The Dirichlet–Ferguson Diffusion

on the space of probability measures over a closed Riemannian manifold

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Summary

- Long-term goals
- "Geometric diffusions"
- Dirichlet forms on Wasserstein spaces
- A candidate for Brownian motion

Long-term Goals

Setting $(X, \mathrm{g}, \mathrm{m})$ closed Riemannian manifold, dim $d \geq 2$, volume meas. m

$\mathscr{P}(X)$ space of Borel probability measures on X

Long-term goals

- Brownian motion on $\mathscr{P}(X)$
  - stochastic Otto calculus [von Renesse–Sturm *AoP* '09]
  - SPDEs on manifolds
- ◦ Curvature of $\mathscr{P}(X)$
- ◦ Representations of $\mathrm{Diff}(X)$

$$\dot\rho_t = -\nabla \mathrm{Ent}_{\mathrm{m}}(\rho_t \mathrm{m})$$

$$\mathscr{P}(X) \xhookleftarrow{\ \delta\ } X$$

$$\partial_t \rho_t = \Delta \rho_t$$

Long-term Goals

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$$\dot\rho_t = -\boldsymbol{\nabla}\mathrm{Ent}_{\mathrm{m}}(\rho_t \mathrm{m})$$

$$\mathscr{P}(X)$$

$$\uparrow \delta$$

$$X$$

$$\partial_t \rho_t = \Delta \rho_t$$

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Long-term goals

- Brownian motion on $\mathscr{P}(X)$
  - stochastic Otto calculus [von Renesse–Sturm *AoP* '09]
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perturbation by "Brownian motion"?

$\dot\rho_t = -\nabla \mathrm{Ent}_{\mathsf{m}}(\rho_t \mathsf{m})$ $\quad (\mu_t)_t$ (?)

$\mathscr{P}(X)$ $\quad \mathscr{P}(X)$

$\uparrow \delta$ $\quad \uparrow \delta$

X $\quad X$

$\partial_t \rho_t = \Delta \rho_t$ $\quad$ stochastic heat equation?

"Geometric Diffusions"

- from Lie Groups to Forms
- Integration by Parts (IbP) and Closability

"Geometric Diffusions": from Lie Groups to Forms

Group	• G connected Lie group	$\mathbb{R}^n$ (translations of $\mathbb{R}^n$)
	▸ $\mathfrak{g} = T_o G$ Lie algebra of G	$\mathbb{R}^n$
	▸ $e\colon \mathfrak{g} \longrightarrow G$ the exponential map	$\mathrm{Id} + \cdot$
Space	• $(\Omega, \mathscr{F}, \mathbf{P})$ a standard (probab.) space s.t.	$(\mathbb{R}^n, \mathcal{B}, \gamma^n)$
	▸ G is acting (freely, transitively) on Ω	$\cdot + \mathrm{Id}$
	▸ $\mathbf{P}$ is *G-quasi-invariant*, i.e. $(g.)_\sharp \mathbf{P} \sim \mathbf{P}$	$\mathrm{d}\gamma^n(\,\cdot\,) \sim \mathrm{d}\gamma^n(\,\cdot\, + v)$
Gradient	• $T_\omega \Omega \subseteq \mathfrak{g}$ a Hilbert space	$T_x \mathbb{R}^n = \mathbb{R}^n$
	▸ $(\nabla_w u)_\omega := \mathrm{d}_t\big\vert_{t=0} u(e^{tw}.\omega)$	$\partial_w u = \mathrm{d}_t\big\vert_{t=0} u(\,\cdot + tw)$
	▸ $\langle (\nabla u)_\omega \mid w \rangle_{T_\omega \Omega} = (\nabla_w u)_\omega$	$\nabla = (\partial_1, \ldots, \partial_n)$
Dirichlet form	• $\mathcal{E}(u,v) := \int_\Omega \langle \nabla u \mid \nabla v \rangle_{T.\Omega} \,\mathrm{d}\mathbf{P}$	Ornstein–Uhlenbeck $\mathrm{d}x_t = -x_t\,\mathrm{d}t + \mathrm{d}W_t$

