_{i}\right|^{2}=\sum_{\beta=1}^{n}\left|\mathbf{e}_{\beta}\cdot\operatorname{grad}(\mathbf{u}_{i})\right|^{2}.$$

From Stokes' formula, we have

$$\int_{\Omega}\sum_{\beta=1}^{n}\left|\mathbf{e}_{\beta}\cdot\operatorname{grad}(\mathbf{u}_{i})\right|^{2}=\sigma_{i}-\alpha\|\operatorname{div}\mathbf{u}_{i}\|^{2}.$$

Therefore, we infer

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_{i})^{2}\big(n-B(n+\alpha)\big)\leqslant\frac{1}{B}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_{i})\left(\sigma_{i}-\alpha\left\|\operatorname{div}(\mathbf{u}_{i})\right\|^{2}\right).$$

Putting

$$B=\frac{\sqrt{\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_{i})(\sigma_{i}-\alpha\|\operatorname{div}(\mathbf{u}_{i})\|^{2})}}{\sqrt{(n+\alpha)\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_{i})^{2}}},$$

we derive

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_{i})^{2}\leqslant\frac{4(n+\alpha)}{n^{2}}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_{i})\left(\sigma_{i}-\alpha\left\|\operatorname{div}(\mathbf{u}_{i})\right\|^{2}\right).\tag{2.16}$$

Since $\alpha\geqslant 0$, we have

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_{i})^{2}\leqslant\frac{4(n+\alpha)}{n^{2}}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_{i})\sigma_{i}.\tag{2.17}$$

On the other hand, taking $f=x^{\beta}$ in the first inequality of Lemma 2.1, $\beta=1,2,\ldots,n$, we infer

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_{i})^{2}\left(1+\alpha\int_{\Omega}(\mathbf{e}_{\beta}\cdot\mathbf{u}_{i})^{2}\right)\leqslant\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_{i})\left\|2\mathbf{e}_{\beta}\cdot\operatorname{grad}(\mathbf{u}_{i})+\alpha\left(\operatorname{grad}(\mathbf{e}_{\beta}\cdot\mathbf{u}_{i})+\operatorname{div}(\mathbf{u}_{i})\mathbf{e}_{\beta}\right)\right\|^{2}.$$

Taking sum on β from 1 to n, we have

$$(n+\alpha)\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_{i})^{2}\leqslant\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_{i})\sum_{\beta=1}^{n}\left\|2\mathbf{e}_{\beta}\cdot\operatorname{grad}(\mathbf{u}_{i})+\alpha\left(\operatorname{grad}(\mathbf{e}_{\beta}\cdot\mathbf{u}_{i})+\operatorname{div}(\mathbf{u}_{i})\mathbf{e}_{\beta}\right)\right\|^{2}.$$

By a simple and direct computation, we infer

$$\sum_{\beta=1}^{n}\left\|2\mathbf{e}_{\beta}\cdot\operatorname{grad}(\mathbf{u}_{i})+\alpha\left(\operatorname{grad}(\mathbf{e}_{\beta}\cdot\mathbf{u}_{i})+\operatorname{div}(\mathbf{u}_{i})\mathbf{e}_{\beta}\right)\right\|^{2}=\left(4+\alpha^{2}\right)\sigma_{i}-\alpha\left(\alpha^{2}-(n+2)\alpha-4\right)\left\|\operatorname{div}(\mathbf{u}_{i})\right\|^{2}.$$

Hence, we have

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_{i})^{2}\leqslant\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_{i})\left(\frac{4+\alpha^{2}}{n+\alpha}\sigma_{i}-\alpha\frac{\alpha^{2}-(n+2)\alpha-4}{n+\alpha}\left\|\operatorname{div}(\mathbf{u}_{i})\right\|^{2}\right).\tag{2.18}$$

For $\alpha\geqslant\frac{n+2+\sqrt{(n+2)^{2}+16}}{2}$, we have $\alpha^{2}-(n+2)\alpha-4\geqslant 0$. Hence,

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_{i})^{2}\leqslant\frac{4+\alpha^{2}}{n+\alpha}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_{i})\sigma_{i}.$$

For $0 \leqslant \alpha < \frac{n+2+\sqrt{(n+2)^2+16}}{2}$, we have $\alpha^2-(n+2)\alpha-4<0$. In this case, from $L=\frac{\{4+(n+2)\alpha-\alpha^2\}n^2}{4(n+\alpha)^2}>0$, (2.16) and (2.18), we have

