Pathwise coupling on spaces with variable Ricci bounds

arXiv:1906.09186

Mathias Braun
braun@iam.uni-bonn.de

With Karen Habermann and Karl-Theodor Sturm

Institute for Applied Mathematics
Rheinische Friedrich-Wilhelms-Universität Bonn

Japanese-German Open Conference on Stochastic Analysis
University of Fukuoka
September 5, 2019

Overview

Ricci lower bounds and Brownian motion

Let (M, g) be a complete, connected Riemannian manifold. Brownian motion on M: any Markov process $(B_t)_{t\geq 0}$ with continuous paths s.t.

$$M^f[B]_t := f(B_t) - f(B_0) - \frac{1}{2}\int_0^t \Delta f(B_s)\,\mathrm{d}s, \qquad t \in [0, \zeta_B],$$

is a local $\mathscr{F}_\bullet^B$-martingale for every $f \in \mathrm{C}^\infty(\mathrm{M})$.

$\mathrm{Ric}_\mathrm{M} \geq K$, $K \in \mathbb{R}$, implies

- stochastic completeness, C_0-property, gradient estimates for the heat semigroup $(\mathsf{P}_t)_{t\geq 0}$ with

$$\mathsf{P}_t f(x) := \mathbf{E}_x[f(B_{2t})], \qquad f \in L^2(\mathrm{M}, \mathrm{vol}_g),$$

- contraction estimates for the dual semigroup $(\mathsf{H}_t)_{t\geq 0}$ with

$$\langle \phi \mid \mathsf{H}_t \mu \rangle := \langle \mathsf{P}_t \phi \mid \mu \rangle, \qquad \phi \in \mathrm{C}_\mathrm{b}(\mathrm{M}),\ \mu \in \mathscr{P}(\mathrm{M}),$$

- spectral gap estimates for $K > 0$.

Question

What does the behavior of Brownian processes tell us about Ric_M, and vice versa?

Coupling of Brownian motions, constant case

[Kendall '86], [Cranston '91], [Wang '97], [von Renesse '04], [Veysseire '11] et al.: estimates for couplings of Brownian motions on M under $\mathrm{Ric}_{\mathrm{M}} \geq K$.

Theorem [von Renesse, Sturm '05], [Arnaudon, Coulibaly, Thalmaier '11]

Given $K \in \mathbb{R}$, we have $\mathrm{Ric}_{\mathrm{M}} \geq K$ if and only if for every $\mu \in \mathscr{P}(\mathrm{M} \times \mathrm{M})$ there exists a coupling $(B_t^1, B_t^2)_{t \geq 0}$ of Brownian motions on M starting in μ which a.s. satisfies

$$\mathsf{d}(B_t^1, B_t^2) \leq \mathrm{e}^{-K(t-s)/2}\, \mathsf{d}(B_s^1, B_s^2) \quad \text{for every } s \leq t. \qquad (\star)$$

Basic argumentation:

$$(\star) \iff W_p(\mathsf{H}_t \mu_1, \mathsf{H}_t \mu_2) \leq \mathrm{e}^{-Kt}\, W_p(\mu_1, \mu_2) \quad \text{for every } \mu_1, \mu_2 \in \mathscr{P}(\mathrm{M}),\ p \in [1, \infty] \qquad \text{Integration}$$

$$\iff |\nabla \mathsf{P}_t f|^q \leq \mathrm{e}^{-qKt}\, \mathsf{P}_t(|\nabla f|^q) \quad \text{for every } f \in W^{1,2}(\mathrm{M}),\ q \in [1, \infty) \qquad \text{Duality}$$

$$\iff \mathrm{Ric}_{\mathrm{M}} \geq K \qquad \text{Bakry–Émery}$$

Questions

- What are appropriate replacements of these results when $\mathrm{Ric}_{\mathrm{M}}(x) \geq k(x)$?
- How about generalizing these results to nonsmooth situations?