Integration by Parts and Closability

Integration by parts: Find ∇_w^* so that

$$\int_\Omega (\nabla_w u)_\omega \, v(\omega) \, \mathrm{d}\mathbf{P}(\omega) = \int_\Omega u(\omega) \, (\nabla_w^* v)_\omega \, \mathrm{d}\mathbf{P}(\omega)$$

Sketch of proof:

$$\int_\Omega u(e^{tw}.\omega) \cdot v(\omega) \, \mathrm{d}\mathbf{P}(\omega) = \int_\Omega u(\omega) \cdot v(e^{-tw}.\omega) \cdot \frac{\mathrm{d}(e^{tw}.)_\sharp \mathbf{P}}{\mathrm{d}\mathbf{P}}(\omega) \, \mathrm{d}\mathbf{P}(\omega)\,,$$

and computing $\mathrm{d}_t\big|_{t=0}$ on both sides

$$\int_\Omega (\nabla_w u)_\omega \cdot v(\omega) \, \mathrm{d}\mathbf{P}(\omega) = -\int_\Omega u(\omega) \cdot (\nabla_w v)_\omega \, \mathrm{d}\mathbf{P}(\omega) + \int_\Omega u(\omega) \cdot v(\omega) \cdot \mathrm{d}_t\big|_{t=0} \frac{\mathrm{d}(e^{tw}.)_\sharp \mathbf{P}}{\mathrm{d}\mathbf{P}}(\omega) \, \mathrm{d}\mathbf{P}(\omega)$$

Thus $$\nabla_w^* u := -\nabla_w u + \mathrm{d}_t\big|_{t=0} \frac{\mathrm{d}(e^{tw}.)_\sharp \mathbf{P}}{\mathrm{d}\mathbf{P}} \cdot u$$

Integration by Parts and Closability

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and computing $\mathrm{d}_t\big|_{t=0}$ on both sides

$$\int_\Omega (\nabla_w u)_\omega \cdot v(\omega) \, \mathrm{d}\mathbf{P}(\omega) = - \int_\Omega u(\omega) \cdot (\nabla_w v)_\omega \, \mathrm{d}\mathbf{P}(\omega) + \int_\Omega u(\omega) \cdot v(\omega) \cdot \mathrm{d}_t\big|_{t=0} \frac{\mathrm{d}(e^{tw}.)_\sharp \mathbf{P}}{\mathrm{d}\mathbf{P}}(\omega) \, \mathrm{d}\mathbf{P}(\omega)$$

Thus

$$\nabla_w^* u := -\nabla_w u + \mathrm{d}_t\big|_{t=0} \frac{\mathrm{d}(e^{tw}.)_\sharp \mathbf{P}}{\mathrm{d}\mathbf{P}} \cdot u$$

On L^2-Wasserstein spaces

- (pseudo-)Wasserstein geometry
- Dirichlet–Ferguson measures
- Dirichlet forms on Wasserstein spaces

(pseudo-)Wasserstein geometry on $\mathscr{P}(X)$

Notation	$w \in \mathfrak{X}^\infty$	$\mathcal{C}^\infty$-vector field on X	$w \in \mathfrak{g}$	(Lie algebra)
	$(\psi^{w,t})_t$	flow of w, $\subseteq \mathrm{Diff}(X)$	$(e^{tw})_t \subseteq G$	(group)
	$\mathscr{P}(X)$	probability measures on X	Ω	(space)

Definition

- *tangent space* to $\mathscr{P}(X)$ at μ

$$T_\mu \mathscr{P}(X) := \overline{\mathfrak{X}^\infty}^{\|\cdot\|_\mu} \text{ where } \|w\|_\mu^2 = \int_X |w|_{\mathrm{g}}^2 \, \mathrm{d}\mu$$

- *directional derivative* of $u\colon \mathscr{P}(X) \to \mathbb{R}$ along w at $\mu \in \mathscr{P}(X)$

$$(\boldsymbol{\nabla}_w u)_\mu := \mathrm{d}_t\big|_{t=0} u\big(\psi^{w,t}_\sharp \mu\big)$$

