TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 358, Number 6, Pages 2625–2635
S 0002-9947(05)04023-7
Article electronically published on October 31, 2005

INEQUALITIES FOR EIGENVALUES OF A CLAMPED PLATE PROBLEM

QING-MING CHENG AND HONGCANG YANG

Abstract. Let D be a connected bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$. Assume that

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$$

are eigenvalues of a clamped plate problem or an eigenvalue problem for the Dirichlet biharmonic operator:

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } D, \\ u|_{\partial D} = \left.\dfrac{\partial u}{\partial n}\right|_{\partial D} = 0. \end{cases}$$

Then, we give an upper bound of the $(k+1)$-th eigenvalue λ_{k+1} in terms of the first k eigenvalues, which is independent of the domain D, that is, we prove the following:

$$\lambda_{k+1} \leq \frac{1}{k}\sum_{i=1}^{k}\lambda_i + \left[\frac{8(n+2)}{n^2}\right]^{1/2} \frac{1}{k}\sum_{i=1}^{k}\Big[\lambda_i(\lambda_{k+1}-\lambda_i)\Big]^{1/2}.$$

Further, a more explicit inequality of eigenvalues is also obtained.

1. Introduction

Let $\mathbb{R}^n$ denote an n-dimensional Euclidean space and let D be a connected bounded domain in $\mathbb{R}^n$. An eigenvalue problem of a fixed membrane or Dirichlet Laplacian on a bounded domain D in $\mathbb{R}^n$ is the following:

$$\begin{cases} \Delta u = -\lambda u, \ \text{in } D, \\ u|_{\partial D} = 0, \end{cases} \tag{1.1}$$

where Δ is the Lapalacian in $\mathbb{R}^n$.

It is well known that this problem has a real and purely discrete spectrum

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty.$$

Received by the editors December 10, 2002 and, in revised form, July 13, 2004.

2000 *Mathematics Subject Classification.* Primary 35P15, 58G25.

Key words and phrases. Eigenvalue, inequality of eigenvalue, biharmonic operator, clamped plate problem.

The first author's research was partially supported by a Grant-in-Aid for Scientific Research from the Japan Society for the Promotion of Science.

The second author's research was partially supported by the NSF of China and the Fund of CAS.

Here each eigenvalue is repeated from its multiplicity. When $n = 2$, in 1955 and 1956, Payne, Pólya and Weinberger proved that, in [10] and [11],

$$\frac{\lambda_2}{\lambda_1} \leq 3 \quad \text{for } D \subset \mathbb{R}^2,$$

and they conjectured

$$\frac{\lambda_2}{\lambda_1} \leq \frac{\lambda_2}{\lambda_1}\Big|_{\text{disk}} \approx 2.5387$$

with equality if and only if D is a disk. For general $n \geq 2$, an analogous statement is

$$\frac{\lambda_2}{\lambda_1} \leq 1 + \frac{4}{n} \quad \text{for } D \subset \mathbb{R}^n,$$

and the conjecture of Payne, Pólya and Weinberger is

$$\frac{\lambda_2}{\lambda_1} \leq \frac{\lambda_2}{\lambda_1}\Big|_{n\text{-ball}}$$

with equality if and only if D is an n-ball. In their excellent papers [3], [4] and [5], Ashbaugh and Benguria solved this important conjecture of Payne, Pólya and Weinberger.

On the other hand, for estimates of higher eigenvalues, Payne, Pólya and Weinberger in [11] proved

$$\lambda_{m+1} - \lambda_m \leq \frac{2}{m}\sum_{i=1}^{m} \lambda_i, \ \ m = 1, 2, \cdots, \tag{1.2}$$

for $D \subset \mathbb{R}^2$. For general $n \geq 2$, we have

$$\lambda_{m+1} - \lambda_m \leq \frac{4}{mn}\sum_{i=1}^{m} \lambda_i, \ \ m = 1, 2, \cdots, \tag{1.3}$$

for $D \subset \mathbb{R}^n$. Although these results introduced by Payne, Pólya and Weinberger have been extended by many authors, a result of Hile and Protter in [7] and a result of the second author in [12] are two main developments. Namely, in 1980, Hile and Protter [7] proved

$$\sum_{i=1}^{m} \frac{\lambda_i}{\lambda_{m+1} - \lambda_i} \geq \frac{mn}{4}, \ \text{ for } m = 1, 2, \cdots. \tag{1.4}$$