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \leqslant \frac{8+(n+2)\alpha}{(n+\alpha)(1+L)}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\sigma_i.$$

Thus, we derive, from the definition of $A(n,\alpha)$,

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \leqslant \frac{A(n,\alpha)}{n+\alpha}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\sigma_i. \tag{2.19}$$

Furthermore, from (2.17) and (2.19), we infer

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \leqslant \min\left\{\frac{4(n+\alpha)}{n^2}, \frac{A(n,\alpha)}{n+\alpha}\right\}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\sigma_i.$$

This completes the proof of Theorem 1.1. □

Remark 2.1. The inequality (1.6) implies Cheng–Yang's inequality (1.5). In order to see this, we need an elementary algebraic inequality.

Lemma 2.2. *Let $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^k$ be two sequences of non-negative real numbers with $\{a_i\}_{i=1}^k$ decreasing and $\{b_i\}_{i=1}^k$ increasing. Then for any fixed $s \geqslant 1$, we have*

$$\left(\sum_{i=1}^{k}a_i^s\right)\left(\sum_{i=1}^{k}a_i^2 b_i\right) \leqslant \left(\sum_{i=1}^{k}a_i^{s+1}\right)\left(\sum_{i=1}^{k}a_i b_i\right). \tag{2.20}$$

Proof. When $k=1$, (2.20) holds trivially. Suppose that (2.20) holds when $k=m$, that is,

$$\left(\sum_{i=1}^{m}a_i^s\right)\left(\sum_{i=1}^{m}a_i^2 b_i\right) \leqslant \left(\sum_{i=1}^{m}a_i^{s+1}\right)\left(\sum_{i=1}^{m}a_i b_i\right). \tag{2.21}$$

Then when $k=m+1$, we have by using (2.21) and the hypothesis on $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^k$ that

$$\begin{aligned}
&\sum_{i=1}^{m+1}a_i^{s+1}\sum_{i=1}^{m+1}a_i b_i-\sum_{i=1}^{m+1}a_i^s\sum_{i=1}^{m+1}a_i^2 b_i\\
&=\sum_{i=1}^{m}a_i^{s+1}\sum_{i=1}^{m}a_i b_i-\sum_{i=1}^{m}a_i^s\sum_{i=1}^{m}a_i^2 b_i+a_{m+1}^{s+1}\sum_{i=1}^{m}a_i b_i-a_{m+1}^2 b_{m+1}\sum_{i=1}^{m}a_i^s+a_{m+1}b_{m+1}\sum_{i=1}^{m}a_i^{s+1}-a_{m+1}^s\sum_{i=1}^{m}a_i^2 b_i\\
&\geqslant a_{m+1}^{s+1}\sum_{i=1}^{m}a_i b_i-a_{m+1}^2 b_{m+1}\sum_{i=1}^{m}a_i^s+a_{m+1}b_{m+1}\sum_{i=1}^{m}a_i^{s+1}-a_{m+1}^s\sum_{i=1}^{m}a_i^2 b_i\\
&=\sum_{i=1}^{m}a_{m+1}a_i\left(b_{m+1}a_i^{s-1}-b_i a_{m+1}^{s-1}\right)(a_i-a_{m+1})\geqslant 0.
\end{aligned}$$

Thus (2.20) holds by induction. □

Now let us get (1.5) by using (1.6). Multiplying (1.6) by $(\sum_{i=1}^k(\sigma_{k+1}-\sigma_i))^2$, we get

$$\left(\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\right)^2\left(\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\right) \leqslant \frac{4(n+\alpha)}{n^2}\left(\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\right)^2\left(\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\sigma_i\right). \tag{2.22}$$

Taking $s=2$, $a_i=(\sigma_{k+1}-\sigma_i)^{1/2}$, $b_i=\sigma_i$ in (2.20), we get

$$\left(\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\right)\left(\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\sigma_i\right) \leqslant \left(\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{3/2}\right)\left(\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{1/2}\sigma_i\right). \tag{2.23}$$

Taking $s=3$, $a_i=(\sigma_{k+1}-\sigma_i)^{1/2}$, $b_i\equiv 1$ in (2.20), we have

$$\left(\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{3/2}\right)\left(\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\right)\leqslant\left(\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\right)\left(\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{1/2}\right). \tag{2.24}$$