- *gradient* $(\boldsymbol{\nabla} u)_\mu \in T_\mu \mathscr{P}(X)$ of $u\colon \mathscr{P}(X) \to \mathbb{R}$ at μ

$$\big\langle (\boldsymbol{\nabla} u)_\mu \,\big|\, w \big\rangle_{T_\mu \mathscr{P}(X)} = (\boldsymbol{\nabla}_w u)_\mu$$

($\exists!$ by Riesz Representation)

Dirichlet–Ferguson measures

Notation	$(X, \mathcal{B}, \mathrm{m})$	(normalized) Riemannian volume measure m
	$(\mathscr{P}(X), \tau_n, \mathcal{B}_n)$	narrow topology (compact), Borel σ-algebra
	$\eta_x := \eta\{x\}$	mass of $\eta \in \mathscr{P}(X)$ at $x \in X$
	$\eta_r^x := (1-r)\eta + r\delta_x$	convex combination of η and δ_x

Definition *Dirichlet–Ferguson measure* [Ferguson *AoS* '73]: $\mathcal{D}_{\mathrm{m}}\mathscr{P} = 1$ and

(M) $$\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathrm{m}}(\eta) \int_X \mathrm{d}\eta(x)\, u(\eta, x, \eta_x) = \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathrm{m}}(\eta) \int_X \mathrm{dm}(x) \int_I \mathrm{d}r\, u(\eta_r^x, x, r)$$

for all measurable semi-bounded $u \colon \mathscr{P}(X) \times X \times [0,1] \to \mathbb{R}$

[Sethuraman *M.Sinica* '95]

Mecke-, Campbell- or Georgii–Nguyen–Zessin-type formula

[D.S.–Lytvynov '17]

Remark A Fourier-transform characterization of $\mathcal{D}_{\mathrm{m}}$ is also available

[D.S. '18b]

Properties of $\mathcal{D}_{\mathsf{m}}$

Universality

- $\psi_{\sharp\sharp}\mathcal{D}_{\mathsf{m}} = \mathcal{D}_{\psi_\sharp \mathsf{m}}$ for all $\psi\colon X \to Y$
- $\mathcal{D}_{\mathsf{m}}$ is an $\varprojlim$ along the marginalizations

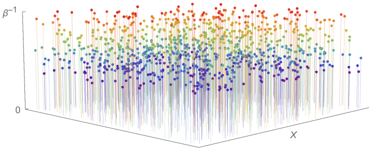
$$\eta \longmapsto (\eta X_1, \dots, \eta X_n)$$

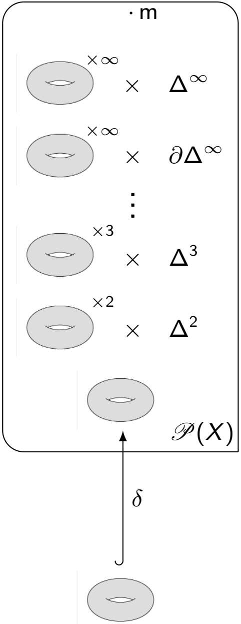
with $\sqcup_i X_i = X$

Support

- $\mathcal{D}_{\mathsf{m}}$ is fully supported
- $\mathcal{D}_{\mathsf{m}}$ is concentrated on

$$X^{\times\infty} \times \Delta^\infty \subsetneq \partial^{\mathrm{Geo}} \mathscr{P}(X)$$





The Dirichlet form

Notation

- *potential energy* $f^\star\colon \mu \longmapsto \mu f := \int_X f \,\mathrm{d}\mu$ for $f \in \mathcal{C}^\infty(X)$

  (linear, τ_n-continuous)

- *algebra of cylinder functions of smooth potential energies*

$$\mathfrak{Z}^\infty := \left\{ \begin{matrix} u\colon \mathscr{P}(X) \to \mathbb{R} : u = F \circ (f_1^\star, \dots, f_k^\star) \ , \\ F \in \mathcal{C}_b^\infty(\mathbb{R}^k)\,, \quad f_i \in \mathcal{C}^\infty(X)\,, \quad i \le k \end{matrix} \right\}$$