It is not hard to check that the inequality (1.4) of Hile and Protter is sharper than the inequality (1.3) of Payne, Pólya and Weinberger. In 1991, Yang [12] obtained very sharp inequalities, that is, he derived

$$\sum_{i=1}^{m}\Big(\lambda_{m+1} - \lambda_i\Big)\Big(\lambda_{m+1} - (1 + \frac{4}{n})\lambda_i\Big) \leq 0, \ \text{ for } m = 1, 2, \cdots. \tag{1.5}$$

According to the inequality, we can infer

$$\lambda_{m+1} \leq \frac{1}{m}(1 + \frac{4}{n})\sum_{i=1}^{m} \lambda_i, \ \text{ for } m = 1, 2, \cdots. \tag{1.6}$$

It is easy to prove that Yang's inequalities (1.5) and (1.6) are sharper than the inequality (1.4) of Hile and Protter (see [1] and [2] for details).

On the other hand, in order to describe vibrations of a clamped plate, we must consider an eigenvalue problem for Dirichlet biharmonic operator, called *a clamped plate problem*:

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } D, \\ u|_{\partial D} = \left.\dfrac{\partial u}{\partial n}\right|_{\partial D} = 0, \end{cases} \tag{1.7}$$

where Δ is the Laplacian in $\mathbb{R}^n$ and Δ^2 is the biharmonic operator in $\mathbb{R}^n$.

For this clamped plate problem, in 1956, Payne, Pólya and Weinberger [11] also established an inequality for the biharmonic operator Δ^2. They obtained

$$\lambda_{k+1} - \lambda_k \leq \frac{8(n+2)}{n^2} \frac{1}{k} \sum_{i=1}^{k} \lambda_i. \tag{1.8}$$

As a generalization of their result, in 1984, Hile and Yeh [8] obtained

$$\sum_{i=1}^{k} \frac{\lambda_i^{1/2}}{\lambda_{k+1} - \lambda_i} \geq \frac{n^2 k^{3/2}}{8(n+2)} \left(\sum_{i=1}^{k} \lambda_i \right)^{-1/2}, \tag{1.9}$$

by making use of an improved method of Hile and Protter [7]. Furthermore, in 1990, Hook [9], Chen and Qian [6] proved, independently, the following inequality:

$$\frac{n^2 k^2}{8(n+2)} \leq \left[\sum_{i=1} \frac{\lambda_i^{1/2}}{\lambda_{k+1} - \lambda_i} \right] \sum_{i=1}^{k} \lambda_i^{1/2}. \tag{1.10}$$

Recently, in [1], a survey paper on recent developments of eigenvalue problems, Ashbaugh pointed out whether one can establish inequalities for eigenvalues of the vibrating clamped plate problem which are analogous inequalities of Yang in the case of the eigenvalue problem of the Laplacian with Dirichlet boundary condition.

In this paper, we shall give an affirmative answer for the problem introduced by Ashbaugh, that is, we obtain the following:

Theorem 1. *Let λ_i denote the i-th eigenvalue of the clamped plate problem*

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } D, \\ u|_{\partial D} = \left.\dfrac{\partial u}{\partial n}\right|_{\partial D} = 0, \end{cases} \tag{1.11}$$

where D is a connected bounded domain in $\mathbb{R}^n$. Then we have

$$\lambda_{k+1} - \frac{1}{k} \sum_{i=1}^{k} \lambda_i \leq \left[\frac{8(n+2)}{n^2} \right]^{1/2} \frac{1}{k} \sum_{i=1}^{k} \left[\lambda_i (\lambda_{k+1} - \lambda_i) \right]^{1/2}. \tag{1.12}$$

From Theorem 1, we can conclude the following more explicit inequality which is weaker than (1.12).

Corollary 1. *Under the assumption of Theorem* 1, *we have*

$$\begin{aligned}\lambda_{k+1} \le & \left[1+\frac{4(n+2)}{n^2}\right]\frac{1}{k}\sum_{i=1}^{k}\lambda_i \\ & + \left\{\left[\frac{4(n+2)}{n^2}\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right]^2 - \frac{8(n+2)}{n^2}\frac{1}{k}\sum_{i=1}^{k}\left(\lambda_i-\frac{1}{k}\sum_{j=1}^{k}\lambda_j\right)^2\right\}^{1/2}.\end{aligned} \tag{1.13}$$

Remark 1. It is obvious that inequalities (1.12) and (1.13) are sharper than the inequality

$$\lambda_{k+1} \le \left[1+\frac{8(n+2)}{n^2}\right]\frac{1}{k}\sum_{i=1}^{k}\lambda_i. \tag{1.14}$$

It is easy to see that inequality (1.14) is better than inequality (1.8) of Payne, Pólya and Weinberger. We shall also discuss the relation between inequality (1.14) and inequality (1.10) introduced by Hook [9], and Chen and Qian [6] in the Remark 2 of Section 2.