It is then easy to obtain (1.5) from (2.22)–(2.24). □

3. Proof of Theorems 1.5–1.8

In this section, we will prove Theorems 1.5–1.8. Before doing this, let us recall the Reilly formula. Let M be an n-dimensional compact manifold M with boundary ∂M. We will often write $\langle,\rangle$ the Riemannian metric on M as well as that induced on ∂M. Let ∇ and Δ be the connection and the Laplacian on M, respectively. Let ν be the unit outward normal vector of ∂M. The shape operator of ∂M is given by $S(X)=\nabla_X\nu$ and the second fundamental form of ∂M is defined as $II(X,Y)=\langle S(X),Y\rangle$, here $X,Y\in T\partial M$. The eigenvalues of S are called the principal curvatures of ∂M and the mean curvature H of ∂M is given by $H=\frac{1}{n-1}\operatorname{tr}S$, here $\operatorname{tr}S$ denotes the trace of S. For a smooth function f defined on an n-dimensional compact manifold M with boundary ∂M, the following identity holds if $h=\frac{\partial f}{\partial\nu}|_{\partial M}$, $z=f|_{\partial M}$ and Ric denotes the Ricci tensor of M (cf. [33], p. 46):

$$\int_M\left((\Delta f)^2-\left|\nabla^2 f\right|^2-\operatorname{Ric}(\nabla f,\nabla f)\right)=\int_{\partial M}\left(\left((n-1)Hh+2\overline{\Delta}z\right)h+II(\overline{\nabla}z,\overline{\nabla}z)\right). \tag{3.1}$$

Here $\nabla^2 f$ is the Hessian of f; $\overline{\Delta}$ and $\overline{\nabla}$ represent the Laplacian and the gradient on ∂M with respect to the induced metric on ∂M, respectively.

Proof of Theorem 1.5. Let u be an eigenfunction of the problem (1.9) corresponding to the first eigenvalue Γ_1. That is,

$$\Delta^2 u=\Gamma_1 u\quad\text{in } M,\qquad u=\frac{\partial u}{\partial\nu}=0\quad\text{on } \partial M.$$

Then we have

$$\Gamma_1=\frac{\int_M(\Delta u)^2}{\int_M u^2}. \tag{3.2}$$

Introducing u into Reilly's formula, it follows that

$$\int_M\left((\Delta u)^2-\left|\nabla^2 u\right|^2\right)=\int_M\operatorname{Ric}(\nabla u,\nabla u)\geqslant(n-1)\int_M|\nabla u|^2. \tag{3.3}$$

From the Schwarz inequality, we have

$$\left|\nabla^2 u\right|^2\geqslant\frac{1}{n}(\Delta u)^2 \tag{3.4}$$

with equality holding if and only if

$$\nabla^2 u=\frac{\Delta u}{n}\langle,\rangle.$$

Thus we have from (3.3) and (3.4) that

$$\int_M(\Delta u)^2\geqslant n\int_M|\nabla u|^2. \tag{3.5}$$

Since u is a non-zero function which vanishes on ∂M, we have from the Poincaré inequality that

$$\int_M|\nabla u|^2\geqslant\lambda_1\int_M u^2 \tag{3.6}$$

with equality holding if and only if u is a first eigenfunction of the Dirichlet Laplacian of M.

Combining (3.2), (3.5) and (3.6), we get $\Gamma_1\geqslant n\lambda_1$. Let us show that the case $\Gamma_1=n\lambda_1$ will not occur. Indeed, if $\Gamma_1=n\lambda_1$, then we must have

$$\left|\nabla^2 u\right|^2=\frac{1}{n}(\Delta u)^2,\qquad\operatorname{Ric}(\nabla u,\nabla u)=(n-1)|\nabla u|^2,\quad\Delta u=-\lambda_1 u. \tag{3.7}$$

Hence

$$\Gamma_1 u = \Delta(\Delta u) = \Delta(-\lambda_1 u) = \lambda_1^2 u \quad \text{in } M \tag{3.8}$$

which implies that $\lambda_1 = n$. Consequently, we get from the Bochner formula that

$$\begin{aligned}\frac{1}{2}\Delta\big(|\nabla u|^2 + u^2\big) &= \big|\nabla^2 u\big|^2 + \big\langle \nabla u, \nabla(\Delta u)\big\rangle + \operatorname{Ric}(\nabla u, \nabla u) + |\nabla u|^2 + u\Delta u \\ &= \frac{(\Delta u)^2}{n} - n\langle \nabla u, \nabla u\rangle + (n-1)|\nabla u|^2 + |\nabla u|^2 - nu^2 = 0.\end{aligned}$$

