Pinning, depinning, and homogenization of interfaces in random media

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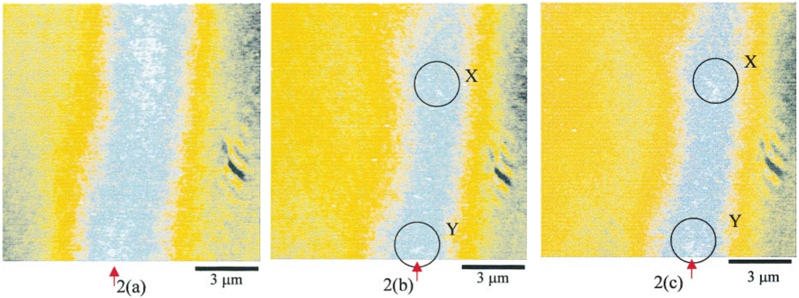
with Nicolas Dirr, Michael Scheutzow, Sebastian Throm, Martin Jesenko

Fukuoka – September 3, 2019

Pinning of a ferroelectric domain wall

increasing applied field ⟶

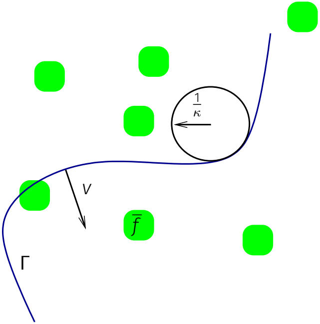


From: T. J. Yang et. al., Direct Observation of Pinning and Bowing of a Single Ferroelectric Domain Wall, *PRL*, 1999

Forced mean curvature flow

Consider an interface moving by forced mean curvature flow:

$$v_\nu(x) = \kappa(x) + \bar{f}(x), \quad x \in \Gamma \subset \mathbb{R}^{n+1}.$$



v_ν: Normal velocity of the interface

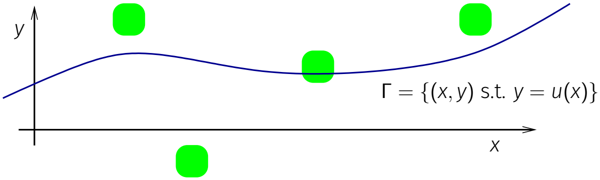
κ: Mean curvature of the interface

$\bar{f}$: Force

Can formally be thought of as a viscous gradient flow from an energy functional

$$\mathcal{H}^n(\Gamma) + \int_{\mathbb{R}^{n+1} \cap E} \bar{f}(x)\,dx, \quad \Gamma = \partial E.$$

The interface as the graph of a function



$$v_\nu(x) = \kappa(x) + \bar{f}(x), \quad x \in \Gamma \subset \mathbf{R}^{n+1}$$

If $\Gamma(t) = \{(x, y) \text{ s.t. } y = u(x, t)\}$, $u \colon \mathbf{R}^n \to \mathbf{R}$, then this is equivalent to

$$u_t(x) = \sqrt{1 + |\nabla u(x)|^2} \, \frac{1}{n} \operatorname{div} \left(\frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) + \sqrt{1 + |\nabla u(x)|^2} \, \bar{f}(x, u(x))$$

Formal approximation for small gradient:

$$u_t(x, t) = \Delta u(x, t) + \bar{f}(x, u(x, t))$$

This describes the time evolution of a nearly flat interface subject to line tension in a quenched environment.

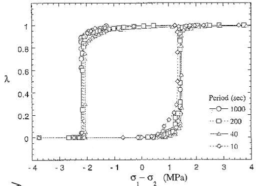
What are we interested in?

Split up the forcing into a heterogeneous part and an external, constant, load F so that

$$\bar{f}(x,y) = -f(x,y) + F,$$

and get

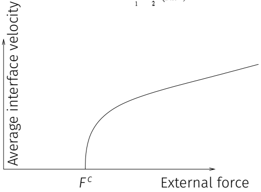
$$u_t(x,t) = \Delta u(x,t) - f(x,u(x,t)) + F.$$



Question

What is the macroscopic behavior of the solution u depending on F?