Overview

Synthetic Ricci lower bounds via optimal transport

Fix a metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, i.e. a complete and separable metric space (X, d) endowed with a locally finite Borel measure $\mathfrak{m}$, and a lower semicontinuous function $k\colon \mathrm{X} \to \mathbb{R}$. The relative entropy on $\mathscr{P}(\mathrm{X})$ is

$$\mathrm{Ent}_{\mathfrak{m}}(\nu) := \int_{\mathrm{X}} \frac{\mathrm{d}\nu}{\mathrm{d}\mathfrak{m}} \log \frac{\mathrm{d}\nu}{\mathrm{d}\mathfrak{m}}\,\mathrm{d}\mathfrak{m} \quad \text{if } \nu \ll \mathfrak{m}, \qquad \mathrm{Ent}_{\mathfrak{m}}(\nu) := \infty \quad \text{else.}$$

Definition [Sturm '15], motivated by [Sturm '06], [Lott, Villani '09]

We say that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ satisfies $\mathrm{CD}(k, \infty)$ if for all $\mu_0, \mu_1 \in \mathscr{P}_2(\mathrm{X}) \cap \mathrm{Dom}(\mathrm{Ent}_{\mathfrak{m}})$, there exists $\boldsymbol{\pi} \in \mathscr{P}(\mathrm{Geo}(\mathrm{X}))$ constituting a W_2-geodesic $((\mathsf{e}_t)_\sharp \boldsymbol{\pi})_{t\in[0,1]}$ connecting μ_0 to μ_1 s.t., for every $t \in [0,1]$,

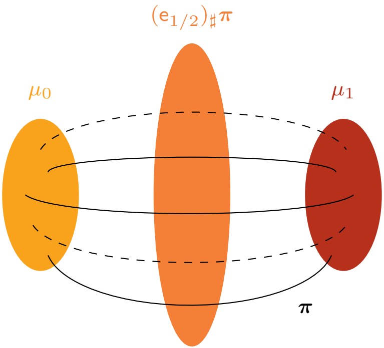
$$\mathrm{Ent}_{\mathfrak{m}}((\mathsf{e}_t)_\sharp \boldsymbol{\pi}) \leq (1-t)\,\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + t\,\mathrm{Ent}_{\mathfrak{m}}(\mu_1) - \iint_0^1 k(\gamma_s)\,\mathrm{g}(s,t)\,\mathsf{d}(\gamma_0, \gamma_1)^2\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}(\gamma).$$

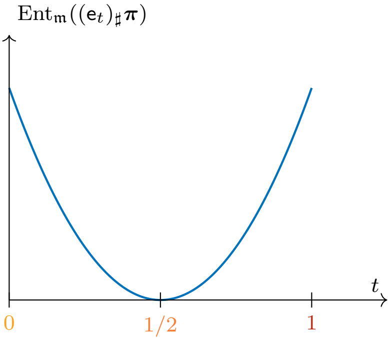
Theorem [von Renesse, Sturm '05]

For a complete, connected Riemannian manifold (M, g) and $\psi \in \mathrm{C}^2(\mathrm{M})$, $(\mathrm{M}, \mathsf{d}_g, \mathrm{e}^{-\psi}\,\mathrm{vol}_g)$ satisfies $\mathrm{CD}(K, \infty)$ with $K \in \mathbb{R}$ if and only if $\mathrm{Ric}_{\mathrm{M}} + \mathrm{Hess}\,\psi \geq K$.

The picture one should have in mind

Mass transport from μ_0 to μ_1 according to the geodesic plan $\boldsymbol{\pi}$ supported on the solid lines...





...and the (convex) graph of the entropy along the geodesic $(\mathsf{e}_t)_\sharp \boldsymbol{\pi}$ for the above mass transport.

“Riemannian-like” structures

To rule out Finsler structures, let

$$\mathscr{E}_0(f) := \frac{1}{2}\int_{\mathrm{X}} \left[\limsup_{y\to x} \frac{|f(x)-f(y)|}{\mathsf{d}(x,y)}\right]^2 \mathrm{d}\mathfrak{m}(x), \qquad f \in \mathrm{Lip}_{\mathrm{bs}}(\mathrm{X}),$$

$$\mathscr{E}(g) := \mathrm{rel}_{L^2(\mathrm{X},\mathfrak{m})}\mathscr{E}_0(g) = \frac{1}{2}\int_{\mathrm{X}} |\nabla g|^2(x)\,\mathrm{d}\mathfrak{m}(x). \qquad \text{Cheeger energy}$$

minimal weak upper gradient $|\nabla g|$

Definition [Ambrosio, Gigli, Savaré ’14]

A $\mathrm{CD}(K,\infty)$ space is termed to satisfy $\mathrm{RCD}(K,\infty)$ if $\mathscr{E}$ is a quadratic form.