2. Proofs of main results

In this section, we shall prove our main results.

Proof of Theorem 1. Let $g = x^p,\ p = 1, \cdots, n$, where $(x^1, x^2, \cdots, x^n)$ are the standard Euclidean coordinates. Let u_i be i-th orthonormal eigenfunction corresponding to eigenvalue λ_i, $i = 1, 2, \cdots, k$, that is, u_i satisfies

$$\begin{cases} \Delta^2 u_i = \lambda_i u_i, & \text{in } D, \\ u_i|_{\partial D} = \left.\dfrac{\partial u_i}{\partial n}\right|_{\partial D} = 0, \\ \displaystyle\int_D u_i u_j = \delta_{ij}, \text{ for any } i, j. \end{cases} \tag{2.1}$$

Defining a function φ_i by

$$\varphi_i = g u_i - \sum_{j=1}^{k} a_{ij} u_j, \tag{2.2}$$

where $a_{ij} = \int_D g u_i u_j = a_{ji}$, then we have

$$\int_D u_j \varphi_i = 0, \quad \text{for any } i, j = 1, \cdots, k. \tag{2.3}$$

Hence, we have

$$\lambda_{k+1} \le \frac{\int_D (\Delta\varphi_i)^2}{\int_D (\varphi_i)^2}. \tag{2.4}$$

From the definition of g, we have

$$\nabla g = (0, \cdots, 0, 1, 0, \cdots, 0), \tag{2.5}$$

where ∇ denotes the gradient operator of $\mathbb{R}^n$.

Next, we shall make an estimate of $\int_D (\Delta\varphi_i)^2$. From (2.1), (2.2) and (2.3), we obtain

$$
\begin{aligned}
\int_D (\Delta\varphi_i)^2 &= \int_D \varphi_i \Delta^2 \varphi_i \\
&= \int_D \varphi_i \left\{ \Delta^2 (g u_i - \sum_{j=1}^{k} a_{ij} u_j) \right\} \\
&= \int_D \varphi_i \left(\Delta^2 (g u_i) - \sum_{j=1}^{k} a_{ij} \lambda_j u_j \right) \\
&= \int_D \varphi_i \Big(4\langle \nabla g, \nabla(\Delta u_i)\rangle + \lambda_i g u_i \Big) \\
&= \int_D \Big\{ 4(g u_i - \sum_{j=1}^{k} a_{ij} u_j)\langle \nabla g, \nabla(\Delta u_i)\rangle + \varphi_i \lambda_i g u_i \Big\} \\
&= \lambda_i \parallel \varphi_i \parallel^2 - 4 \sum_{j=1}^{k} a_{ij} b_{ij} + \int_D 4 g u_i \langle \nabla g, \nabla(\Delta u_i)\rangle,
\end{aligned} \tag{2.6}
$$

where

$$
b_{ij} = \int_D \langle \nabla g, \nabla u_i \rangle (\Delta u_j) = -b_{ji} \tag{2.7}
$$

and

$$
\|\varphi_i\|^2 = \int_D {\varphi_i}^2.
$$

By a simple calculation, we have, from (2.5),

$$
\begin{aligned}
&4 \int_D g u_i \langle \nabla g, \nabla(\Delta u_i)\rangle \\
&= -2 \int_D \Delta u_i \langle \nabla u_i, \nabla g^2 \rangle - 2 \int_D \Delta u_i (u_i \Delta g^2) \\
&= \int_D \big(4|\nabla_p u_i|^2 + 2|\nabla u_i|^2 \big) = 4\|\nabla_p u_i\|^2 + 2\|\nabla u_i\|^2.
\end{aligned} \tag{2.8}
$$

Then, according to (2.4), (2.6) and (2.8), we obtain

$$
(\lambda_{k+1} - \lambda_i) \parallel \varphi_i \parallel^2 \leq \big(2\|\nabla u_i\|^2 + 4\|\nabla_p u_i\|^2 \big) - 4 \sum_{j=1}^{k} a_{ij} b_{ij}. \tag{2.9}
$$