- Hysteresis: There exists a stationary solution up to a critical F^c
- Ballistic movement: $\overline{v} \;\; = \frac{u(t)}{t} \to const > 0.$
- Critical behavior: $|\overline{v}| = |F - F^c|^\alpha$



The periodic case

$$u_t(x,t) = \Delta u(x,t) - f(x, u(x,t)) + F$$

$$u\colon T^n \times \mathbf{R}^+ \to \mathbf{R}, \quad f \in C^2(T^n \times \mathbf{R}, \mathbf{R}), \quad f(x,y) = f(x, y{+}1), \quad \int_{T^n \times [0,1]} f = 0$$

Theorem (Dirr-Yip)

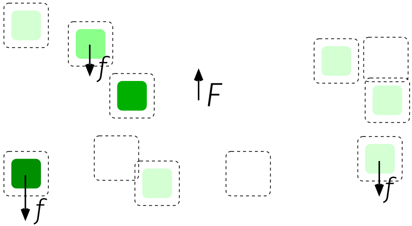
- *There exists $F^c \geq 0$ s.t. the evolution equation admits a stationary solution for all $F \leq F^c$.*
- *For $F > F^c$ there exists a unique time-space periodic ('pulsating wave') solution (i.e., u(x, t+T) = u(x, t) +1).*
- *If critical stationary solutions (i.e., stationary solutions at $F = F^c$) are non-degenerate, then $|\overline{v}| = \frac{1}{T} = |F - F^c|^{1/2} + o(|F - F^c|^{1/2})$*

Existence of pulsating wave solutions can also be shown for MCF-graph case, forcing small in C^1 (Dirr-Karali-Yip).

Random environment

$$u_t(x,t,\omega) = \Delta u(x,t,\omega) - f(x,u(x,t,\omega),\omega) + F,$$

$$u\colon \mathbf{R}^n \times \mathbf{R}^+ \times \Omega \to \mathbf{R}, \quad f\colon \mathbf{R}^n \times \mathbf{R} \times \Omega \to \mathbf{R}, \quad u(x,0) = 0.$$



Poisson process to scatter obstacles

$$f(x,y,\omega) = \sum_{k\in\mathbf{N}} f_k(\omega)\varphi(x - x_k(\omega), y - y_k(\omega)), \quad \varphi \in C^\infty(\mathbf{R}^n \times \mathbf{R}, [0,\infty)),$$

$$\varphi(x,y) = 0 \text{ if } ||(x,y)||_\infty > r_1, \quad \varphi(x,y) = 1 \text{ if } ||(x,y)||_\infty \le r_0, \quad y_k > r_1.$$

Existence of a stationary solution

Do solutions of the evolution equation become pinned by the obstacles for sufficiently small driving force, even though there are arbitrarily large areas with arbitrarily weak obstacles?

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Do solutions of the evolution equation become pinned by the obstacles for sufficiently small driving force, even though there are arbitrarily large areas with arbitrarily weak obstacles?

Theorem (Dirr-D.-Scheutzow)
Let (x_k, y_k) *be distributed according to a* $n+1$*-d Poisson process on* R^{n+1} *with intensity* λ, f_k *be strictly positive and independent of* (x_k, y_k)*. Then there exists* $F^* > 0$ *and* $v\colon \mathrm{R}^n \times \Omega \to \mathrm{R}$, $v > 0$ *so that, a.s., for all* $F < F^*$,

$$0 > \Delta v(x, \omega) - f(x, v(x, \omega), \omega) + F.$$

This implies that v is a supersolution to the stationary equation, and thus provides a barrier that a solution starting with zero initial condition can not penetrate by the comparison principle.

Related results: pinning with $\pm$-Obstacles, localized rate-independent dissipation, mean curvature.

A percolation problem

Let $\mathcal{Z} = \mathsf{Z}^n \times \mathsf{N}$.

We consider site percolation on $\mathcal{Z}$: let $p \in (0,1)$.

Each site is declared *good* with probability p, independent for all sites.