Theorem (Closability [D.S. '18a])

The form $\mathcal{E}(u,v) := \int_{\mathscr{P}} \left\langle (\nabla u)_\eta \,\middle|\, (\nabla v)_\eta \right\rangle_{T_\eta \mathscr{P}} \mathrm{d}\mathcal{D}_{\mathrm{m}}(\eta)\,, \quad u, v \in \mathfrak{Z}^\infty, \quad$ *is closable.*

Its closure $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ is a regular recurrent strongly local (symmetric) Dirichlet form, properly associated with a recurrent (reversible) Markov diffusion process $\eta_\bullet$ on $\mathscr{P}(X)$.

Integration by Parts I: No Generator on Cylinder F's

$$
\begin{aligned}
\mathcal{E}(u,v) &:= \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathrm{m}}(\eta) \left\langle (\boldsymbol{\nabla} u)_\eta \,\middle|\, (\boldsymbol{\nabla} v)_\eta \right\rangle_{T_\eta \mathscr{P}} \\
&= \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathrm{m}}(\eta) \int_X \mathrm{d}\eta(x)\, (\boldsymbol{\nabla} u)_\eta(x)\, (\boldsymbol{\nabla} v)_\eta(x) && \text{(M)} \\
&= \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathrm{m}}(\eta) \int_X \mathrm{dm}(x) \int_I \mathrm{d}r\, r^{-1} \big(\nabla u(\eta_r^\bullet)\big)_x\, r^{-1} \big(\nabla v(\eta_r^\bullet)\big)_x && (\mathrm{IbP}_X) \\
&= -\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathrm{m}}(\eta) \int_X \mathrm{dm}(x) \int_I \mathrm{d}r\, u(\eta_r^x)\, r^{-2} \big(\Delta v(\eta_r^\bullet)\big)_x && \text{(M)} \\
&= -\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathrm{m}}(\eta)\, u(\eta) \int_X \mathrm{d}\eta(x)\, \frac{\big(\Delta v(\eta + \eta_x \delta_\bullet - \eta_x \delta_x)\big)_x}{(\eta_x)^2}
\end{aligned}
$$

Lack of generator on $\mathfrak{Z}^\infty$

The operator $(\mathbf{L}, \mathfrak{Z}^\infty)$ is *not* $L^2(\mathcal{D}_{\mathrm{m}})$-valued.

$$
(-\mathbf{L}u)_\eta \overset{!}{=} \sum_{x \in \eta} \frac{\big(\Delta u(\eta + \eta_x \delta_\bullet - \eta_x \delta_x)\big)_x}{\eta_x} \quad .
$$

Integration by Parts II: Directional Derivatives

Notation

- *red. potential energy* $\hat{f}^\star \colon \mu \mapsto \int_X \hat{f}(x, \mu_x)\,\mathrm{d}\mu(x)$ for $\hat{f} \in \mathcal{C}^\infty(X \times I)$ (non-linear, not τ_n-continuous)
- $$\widehat{\mathfrak{Z}}^\infty_\varepsilon := \left\{ \begin{matrix} u \colon \mathscr{P}(X) \to \mathbb{R} : u = F \circ \left(\hat{f}_1^\star, \dots, \hat{f}_k^\star\right), \\ F \in \mathcal{C}_b^\infty(\mathbb{R}^k), \quad \hat{f}_i \in \mathcal{C}_c^\infty(X \times (\varepsilon, 1]), \quad i \leq k \end{matrix} \right\}, \qquad \varepsilon \geq 0$$
- $$\mathbf{B}_\varepsilon[w](\eta) := \sum_{x : \eta_x > \varepsilon} \mathrm{div}_x^{\mathrm{m}}\, w, \qquad \eta \in \mathscr{P}(X), \quad w \in \mathfrak{X}^\infty$$

Theorem ($\mathcal{D}_{\mathrm{m}}$-Martingale IbP on $\mathscr{P}(X)$ [D.S. '18a])