Theorem [Ambrosio, Gigli, Savaré ’14]

If $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,\infty)$ space, $\mathscr{E}$ is a strongly local, quasi-regular Dirichlet form. Given any $\mu \in \mathscr{P}(\mathrm{X})$, uniquely in law there exists a conservative Markov process $(B_t)_{t\geq 0}$ with continuous sample paths and initial distribution μ, which is associated to $\mathscr{E}$ [Brownian motion].

Overview

Main result

Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space, $K \in \mathbb{R}$, and $k \colon \mathrm{X} \to \mathbb{R}$ a lower semicontinuous function, bounded from below by K. Define

$$\underline{k}(x, y) := \liminf_{\varepsilon \to 0} \left\{ \int_0^1 k(\gamma_s)\, \mathrm{d}s : \gamma \in \mathrm{Geo}(\mathrm{X}),\ \gamma_0 \in B_\varepsilon(x),\ \gamma_1 \in B_\varepsilon(y) \right\}.$$

Theorem [Br., Habermann, Sturm '19]

The metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ satisfies $\mathrm{CD}(k, \infty)$ if and only if for every $\mu \in \mathscr{P}(\mathrm{X} \times \mathrm{X})$ there exists a coupling $(B_t^1, B_t^2)_{t \geq 0}$ of Brownian motions on X which starts in μ s.t. a.s.,

$$\mathsf{d}(B_t^1, B_t^2) \leq \mathrm{e}^{-\int_s^t \underline{k}(B_r^1, B_r^2)/2\, \mathrm{d}r}\, \mathsf{d}(B_s^1, B_s^2) \quad \text{for every } s \leq t.$$

Already known before for constant k by [Sturm '15]. Problem: variable estimate requires control over whole paths, not only over endpoints.

Contraction and integration

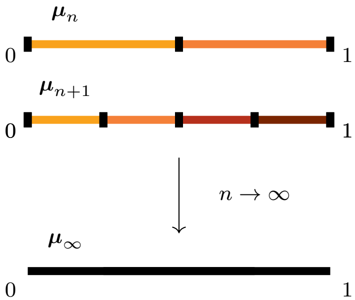
For $\mu_1, \mu_2 \in \mathscr{P}_p(\mathrm{X})$ and $p \in [1, \infty)$, replace

$$\mathrm{e}^{Kt}\, W_p(\mathsf{H}_t\mu_1, \mathsf{H}_t\mu_2) \leq \mathrm{e}^{Ks}\, W_p(\mathsf{H}_s\mu_1, \mathsf{H}_s\mu_2) \quad \text{for every } s \leq t$$

by nonincreasingness of

$$W_p^k(\mu_1, \mu_2, t) := \inf \mathbf{E}\Big[\mathrm{e}^{\int_0^{2t} p\underline{k}(B_r^1, B_r^2)/2\, \mathrm{d}r}\, \mathsf{d}^p(B_{2t}^1, B_{2t}^2)\Big]^{1/p}.$$

The infimum is taken over all couplings $(B_t^1, B_t^2)_{t \geq 0}$ of Brownian motions on X with initial distributions μ_1 and μ_2, respectively.