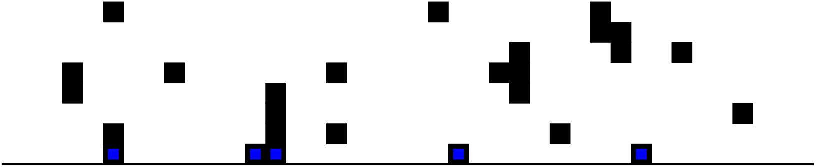
Theorem (Dirr-D.-Grimmett-Holroyd-Scheutzow)

There exists $p_c < 1$ such that if $p > p_c$, then a random non-negative discrete 1-Lipschitz function $w\colon \mathsf{Z}^n \to \mathsf{N}$ exists a.s. with $(x, w(x))$ good for all $x \in \mathsf{Z}^n$.

Idea:

Blocking argument. Define Λ-path: Finite sequence of distinct sites x_i from a to b so that $x_i - x_{i-1} \in \{\pm e_{n+1}\} \cup \{-e_{n+1} \pm e_j : j = 1, \dots, n\}$. Admissible if going up only to closed sites.

Which sites on the positive side are reachable from anywhere below?



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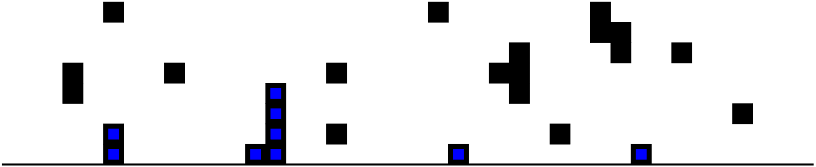

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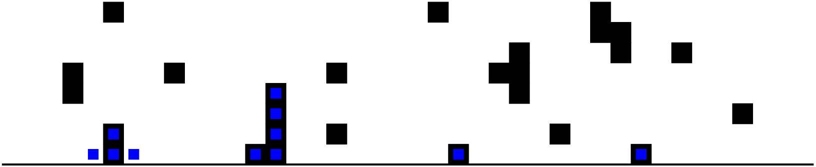

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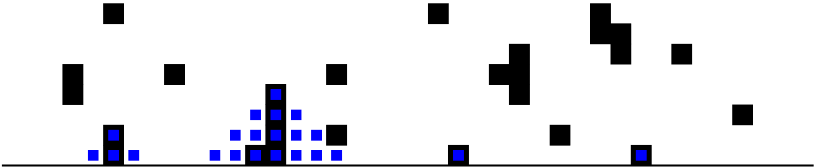

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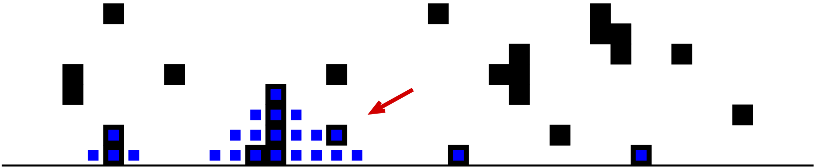

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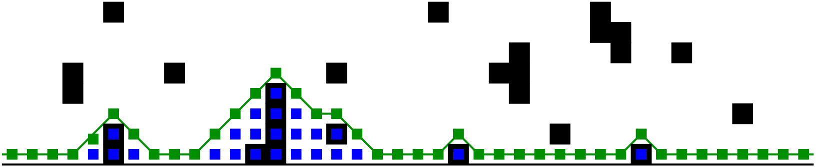
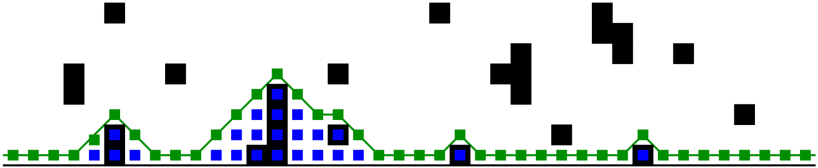
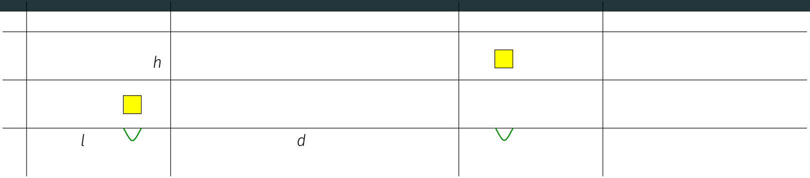