Let $\mathcal{B}_\varepsilon$ be the σ-algebra generated by $\widehat{\mathfrak{Z}}^\infty_\varepsilon$. Then, $\mathcal{B}_\bullet := (\mathcal{B}_\varepsilon)_{\varepsilon \in I}$ is a filtration on $\mathscr{P}(X)$,

$$\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathrm{m}}\, u\, \nabla_w v = -\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathrm{m}}\, v\, \nabla_w u - \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathrm{m}}\, u\, v\, \mathbf{B}_\varepsilon[w], \qquad u, v \in \widehat{\mathfrak{Z}}^\infty_\varepsilon, \quad w \in \mathfrak{X}^\infty,$$

and $\mathbf{B}_\bullet$ is a centered square-integrable $\mathcal{D}_{\mathrm{m}}$-martingale adapted to $\mathcal{B}_\bullet$.

Geometric diffusions in spaces of probability measures

- Generator(s)
- SPDEs
- The Dirichlet–Ferguson diffusion

Generator I: Diffusion + "Boundary term"

Corollary (Generator on $\widehat{\mathfrak{Z}}_0^\infty$)

The form $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ is generated by the Friedrichs extension of $(\mathbf{L}, \widehat{\mathfrak{Z}}_0^\infty)$,

$$\mathbf{L}u = \sum_{x\in\eta} \frac{\left(\Delta u(\eta + \eta_x\delta_\bullet - \eta_x\delta_x)\right)_x}{\eta_x} = \mathbf{L}_1 u + \mathbf{L}_2 u \qquad u \in \widehat{\mathfrak{Z}}_0^\infty$$

$$\mathbf{L}_1 u = \frac{1}{2}\sum_{i,j}^{k} \partial^2_{ij} F \circ (\hat{f}_1^\star, \dots, \hat{f}_k^\star) \cdot \left\langle \nabla \hat{f}_i \,\middle|\, \nabla \hat{f}_j \right\rangle^\star \qquad (\textit{diffusion term})$$

$$\mathbf{L}_2 u = \frac{1}{2}\sum_{i}^{k} \partial_i F \circ (\hat{f}_1^\star, \dots, \hat{f}_k^\star) \cdot \mathbf{B}_0[\nabla \hat{f}_i] \qquad (\textit{boundary term})$$

Remark

- $\mathbf{L}_1$ extends to and fixes $\mathfrak{Z}^\infty$
- $(\mathbf{L}_1, \mathfrak{Z}^\infty)$ is the "diffusion part" of other measure-valued stochastic process
  - Wasserstein Diffusion [von Renesse–Sturm *AoP* '09]
  - Modified Massive Arratia Flow [Konarovskyi *AoP* '17]

Generator II: SPDEs Literature Comparison

$\mathscr{P}(\mathbb{S}^1)$	$\mathrm{d}\mu_t = \mathrm{div}(\sqrt{\mu_t}\,\mathrm{d}W_t) + \frac{1}{2}\,\mathbf{L}_2^{WD}\,\mu_t\,\mathrm{d}t$	Wasserstein Diffusion [von Renesse–Sturm *AoP* '09]
$\mathscr{P}(I)$	$\mathrm{d}\mu_t = \mathrm{div}(\sqrt{\mu_t}\,\mathrm{d}W_t) + \frac{1}{2}\sum_{x\in\mu_t}\delta_x''\,\mathrm{d}t$	Modif. Massive Arratia Flow [Konarovskyi–von Renesse *CPAM* '19]
$\mathscr{P}_2(\mathbb{R}^n)$	$-\mathrm{d}\mu_t = \mathrm{div}(\mu_t\,\mathrm{d}W_t) - \frac{1}{2}\Delta\mu_t\,\mathrm{d}t$	"Brownian motion" [Chow–Gangbo *JDE* '19+]

Corollary (SPDE)

The diffusion process $\eta_\bullet$ satisfies (with $W_\bullet$ a Brownian motion) the SPDE

$\mathscr{P}(X)$	$\mathrm{d}\eta_t = \mathrm{div}(\sqrt{\eta_t}\,\mathrm{d}W_t) + \frac{1}{2}\sum_{x\in\eta_t}\Delta\delta_x\,\mathrm{d}t$	*Dirichlet–Ferguson* [D.S. '18a]

tested against $\hat{f}^\star,\ \hat{f} \in \mathcal{C}_c^\infty(X \times (0,1])$.