On the other hand, since

$$
\int_D u_i \langle \nabla(g u_i), \nabla g \rangle = \frac{1}{2}
$$

holds, we have

$$
\begin{aligned}
&\int_D \varphi_i(-2\langle\nabla g, \nabla u_i\rangle) \\
=&-2\int_D (gu_i - \sum_{j=1}^k a_{ij}u_j)\langle\nabla g, \nabla u_i\rangle \\
=&2\int_D u_i\langle\nabla(gu_i), \nabla g\rangle + 2\int_D \sum_{j=1}^k a_{ij}u_j\langle\nabla g, \nabla u_i\rangle \\
=&1 + 2\sum_{j=1}^k a_{ij}c_{ij},
\end{aligned}
\tag{2.10}
$$

where

$$
c_{ij} = \int_D u_j\langle\nabla g, \nabla u_i\rangle = -c_{ji}. \tag{2.11}
$$

Because of

$$
\lambda_i a_{ij} = \int_D (\Delta^2 u_i) g u_j = \lambda_j a_{ij} + 4\int_D \langle\nabla g, \nabla u_j\rangle(\Delta u_i) = \lambda_j a_{ij} - 4b_{ij},
$$

we have

$$
-(\lambda_i - \lambda_j)a_{ij} = 4b_{ij} = -4b_{ji}. \tag{2.12}
$$

For any constant $\alpha > 0$, we have, from (2.3), (2.5) and (2.10),

$$
\begin{aligned}
&1 + 2\sum_{j=1}^k a_{ij}c_{ij} \\
=&\int_D \varphi_i\left(-2\langle\nabla g, \nabla u_i\rangle + 2\sum_{j=1}^k c_{ij}u_j\right) \\
\le&\int_D \left\{\alpha\varphi_i^2 + \frac{1}{\alpha}\left(-\langle\nabla g, \nabla u_i\rangle + \sum_{j=1}^k c_{ij}u_j\right)^2\right\} \\
=&\alpha\|\varphi_i\|^2 + \frac{1}{\alpha}\left(\|\nabla_p u_i\|^2 - \sum_{j=1}^k c_{ij}^2\right).
\end{aligned}
\tag{2.13}
$$

Multiplying (2.13) by $(\lambda_{k+1}-\lambda_i)$, we infer, from (2.9),

$$
\begin{aligned}
&(1+2\sum_{j=1}^{k} a_{ij}c_{ij})(\lambda_{k+1}-\lambda_i)\\
&\leq(\lambda_{k+1}-\lambda_i)\left\{\alpha\|\varphi_i\|^2+\frac{1}{\alpha}\left(\|\nabla_p u_i\|^2-\sum_{j=1}^{k} c_{ij}^2\right)\right\}\\
&\leq\alpha\left(2\|\nabla u_i\|^2+4\|\nabla_p u_i\|^2-4\sum_{j=1}^{k} a_{ij}b_{ij}\right)\\
&\quad+\frac{\lambda_{k+1}-\lambda_i}{\alpha}\left(\|\nabla_p u_i\|^2-\sum_{j=1}^{k} c_{ij}^2\right).
\end{aligned}
\tag{2.14}
$$

Putting $\alpha=(\lambda_{k+1}-\lambda_i)^{1/2}\alpha_1$, $\alpha_1=(2n+4)^{-1/2}$, we have

$$
\begin{aligned}
&\lambda_{k+1}-\lambda_i+2\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_i)a_{ij}c_{ij}\\
&\leq\alpha_1(\lambda_{k+1}-\lambda_i)^{1/2}\left(2\|\nabla u_i\|^2+4\|\nabla_p u_i\|^2-4\sum_{j=1}^{k} a_{ij}b_{ij}\right)\\
&\quad+\frac{1}{\alpha_1}(\lambda_{k+1}-\lambda_i)^{1/2}\left(\|\nabla_p u_i\|^2-\sum_{j=1}^{k} c_{ij}^2\right).
\end{aligned}
\tag{2.15}
$$

Taking sum on i from 1 to k for (2.15), we have

$$
\begin{aligned}
&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)+2\sum_{i=1}^{k}\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_i)a_{ij}c_{ij}\\
&\leq\alpha_1\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{1/2}\left(2\|\nabla u_i\|^2+4\|\nabla_p u_i\|^2-4\sum_{j=1}^{k} a_{ij}b_{ij}\right)\\
&\quad+\frac{1}{\alpha_1}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{1/2}\left(\|\nabla_p u_i\|^2-\sum_{j=1}^{k} c_{ij}^2\right).
\end{aligned}
\tag{2.16}
$$