Duality

Replace $\mathrm{e}^{-qKt}\,\mathsf{P}_t(|\nabla f|^q)$ by $\mathsf{P}_t^{qk}(|\nabla f|^q)$, where Brownian motion on X

$$\mathsf{P}_t^{\kappa} g(x) := \mathbf{E}_x\Big[\mathrm{e}^{-\int_0^{2t}\kappa(B_s)/2\,\mathrm{d}s}\,g(B_{2t})\Big], \qquad g \in L^2(\mathrm{X},\mathfrak{m}),$$

i.e. $(\mathsf{P}_t^{qk})_{t\geq 0}$ corresponds to the Schrödinger semigroup associated to

$$\mathscr{E}^{qk}(f) := 2\,\mathscr{E}(f) + q\int_{\mathrm{X}} k\,f^2\,\mathrm{d}\mathfrak{m}.$$

Theorem [Br., Habermann, Sturm '19]

Given $(\mathrm{X},\mathsf{d},\mathfrak{m})$ as above and $p,q\in(1,\infty)$ with $1/p+1/q=1$, the following are equivalent:

- for every $f\in\mathrm{Dom}(\mathscr{E})$ and $t\geq 0$, we have

$$|\nabla \mathsf{P}_t f|^q \leq \mathsf{P}_t^{qk}(|\nabla f|^q) \quad \mathfrak{m}\text{-a.e.},$$

- for every $\mu_1,\mu_2\in\mathscr{P}_p(\mathrm{X})$, $W_p^k(\mu_1,\mu_2,t)$ is nonincreasing in $t\geq 0$.

Already proved in the constant case in [Kuwada '10] [Kuwada's duality].

Equivalent characterizations of variable Ricci bounds

Theorem [Br., Habermann, Sturm '19]

Given $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ and k as above, $p \in (1, \infty)$ and $q \in [1, \infty)$, the following are equivalent:

- the $\mathrm{CD}(k, \infty)$ condition holds,
- the q-Bochner inequality $\Delta |\nabla f|^q / q - |\nabla f|^{q-2} \langle \nabla f, \nabla \Delta f \rangle \geq k \, |\nabla f|^q$ holds in a weak form,
- for every $f \in \mathrm{Dom}(\mathscr{E})$ and $t \geq 0$, we have

$$|\nabla \mathsf{P}_t f|^q \leq \mathsf{P}_t^{qk}(|\nabla f|^q) \quad \mathfrak{m}\text{-a.e.},$$

- for every $\mu_1, \mu_2 \in \mathscr{P}_p(\mathrm{X})$, $W_p^k(\mu_1, \mu_2, t)$ is nonincreasing in $t \geq 0$,
- for every $\mu \in \mathscr{P}(\mathrm{X} \times \mathrm{X})$ there exists a coupling $(B_t^1, B_t^2)_{t \geq 0}$ of Brownian motions on X which starts in μ s.t. a.s.,

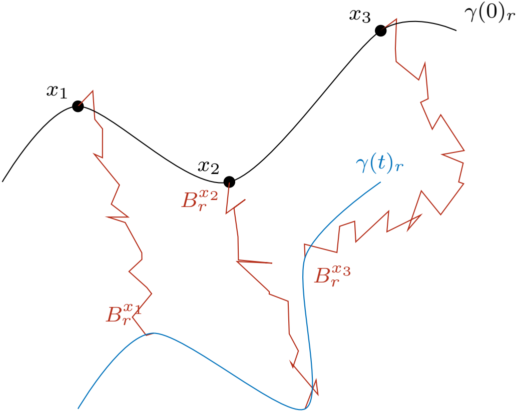
$$\mathsf{d}(B_t^1, B_t^2) \leq \mathrm{e}^{-\int_s^t \underline{k}(B_r^1, B_r^2)/2 \, \mathrm{d}r} \, \mathsf{d}(B_s^1, B_s^2) \quad \text{for every } s \leq t.$$

Moreover, any of these properties yields the other ones for arbitrary exponents $p, q \in [1, \infty)$ [self-improvement of Bochner's inequality].

Overview

Open problems

- Relaxations on k, e.g. removing the lower bound? Other "forms" of k?
- Does the previous process $(B^1_t, B^2_t)_{t\geq 0}$ satisfy the Markov property?
- Study the heat flow on 1-forms in the framework of [Gigli '18]. Estimates in terms of $|\mathsf{P}_{1,t}\mathrm{d}f| \leq \mathsf{P}^k_t|\mathrm{d}f|$?



Thank you for your kind attention!

At Gokuku Shrine, Fukuoka

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