Process I: Dirichlet–Ferguson Diffusion

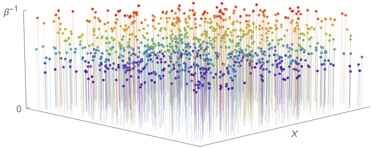
Initial condition: $\mu_0 := \sum_i^N s_0^i \, \delta_{x_0^i}\,, \qquad N \in \mathbb{N} \cup \{+\infty\}\,, \quad \sum_i^N s_0^i = 1\,, \quad s_0^i > 0$

Dirichlet–Ferguson Diffusion

Diffusion process on $\mathscr{P}(X)$ with $\dim X \geq 2$:

$$\eta_t = \sum_i^N s_0^i \, \delta_{x_t^i}\,, \qquad x_t^i = y^i_{t/s_0^i}\,, \qquad y^i_\bullet \text{ i.i.d. Brownian motions on } X$$

- since $\dim X \geq 2$, particles collide in finite time with probability 0
- if $\operatorname{supp} \eta_0 = X$, then $\operatorname{supp} \eta_t = X$ for each $t > 0$



Recap

- **(pre-)Dirichlet form:**

$$\mathcal{E}(u,v) := \int_{\mathscr{P}} \left\langle (\nabla u)_\eta \,\middle|\, (\nabla v)_\eta \right\rangle_{T_\eta \mathscr{P}} \mathrm{d}\mathcal{D}_{\mathrm{m}}(\eta)\,, \qquad u,v \in \mathfrak{Z}^\infty,$$

- **Process:** (started at purely atomic measures)

$$\mu_t = \sum_i^N s_0^i\, \delta_{x_t^i}\,, \qquad x_t^i = y^i_{t/s_0^i}\,, \qquad y^i_\bullet \text{ i.i.d. Brownian motions on } X$$

- **SPDE:**

$$\mathrm{d}\eta_t = \mathrm{div}(\sqrt{\eta}_t\, \mathrm{d}W_t) + \tfrac{1}{2} \sum_{x\in\eta_t} \Delta\delta_x\, \mathrm{d}t$$

- **Properties:** reversible recurrent diffusion, not ergodic, short-time heat kernel upper estimates via Rademacher Theorem on $\mathscr{P}_2$ [D.S. '18c], good points are purely atomic measures, essential self-adjointness of generator, ...
- **Questions:** Feller property, Ray–Knight compactification, ...

ご清聴ありがとうございました。

Thank you very much for your kind attention!

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Process II: Modified Massive Arratia Flow

Initial condition: $\mu_0 := \sum_i^N s_0^i \, \delta_{x_0^i}$, $N \in \mathbb{N} \cup \{+\infty\}$, $\sum_i^N s_0^i = 1$, $s_0^i > 0$

Modified Massive Arratia Flow (MMAF) (Konarovskyi *AoP* '17, K.–v. Renesse '18)

Sticky particle diffusion process on $\mathscr{P}(I)$:

$$\mu_t = \sum_i^N s_t^i \, \delta_{x_t^i}, \qquad x_t^i = y_{t/s_t^i}^i, \qquad y_\bullet^i \text{ i.i.d. Brownian motions on } X$$

two Brownian paths collide $\implies$ massive particles stick together:

at each first collision time s.t. $y_t^i = y_t^j$ set

$s_t^{i\wedge j} = s_{t-}^i + s_{t-}^j$ and $s_r^{i\vee j} = 0$ for all $r > 0$

- since $\dim I = 1$, particles collide in finite time with probability 1
- $\operatorname{supp} \mu_t$ is finite for each $t > 0$
- $\sum_{x \in \mu_t} f(x)$ converges for every $f \in \mathcal{C}(I)$ and $t > 0$