Defining

$$
A=\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{1/2}\left[\alpha_1\left(4\|\nabla_p u_i\|^2+2\|\nabla u_i\|^2\right)+\frac{1}{\alpha_1}\|\nabla_p u_i\|^2\right],
\tag{2.17}
$$

we have

$$
\begin{aligned}
&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)+2\sum_{i=1}^{k}\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_i)a_{ij}c_{ij}\\
&\leq A-4\alpha_1\sum_{i=1}^{k}\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_i)^{1/2}a_{ij}b_{ij}-\frac{1}{\alpha_1}\sum_{i=1}^{k}\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_i)^{1/2}c_{ij}^2.
\end{aligned}
\tag{2.18}
$$

Since $a_{ij} = a_{ji},\ c_{ij} = -c_{ji}$, we have

$$2\sum_{i=1}^{k}\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_i)a_{ij}c_{ij} = -\sum_{i=1}^{k}\sum_{j=1}^{k}(\lambda_i-\lambda_j)a_{ij}c_{ij}.$$

From (2.12), we have

$$4b_{ij} = -(\lambda_i-\lambda_j)a_{ij}.$$

Thus, we can obtain

$$\begin{aligned}
&-4\alpha_1\sum_{i=1}^{k}\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_i)^{1/2}a_{ij}b_{ij}\\
=&\alpha_1\sum_{i=1}^{k}\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_i)^{1/2}(\lambda_i-\lambda_j)a_{ij}^2\\
=&\frac{\alpha_1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\Big\{(\lambda_{k+1}-\lambda_i)^{1/2}-(\lambda_{k+1}-\lambda_j)^{1/2}\Big\}(\lambda_i-\lambda_j)a_{ij}^2\\
=&-\frac{\alpha_1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{1}{(\lambda_{k+1}-\lambda_i)^{1/2}+(\lambda_{k+1}-\lambda_j)^{1/2}}(\lambda_i-\lambda_j)^2a_{ij}^2
\end{aligned}\tag{2.19}$$

and

$$\begin{aligned}
&-\frac{1}{\alpha_1}\sum_{i=1}^{k}\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_i)^{1/2}c_{ij}^2\\
=&-\frac{1}{2\alpha_1}\sum_{i=1}^{k}\sum_{j=1}^{k}\Big\{(\lambda_{k+1}-\lambda_i)^{1/2}+(\lambda_{k+1}-\lambda_j)^{1/2}\Big\}c_{ij}^2.
\end{aligned}\tag{2.20}$$

Since

$$\begin{aligned}
&\sum_{i=1}^{k}\sum_{j=1}^{k}(\lambda_i-\lambda_j)a_{ij}c_{ij}\\
\le&\frac{\alpha_1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{1}{(\lambda_{k+1}-\lambda_i)^{1/2}+(\lambda_{k+1}-\lambda_j)^{1/2}}(\lambda_i-\lambda_j)^2a_{ij}^2\\
&+\frac{1}{2\alpha_1}\sum_{i=1}^{k}\sum_{j=1}^{k}\Big\{(\lambda_{k+1}-\lambda_i)^{1/2}+(\lambda_{k+1}-\lambda_j)^{1/2}\Big\}c_{ij}^2
\end{aligned}\tag{2.21}$$

holds, we infer, from (2.18), (2.19), (2.20) and (2.21),

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\le A.$$

That is,

$$k\lambda_{k+1}-\sum_{i=1}^{k}\lambda_i\le A.\tag{2.22}$$

From the definition of A and $\alpha_1 = (2n+4)^{-\frac{1}{2}}$, we obtain, by taking the sum on p from 1 to n for (2.22),

$$\begin{aligned} n(k\lambda_{k+1} - \sum_{i=1}^{k} \lambda_i) &\leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} \left[\alpha_1(2n+4) + \frac{1}{\alpha_1}\right] \|\nabla u_i\|^2 \\ &= [8(n+2)]^{1/2} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} \|\nabla u_i\|^2. \end{aligned} \tag{2.23}$$

Since

$$\|\nabla u_i\|^2 = \int_D |\nabla u_i|^2 = \int_D u_i(-\Delta u_i) \leq \left(\|u_i\|^2 \|\Delta u_i\|^2\right)^{1/2} = (\lambda_i)^{1/2}, \tag{2.24}$$

we have

$$k\lambda_{k+1} - \sum_{i=1}^{k} \lambda_i \leq \frac{[8(n+2)]^{1/2}}{n} \sum_{i=1}^{k} \Big[(\lambda_{k+1} - \lambda_i)\lambda_i\Big]^{1/2}. \tag{2.25}$$

This completes the proof of Theorem 1.