Proof of Lipschitz-Percolation Theorem



- Define $G := \{b \in \mathcal{Z} :$ there ex. path to b from some $a \in \mathbf{Z}^n \times \{\dots, -1, 0\}\}$.
- We have $\mathrm{P}(he_{n+1} \in G) \leq C(cq)^h$, thus there are only finitely many sites in G above each $x \in \mathbf{Z}^n$.
- Define $w(x) := \min\{t > 0 : (x, t) \notin G\}$.
- Properties of w follow from the definition of admissible paths.

Proof of the pinning theorem



- We rescale such that a cuboid of size $l^n \times h$ contains an obstacle (x_k, y_k) of strength f_0 with probability $p > p_c$.
- Explicit construction of the supersolution
  - Inside of the obstacles: $\Delta v_{\text{in}} = F_1 < \frac{f_0}{2}$, $v_{\text{in}} = 0$ on $\partial B_{r_0}(x_k)$.
  - Outside: $\min_k\{v_{\text{out}}(x - x_k)\}$, where $\Delta v_{\text{out}} = -F_2$ on $B_{r_l}(0) \setminus B_{r_0}(0)$, $v_{\text{out}} = 0$ on $\partial B_{r_0}(0)$, $\nabla v_{\text{out}} \cdot \nu = 0$ on $\partial B_{r_l}(0)$
  - Glue together using v_{glue} with non-vanishing gradient only on the gaps, otherwise $v_{\text{glue}} = y_k$.
  - Scaling:
$$CF_1 > F_2(h^{-1/n} + d)^n \quad \text{and} \quad F_2 \geq C' \frac{h}{d^2}.$$
- Also for MCF, taking care of the nonlinearity when considering the addition of v_{glue}.

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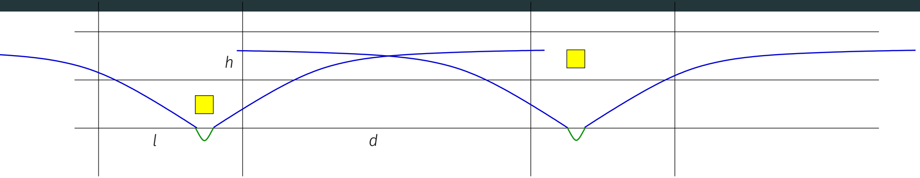

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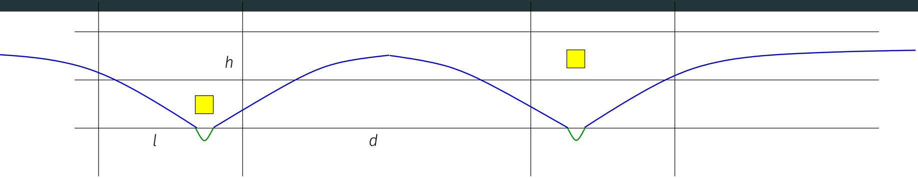

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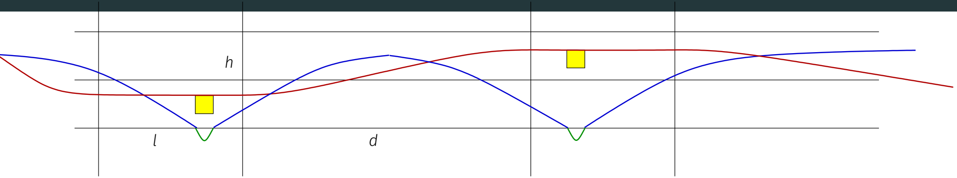