Integration by Parts III: Partial Quasi-Invariance

Definition

- *partial G-quasi-invariance* (Kondrat'ev–Lytvynov–Vershik *JFA* '15)
  - G group acting measurably on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$
  - $(\mathscr{F}_t)_{t\geq 0}$ a filtration of $(\Omega, \mathscr{F})$ s.t. $g.\mathscr{F}_s \subseteq \mathscr{F}_t$ for some $s \leq t$
  - for $g \in G$ there exists the Radon–Nikodým derivative $R_t[g]$

$$\int_\Omega u \, \mathrm{d}(g.)_\sharp \mathbf{P} = \int_\Omega u \, R_t[g] \, \mathrm{d}\mathbf{P} \,, \qquad u \colon (\Omega, \mathscr{F}_t) \longrightarrow \mathbb{R}$$

- $$\mathbf{R}_\varepsilon[\psi](\eta) := \prod_{x : \eta_x > \varepsilon} \frac{\mathrm{d}\psi_\sharp \mathrm{m}}{\mathrm{d}\mathrm{m}}(x) \,, \qquad \eta \in \mathscr{P}(X) \,, \quad \psi \in \mathrm{Diff}(X)$$

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- $$\mathbf{R}_\varepsilon[\psi](\eta) := \prod_{x : \eta_x > \varepsilon} \frac{\mathrm{d}\psi_\sharp \mathrm{m}}{\mathrm{dm}}(x) \,, \qquad \eta \in \mathscr{P}(X) \,, \quad \psi \in \mathrm{Diff}(X)$$

Theorem (partial $\mathrm{Diff}(X)_\sharp$-quasi-invariance of $\mathcal{D}_\mathrm{m}$)

The measure $\mathcal{D}_\mathrm{m}$ is partially $\mathrm{Diff}(X)_\sharp$-quasi-invariant w.r.t. $\mathcal{B}_\bullet$, and

$$\mathrm{d}_t\big|_{t=0} \mathbf{R}_\varepsilon[\psi^{w,t}](\eta) = \mathbf{B}_\varepsilon[w](\eta) \,, \qquad \eta \in \mathscr{P}(X) \,, \quad w \in \mathfrak{X}^\infty$$

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- $$\mathbf{R}_\varepsilon[\psi](\eta) := \prod_{x : \eta_x > \varepsilon} \frac{\mathrm{d}\psi_\sharp \mathrm{m}}{\mathrm{dm}}(x), \qquad \eta \in \mathscr{P}(X), \quad \psi \in \mathrm{Diff}(X)$$

Theorem (partial $\mathrm{Diff}(X)_\sharp$-quasi-invariance of $\mathcal{D}_\mathrm{m}$)

The measure $\mathcal{D}_\mathrm{m}$ is partially $\mathrm{Diff}(X)_\sharp$-quasi-invariant w.r.t. $\mathcal{B}_\bullet$, and

$$\mathrm{d}_t\big|_{t=0} \mathbf{R}_\varepsilon[\psi^{w,t}](\eta) = \mathbf{B}_\varepsilon[w](\eta), \qquad \eta \in \mathscr{P}(X), \quad w \in \mathfrak{X}^\infty$$

Generator III: Martingale Problem

Definition *weak atomic topology* τ_a (Ethier–Kurtz *SPA* '94)

the coarsest topology making all functions in $\widehat{\mathfrak{Z}}_0^\infty$ continuous

Theorem the form $\big(\mathcal{E}, \mathscr{D}(\mathcal{E})\big)$ is (quasi-)regular w.r.t. τ_a

Corollary (Martingale problem)

The Markov process $\eta_\bullet$ associated with $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ has τ_a-continuous sample paths. Furthermore, for each $u \in \widehat{\mathfrak{Z}}_0^\infty$ the process

$$M_t^u := u(\eta_t) - u(\eta_0) - \int_0^t (\mathbf{L}u)_{\eta_s}\, \mathrm{d}s$$

is a continuous martingale with quadratic variation process

$$[M^u]_t = \int_0^t \|(\boldsymbol{\nabla} u)_{\eta_s}\|^2_{T_{\eta_s}\mathscr{P}}\, \mathrm{d}s$$