We now prove Corollary 1.

Proof of Corollary 1. Let

$$\Lambda_k = \frac{1}{k} \sum_{i=1}^{k} \lambda_i, \qquad T_k = \frac{1}{k} \sum_{i=1}^{k} \lambda_i^2. \tag{2.26}$$

It follows from (2.25) that

$$\begin{aligned} &(\lambda_{k+1} - \Lambda_k)^2 \\ \leq &\frac{8(n+2)}{n^2} \left[\frac{1}{k} \sum_{i=1}^{k} \Big\{(\lambda_{k+1} - \lambda_i)\lambda_i\Big\}^{1/2}\right]^2 \\ \leq &\frac{8(n+2)}{n^2} \frac{1}{k} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\lambda_i \\ = &\frac{8(n+2)}{n^2} (\lambda_{k+1}\Lambda_k - T_k). \end{aligned} \tag{2.27}$$

Therefore,

$$\begin{aligned} &\left\{\lambda_{k+1} - \left[1 + \frac{4(n+2)}{n^2}\right] \Lambda_k\right\}^2 \\ \leq &\left[\frac{8(n+2)}{n^2} + \frac{16(n+2)^2}{n^4}\right] \Lambda_k^2 - \frac{8(n+2)}{n^2} T_k. \end{aligned} \tag{2.28}$$

Namely,

$$\begin{aligned} &\lambda_{k+1} - \left[1 + \frac{4(n+2)}{n^2}\right] \Lambda_k \\ \leq &\left\{\left[\frac{4(n+2)}{n^2} \Lambda_k\right]^2 - \frac{8(n+2)}{n^2} \frac{1}{k} \sum_{j=1}^{k} (\lambda_j - \Lambda_k)^2\right\}^{1/2}. \end{aligned} \tag{2.29}$$

This finishes the proof of Corollary 1.

Remark 2. In order to prove our Theorem 1, we introduced a factor $(\lambda_{k+1}-\lambda_i)$ in the formula (2.14). From the assertions from formulas (2.14) to (2.22), we know that unwanted terms between both sides of the inequalities are eliminated perfectly. If we do not introduced the factor $(\lambda_{k+1}-\lambda_i)$ in the formula (2.14), we shall obtain the inequality (2.10) of Hook, and Chen and Qian. In fact, putting $\alpha = \frac{(\lambda_{k+1}-\lambda_i)\alpha_2}{\sum_{l=1}^k \lambda_l^{\frac{1}{2}}}$, $\alpha_2 = \frac{nk}{4(n+2)}$ and taking the sum on i from 1 to k for (2.13), we have, from (2.9),

$$\begin{aligned} & k + 2\sum_{i,j=1}^k a_{ij}c_{ij} \\ \leq & \frac{\alpha_2}{\sum_{l=1}^k \lambda_l^{\frac{1}{2}}}\{2\sum_{i=1}^k(\|\nabla u_i\|^2 + 4\|\nabla_p u_i\|^2) + \sum_{i,j=1}^k(\lambda_i-\lambda_j)a_{ij}^2\} \\ & + \frac{\sum_{l=1}^k \lambda_l^{\frac{1}{2}}}{\alpha_2}\sum_{i=1}^k \frac{1}{\lambda_{k+1}-\lambda_i}\left(\|\nabla_p u_i\|^2 - \sum_{j=1}^k c_{ij}^2\right), \end{aligned}$$

where we used the formula (2.12). From the antisymmetry of c_{ij} and $(\lambda_i-\lambda_j)a_{ij}^2$, we have

$$2\sum_{i,j=1}^k a_{ij}c_{ij} = 0 \text{ and } \sum_{i,j=1}^k (\lambda_i-\lambda_j)a_{ij}^2 = 0.$$