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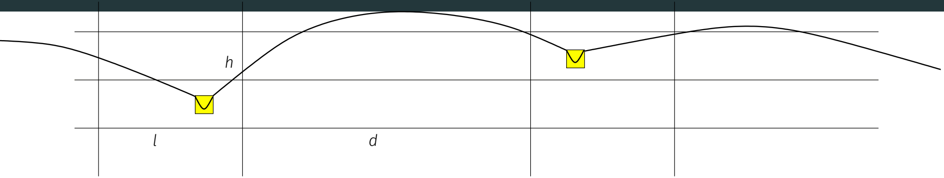

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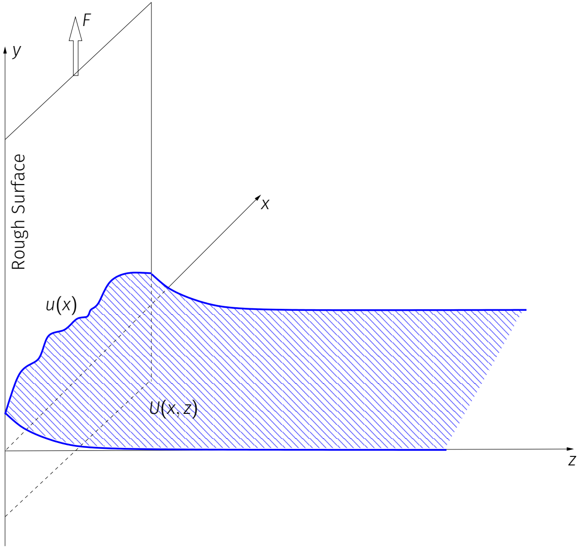
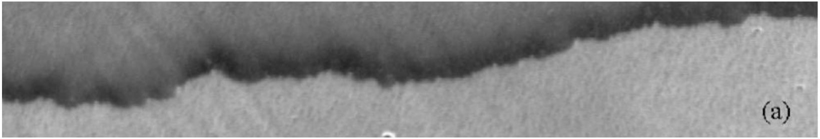
Experiment



Experiments

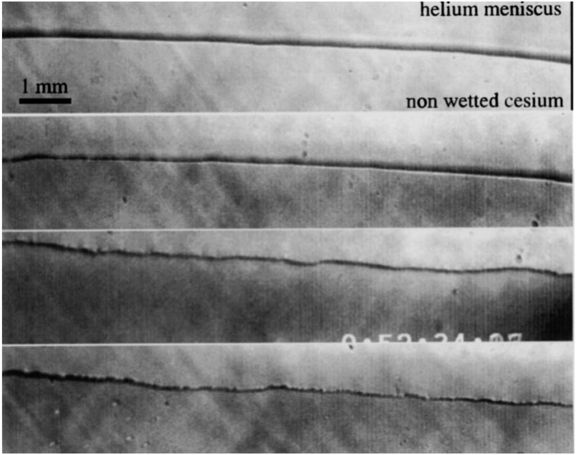
Aus: S. Moulinet et. al., Roughness and dynamics of a contact line of a viscous fluid on a disordered substrate, *Eur. Phys. J. E*, 2002



Aus: A. Prevost et. al., Dynamics of a helium-4 meniscus on a strongly disordered cesium substrate, *Phys. Rev. B* 2002

Second model: evolution of contact lines (cont.)

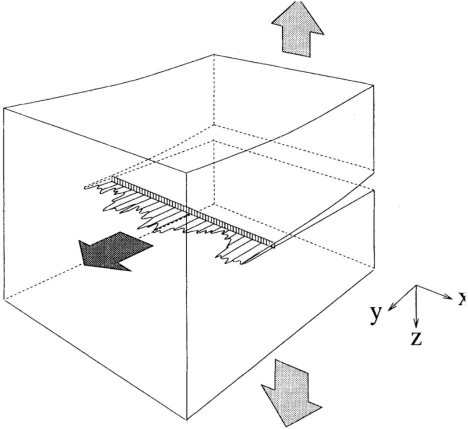
Experiments



From: E. Rolley et. al., Roughness of the Contact Line on a Disordered Substrate, *Phys. Rev. Lett.*, 1998