Since $(|\nabla u|^2 + u^2)|_{\partial M} = 0$, we conclude from the maximum principle that $|\nabla u|^2 + u^2 = 0$ on M. This is a contradiction and completes the proof of Theorem 1.5. □

Proof of Theorem 1.6. Let w be an eigenfunction of the problem (1.8) corresponding to the first eigenvalue Λ_1. That is,

$$\Delta^2 w = -\Lambda_1 \Delta w \quad \text{in } M, \qquad w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial M. \tag{3.9}$$

Then we have

$$\Lambda_1 = \frac{\int_M (\Delta w)^2}{\int_M |\nabla w|^2}. \tag{3.10}$$

As in the proof of Theorem 1.5, we have by introducing w into Reilly's formula that

$$\int_M (\Delta w)^2 \geqslant n \int_M |\nabla w|^2 \tag{3.11}$$

with equality holding if and only if

$$\nabla^2 w = \frac{\Delta w}{n}\langle,\rangle \quad \text{and} \quad \operatorname{Ric}(\nabla w, \nabla w) = (n-1)|\nabla w|^2. \tag{3.12}$$

Combining (3.10) and (3.11), we get $\Lambda_1 \geqslant n$. If $\Lambda_1 = n$, then (3.12) holds. Since

$$w|_{\partial M} = \frac{\partial w}{\partial \nu}\bigg|_{\partial M} = 0, \tag{3.13}$$

we have

$$\Delta w|_{\partial M} = \nabla^2 w(\nu, \nu), \tag{3.14}$$

which, combining with $\nabla^2 w = \frac{\Delta w}{n}\langle,\rangle$, implies that

$$\Delta w|_{\partial M} = 0. \tag{3.15}$$

It then follows from the divergence theorem that

$$\int_M \big|\nabla(\Delta w + nw)\big|^2 = -\int_M (\Delta w + nw)\Delta(\Delta w + nw) = -\int_M (\Delta w + nw)\big(\Delta^2 w + n\Delta w\big) = 0.$$

Hence

$$\Delta w + nw = 0 \quad \text{in } M. \tag{3.16}$$

Consequently, we get

$$\begin{aligned}\frac{1}{2}\Delta\big(|\nabla w|^2 + w^2\big) &= \big|\nabla^2 w\big|^2 + \big\langle \nabla w, \nabla(\Delta w)\big\rangle + \operatorname{Ric}(\nabla w, \nabla w) + |\nabla w|^2 + w\Delta w \\ &= \frac{(\Delta w)^2}{n} - n\langle \nabla w, \nabla w\rangle + (n-1)|\nabla w|^2 + |\nabla w|^2 - nw^2 = 0.\end{aligned}$$

We then conclude from $(|\nabla w|^2 + w^2)|_{\partial M} = 0$ and the maximum principle that $|\nabla w|^2 + w^2 = 0$. This is a contradiction and so $\Lambda_1 > n$. The proof of Theorem 1.6 is completed. □

Proof of Theorem 1.7. Let f be the eigenfunction of the problem (1.10) corresponding to the first eigenvalue p_1. That is,

$$\begin{cases} \Delta^2 f = p_1 f & \text{in } M, \\ f = \frac{\partial^2 f}{\partial \nu^2} = 0 & \text{on } \partial M. \end{cases} \tag{3.17}$$

Multiplying (3.17) by f and integrating on M, we have from the divergence theorem that

$$p_1 \int_M f^2 = \int_M f \Delta^2 f = -\int_M \langle \nabla f, \nabla(\Delta f) \rangle = \int_M (\Delta f)^2 - \int_{\partial M} h \Delta f, \tag{3.18}$$

where $h = \frac{\partial f}{\partial \nu}|_{\partial M}$. Since $f|_{\partial M} = \frac{\partial^2 u}{\partial \nu^2}|_{\partial M} = 0$, we have

$$\Delta f|_{\partial M} = (n-1)Hh, \tag{3.19}$$

where H is the mean curvature of ∂M.