Making use of the similar assertion in the proof of Theorem 1, we have

$$nk \leq 2(n+2)\alpha_2 + \frac{1}{\alpha_2}\left[\sum_{i=1}\frac{\lambda_i^{1/2}}{\lambda_{k+1}-\lambda_i}\right]\sum_{i=1}^k \lambda_i^{1/2} - \frac{\sum_{l=1}^k \lambda_l^{\frac{1}{2}}}{\alpha_2}\sum_{i,j=1}^k \frac{c_{ij}^2}{\lambda_{k+1}-\lambda_i}. \tag{2.30}$$

Hence, we have

$$\frac{n^2k^2}{8(n+2)} \leq \left[\sum_{i=1}\frac{\lambda_i^{1/2}}{\lambda_{k+1}-\lambda_i}\right]\sum_{i=1}^k \lambda_i^{1/2} - \sum_{l=1}^k \lambda_l^{\frac{1}{2}}\sum_{i,j=1}^k \frac{c_{ij}^2}{\lambda_{k+1}-\lambda_i}. \tag{2.31}$$

Therefore, in order to infer the inequality (1.10), we must throw away the unwanted term in (2.31). Thus, we know that the inequality in Theorem 1 should be sharper than (1.10). If we multiply (2.13) by a factor $(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}$ and use a similar argument, we shall obtain the inequality (1.14).

Acknowledgements

This work was mainly done while the second author was visiting Saga University. He was supported in part by a Grant-in-Aid for Scientific Research from the Japan Society for the Promotion of Science. He would like to express his gratitude to Professor Qing-Ming Cheng for his kind hospitality.

References

[1] M. S. Ashbaugh, Isoperimetric and universal inequalities for eigenvalues, in Spectral theory and geometry (Edinburgh, 1998), E. B. Davies and Yu Safalov eds., London Math. Soc. Lecture Notes, vol. **273,** Cambridge Univ. Press, Cambridge, 1999, pp. 95-139. MR1736867 (2001a:35131)

[2] M. S. Ashbaugh, Universal eigenvalue bounds of Payne-Pólya-Weinberger, Hile-Protter and H.C. Yang, Proc. Indian Acad. Sci. Math. Sci. **112** (2002), 3-30. MR1894540 (2004c:35302)

[3] M. S. Ashbaugh and R. D. Benguria, Proof of the Payne-Pólya-Weinberger conjecture, Bull. Amer. Math. Soc. **25** (1991), 19-29. MR1085824 (91m:35173)
[4] M. S. Ashbaugh and R. D. Benguria, A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions, Ann. of Math. **135** (1992), 601-628. MR1166646 (93d:35105)
[5] M. S. Ashbaugh and R. D. Benguria, A second proof of the Payne-Pólya-Weinberger conjecture, Commun. Math. Phys. **147** (1992), 181-190. MR1171765 (93k:33002)
[6] Z.-C. Chen and C.-L. Qian, Estimates for discrete spectrum of Laplacian operator with any order, J. China Univ. Sci. Tech. **20** (1990), 259-266. MR1077287 (92c:35087)
[7] G. N. Hile and M. H. Protter, Inequalities for eigenvalues of the Laplacian, Indiana Univ. Math. J. **29** (1980), 523-538. MR0578204 (82c:35052)
[8] G. N. Hile and R. Z. Yeh, Inequalities for eigenvalues of the biharmonic operator, Pacific J. Math. **112** (1984), 115-133. MR0739143 (85k:35170)
[9] S. M. Hook, Domain independent upper bounds for eigenvalues of elliptic operator, Trans. Amer. Math. Soc. **318** (1990), 615-642. MR0994167 (90h:35075)
[10] L. E. Payne, G. Polya, and H. F. Weinberger, Sur le quotient de deux fréquences propres consécutives, Comptes Rendus Acad. Sci. Paris **241** (1955), 917-919. MR0073046 (17:372d)
[11] L. E. Payne, G. Polya, and H. F. Weinberger, On the ratio of consecutive eigenvalues, J. Math. and Phys. **35** (1956), 289–298. MR0084696 (18:905c)
[12] H. C. Yang, An estimate of the differance between consecutive eigenvalues, preprint IC/91/60 of ICTP, Trieste, 1991.

Department of Mathematics, Faculty of Science and Engineering, Saga University, Saga 840-8502, Japan
E-mail address: cheng@ms.saga-u.ac.jp

Academy of Mathematics and Systematical Sciences, CAS, Beijing 100080, People's Republic of China
E-mail address: yanghc@math03.math.ac.cn