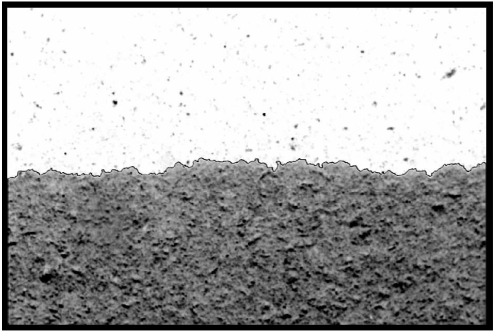
Second model: fracture in heterogeneous media

Experiment



Aus: J. Schmittbuhl et. al., Interfacial Crack Pinning: Effect of Nonlocal Interactions, *Phys. Rev. Lett.* 1995

Experimental observation



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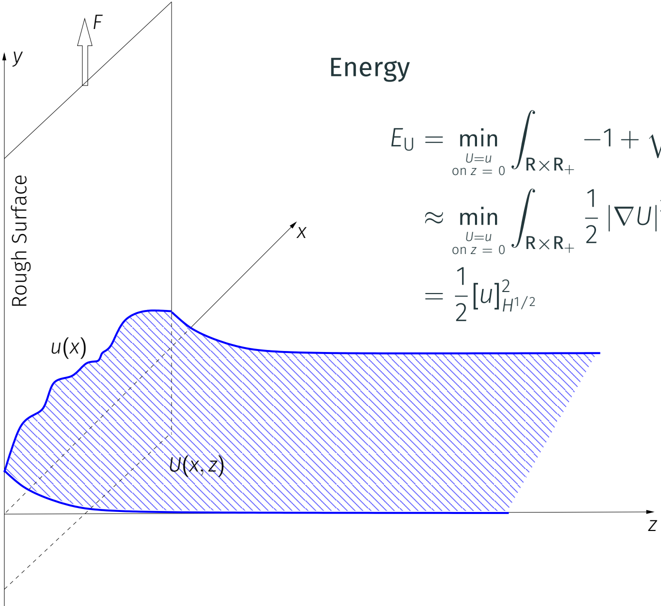
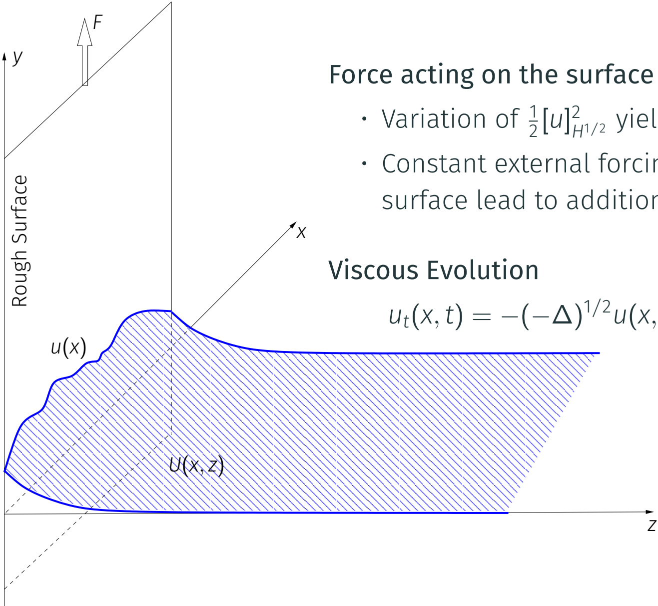
Second model: A formal derivation



Energy

$$
\begin{aligned}
E_{\mathrm{U}} &= \min_{\substack{U=u \\ \text{on } z=0}} \int_{\mathbf{R}\times\mathbf{R}_+} -1 + \sqrt{1+|\nabla U|^2}\,\mathrm{d}x\mathrm{d}z \\
&\approx \min_{\substack{U=u \\ \text{on } z=0}} \int_{\mathbf{R}\times\mathbf{R}_+} \frac{1}{2}|\nabla U|^2\,\mathrm{d}x\mathrm{d}z \\
&= \frac{1}{2}[u]^2_{H^{1/2}}
\end{aligned}
$$