Substituting (3.19) into (3.18), we get

$$p_1 = \frac{\int_M (\Delta f)^2 - (n-1)\int_{\partial M} Hh^2}{\int_M f^2}. \tag{3.20}$$

Introducing f into Reilly's formula, we have

$$\int_M \left((\Delta f)^2 - \left|\nabla^2 f\right|^2\right) = \int_M \mathrm{Ric}(\nabla f, \nabla f) + (n-1)\int_{\partial M} Hh^2 \geqslant (n-1)\int_M |\nabla f|^2 + (n-1)\int_{\partial M} Hh^2. \tag{3.21}$$

It follows from the Schwarz inequality that

$$\left|\nabla^2 f\right|^2 \geqslant \frac{1}{n}(\Delta f)^2 \tag{3.22}$$

with equality holding if and only if

$$\nabla^2 f = \frac{\Delta f}{n}\langle , \rangle.$$

Combining (3.21) and (3.22), one gets

$$\int_M (\Delta f)^2 \geqslant n \int_M |\nabla f|^2 + n \int_{\partial M} Hh^2. \tag{3.23}$$

Since $H \geqslant 0$, we have from (3.20) and (3.23) that

$$p_1 \geqslant \frac{\int_M |\nabla f|^2}{\int_M f^2}. \tag{3.24}$$

On the other hand, since f is a nonzero function which vanishes on ∂M, we have from the Poincaré inequality that

$$\int_M |\nabla f|^2 \geqslant \lambda_1 \int_M f^2 \tag{3.25}$$

with equality holding if and only if f is a first eigenfunction of the Dirichlet Laplacian of M. Thus we conclude that $p_1 \geqslant n\lambda_1$. This finishes the proof of the first part of Theorem 1.7. Assume now that $p_1 = n\lambda_1$. In this case, (3.25) should take equality sign which implies that f is a first eigenfunction corresponding to the first eigenvalue λ_1 of the Dirichlet Laplacian of M. That is, we have

$$\Delta f = -\lambda_1 f \quad \text{in } M, \qquad f|_{\partial M} = 0. \tag{3.26}$$

It then follows that

$$\Delta^2 f = -\lambda_1 \Delta f = \lambda_1^2 f \quad \text{in } M \tag{3.27}$$

which, combining with (3.17) gives $\lambda_1 = n$. We then conclude from Reilly's theorem as stated before that M is isometric to an n-dimensional unit semi-sphere. Consider now the n-dimensional unit semi-sphere $S_+^n(1)$ given by

$$S_+^n(1) = \left\{ (x_1, \dots, x_{n+1}) \in \mathbf{R}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} x_i^2 = 1,\ x_{n+1} \geqslant 0 \right\}.$$

It is easy to see that the function x_{n+1} on $S_+^n(1)$ is an eigenfunction of the problem (1.11) corresponding to the eigenvalue n^2 and the first Dirichlet eigenvalue λ_1 of $S_+^n(1)$ is n. Thus the first eigenvalue of the problem (1.11) of $S_+^n(1)$ is $n\lambda_1$. This completes the proof of Theorem 1.7. □

Proof of Theorem 1.8. The discussion is similar to the proof of Theorem 1.7. For the sake of completeness, we include it. Let g be the eigenfunction of the problem (1.12) corresponding to the first eigenvalue q_1:

$$\begin{cases} \Delta^2 g = -q_1 \Delta g & \text{in } M, \\ g = \frac{\partial^2 g}{\partial \nu^2} = 0 & \text{on } \partial M. \end{cases} \tag{3.28}$$

Multiplying (3.28) by g and integrating on M, we have from the divergence theorem that

$$p_1 \int_M |\nabla g|^2 = \int_M (\Delta g)^2 - \int_{\partial M} s\Delta g, \tag{3.29}$$

where $s = \frac{\partial g}{\partial \nu}|_{\partial M}$. Also, we have

$$\Delta g|_{\partial M} = (n-1)Hs. \tag{3.30}$$

Hence

$$q_1 = \frac{\int_M (\Delta g)^2 - (n-1)\int_{\partial M} Hs^2}{\int_M |\nabla g|^2}. \tag{3.31}$$

Introducing g into Reilly's formula and using Schwarz inequality, we have

$$\int_M (\Delta g)^2 \geqslant n \int_M |\nabla g|^2 + n \int_{\partial M} Hs^2. \tag{3.32}$$

Consequently, we get $q_1 \geqslant n$. In the case that $q_1 = n$, we must have $Hs = 0$ and so we know from (3.30) that $\Delta g|_{\partial M} = 0$. Observe that Δg is not a zero function on M since otherwise g would be identically zero by the maximum principle. Thus Δg is an eigenfunction corresponding the eigenvalue n of the Dirichlet Laplacian of M. It then follows from Reilly's theorem that M is isometric to an n-dimensional Euclidean semi-sphere. One can check that the function x_{n+1} given in the proof of Theorem 1.7 is an eigenfunction of the problem (1.11) for the n-dimensional unit semi-sphere corresponding to the eigenvalue n. This completes the proof of Theorem 1.8. □

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