Force acting on the surface

- Variation of $\frac{1}{2}[u]^2_{H^{1/2}}$ yields $-(-\Delta)^{1/2}u$
- Constant external forcing and rough surface lead to additional force $\bar{f}(x, y)$

Viscous Evolution

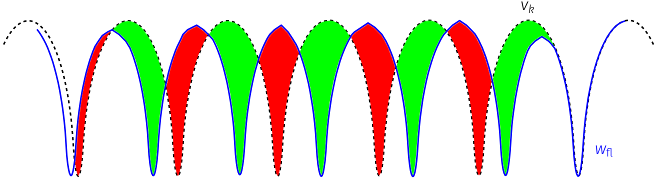
$$u_t(x,t) = -(-\Delta)^{1/2}u(x,t) + \bar{f}(x, u(x,t))$$

Second model: Non-local issues

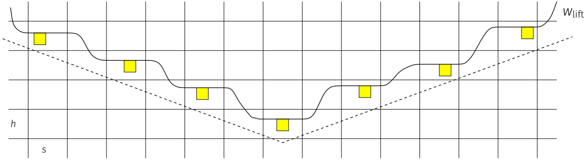
The non-locality of the fractional Laplacian introduces new issues

$$-(-\Delta)^{\alpha}u(x) = \int_{\mathbf{R}^n} \frac{u(y)-u(x)}{|x-y|^{n+2\alpha}}\,\mathrm{d}y$$

- Piecewise constructoion is no longer possible



- Growth of the lifting function v_{glue} is linear in the case of Lipschitz percolation. We again need the stronger percolation result.



Theorem: *Flat percolation clusters (D.-Scheutzow, 2015)*
Consider again site percolation on $\mathcal{Z}$ with parameter p. Let $H : \mathbb{N}_0 \to \mathbb{N}_0$ be a non-decreasing function satisfying

i) $H(0) = 0$
ii) $H(1) \geq 1$
iii) $\liminf_{k\to\infty} \frac{H(k)}{\log k} > 0$,

Then there exists $p_H = p_H(n) \in (0,1)$ such that for any $p \in (p_H, 1]$ and almost any realization of the site percolation model we can find a (random) function $w : \mathbb{Z}^n \to \mathbb{N}$ such that

i) $|w(x) - w(y)| \leq H(\|x - y\|)$ for all $x, y \in \mathbb{Z}^n$
ii) $(x, w(x))$ is open for every $x \in \mathbb{Z}^n$.

Depinning, pinning sites on lattice

$$u_t = \Delta u - f(x, u(x,t), \omega) + F$$

$$\text{with } f(x, y, \omega) = f_{ij}(\omega)\varphi(x - i, y - j), \quad i, j \in \mathbb{Z}, \quad f_{ij} \text{ iid.}$$

Can we exclude pinning for unbounded obstacles, if the probability of finding a large obstacle is sufficiently small and the driving force is sufficiently high?

Depinning, pinning sites on lattice

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Theorem (Dirr-Coville-Luckhaus)
Let f_{ij} be iid random variables so that $\beta := \mathbf{E}\exp\{\lambda f_{00}\} < \infty$ for some $\lambda > 0$. Then there exists $F^{**} > 0$ so that a.s. no stationary solution $v > 0$ for the evolution equation at $F > F^{**}$ exists.

Proof by asserting that every possible stationary solution with Dirichlet boundary conditions $u(-L) = 0, u(L) = 0$ becomes large as $L \to \infty$. (The pinning sites are not strong enough to keep the solution flat.)

Depinning, pinning sites on lattice

Theorem (*Ballistic propagation*, D.-Scheutzow)
Let $u(x,t,\omega)$ be a solution of the evolution equation for $n=1$ with f_{ij} iid so that $\beta := \mathsf{E}\exp\{\lambda f_{00}\} < \infty$ for some $\lambda > 0$. Then there exists $V_{\lambda,\beta}\colon [0,\infty) \to [0,\infty)$, non-decreasing, not identiacally zero, depending only on λ und β, such that

$$\mathsf{E}\,\frac{1}{t}\int_0^1 u(\xi,t)\,\mathrm{d}\xi \geq V(F) \quad \textit{for all } t \geq 0 \quad \textit{and} \quad \limsup_{t\to\infty} \frac{1}{t}u(0,t) \geq V(F) \textit{ a.s.}$$

and an analogous result holds for the lattice differential equation with discrete Laplacian on Z^n.

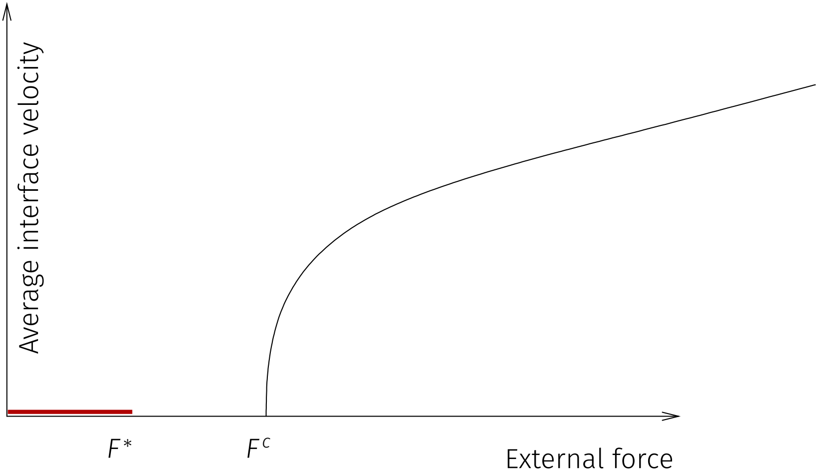
Idea: Discretize and look at the random variables

$$Y_k := \sum \exp\Big\{\lambda \sum_{\substack{i\in Q_k\\ r\notin Q_k\\ ||i-r||_1=1}} (w_r - w_i) - \mu \sum_{i\in Q_k} \big(\Delta_1 w_i - \bar{f}_i(w_i,\omega) + F\big)\Big\}$$

with the first sum taken over admissible discrete paths, $\mu > \lambda$, Q_k boxes of side-length k.

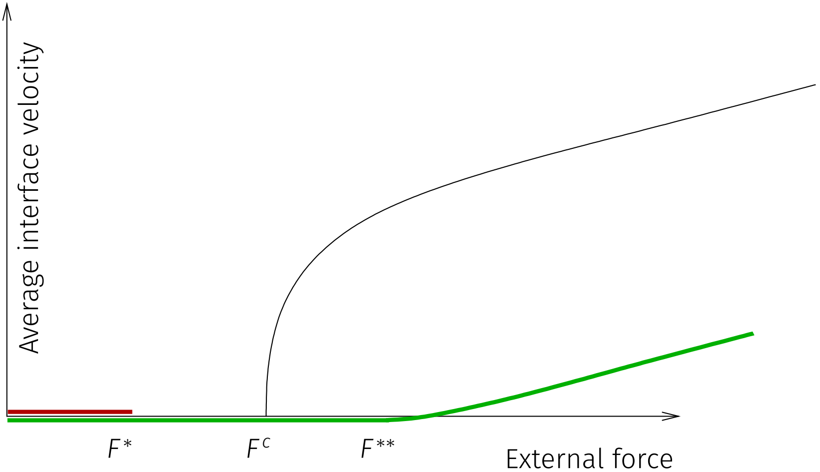
Summary of the results

Obstacles scattered by Poisson process, any strength



Summary of the results (cont.)

Some special cases and obstacles with exponential tails



Many open questions

- Homogenization in the general setting, i.e., $\frac{1}{t}\mathsf{E}(u(0,t)) \to \bar{v}$?
- Exclusion of an intermediate sub-ballistic regime (c/f Bodineau-Teixeira)?
- More general random fields.
- Power-law depinning behavior.
- Infinite Pinning.

Thank you for your attention.
ご清聴ありがとうございました。