A variational characterization of the sine_β process

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The 1D log gas

- random points $\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ distributed via Gibbs measure

$$\mathrm{d}\mathbb{P}_{N,\beta}(\boldsymbol{x}) = \frac{1}{Z_{N,\beta}} \exp\Big[-\frac{\beta}{2} H_N(\boldsymbol{x}) \Big] \mathrm{d}\boldsymbol{x}$$

  with β inverse temperature and Hamiltonian

$$H_N(\boldsymbol{x}) = N \sum_{i=1}^{N} \frac{|x_i|^2}{2} - \sum_{i \neq j} \log |x_i - x_j|$$

- connections to random matrix theory
  $\beta = 1, 2, 4$: EVs of Gaussian orthogonal, unitary, symplectic ensemble

- **Question:** Limiting behaviour as $N \to \infty$?

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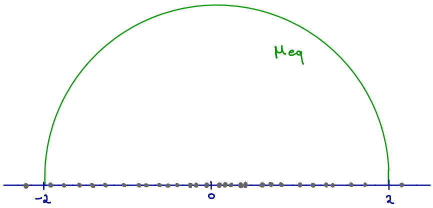
Macroscopic behaviour

- convergence of empirical measure:

  $\mathbb{P}_{N,\beta}$ a.s. we have:

$$\mu_N(\boldsymbol{x}) := \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i} \xrightarrow{N\to\infty} \mu_{\text{eq}}$$



- μ_{eq} minimizes macroscopic energy

$$I(\mu) := \int\int -\log|x-y|\mathrm{d}\mu(x)\mathrm{d}\mu(y) + \int \frac{|x|^2}{2}\mathrm{d}\mu(x)$$

- large deviation principle: [Ben Arous –Gionnet '97]

$$\mathbb{P}_{N,\beta}\Big[\mu_N \approx \mu\Big] \asymp \exp\Big(-N^2\big(I(\mu) - I(\mu_{\text{eq}})\big)\Big)$$

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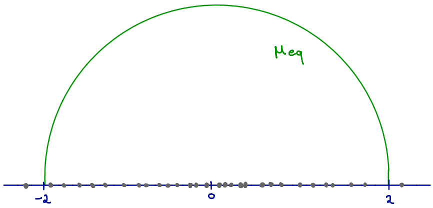

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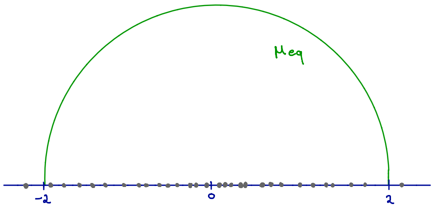

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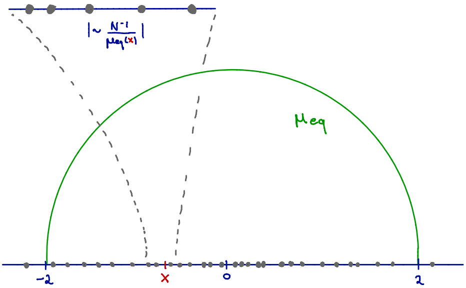
Microscopic behaviour

- blow up configuration around x:

$$C_{N,x}(\boldsymbol{x}) := \sum_{i=1}^{N} \delta_{N(x_i - x)}$$

with $\boldsymbol{x} \sim \mathbb{P}_{N,\beta}$



- [Valkò–Viràg '09]:

$$\text{law}(C_{N,x}) \overset{N\to\infty}{\longrightarrow} \text{sine}_\beta(\mu_{\text{eq}}(x))$$

sine$_\beta$ is stationary point process

- dependence on x only through intensity of equilibrium, universal bulk behaviour
- sine$_\beta$ minimizes free energy $\mathcal{F}_\beta = \mathcal{E} + \beta\mathcal{W}$ with $\mathcal{E}$ (specific) entropy, $\mathcal{W}$ renormalized energy
- large deviation principle: [Leblé–Serfaty '17]

$$\mathbb{P}_{N,\beta}\Big[\int_{-2}^{2} C_{N,x}\mathrm{d}x \approx P\Big] \asymp \exp\Big(-N\big(\bar{\mathcal{F}}_\beta(P) - \min \bar{\mathcal{F}}_\beta\big)\Big)$$

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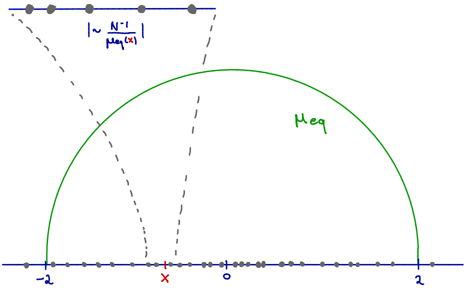

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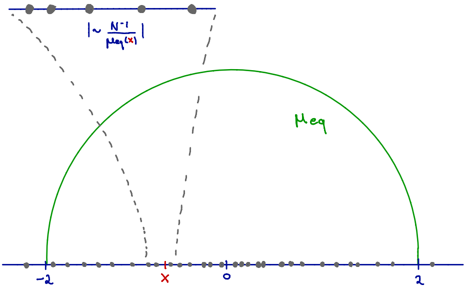

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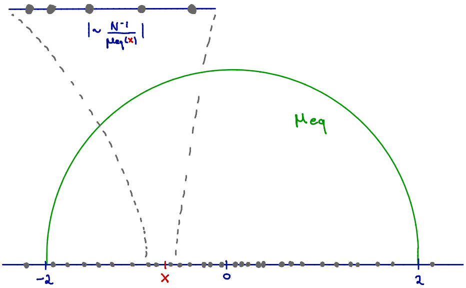

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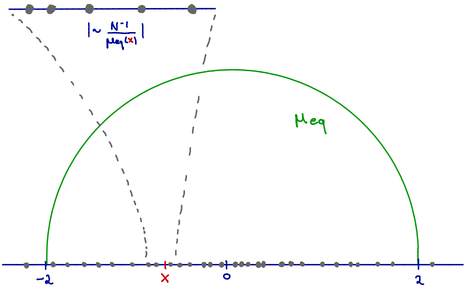

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Microscopic contribution to the energy

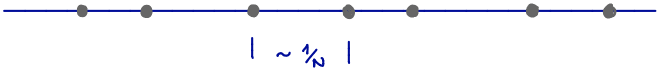
- consider fluctuations $f_N^{\boldsymbol{x}} = \sum_{i=1}^N \delta_{x_i} - N\mu_{\mathrm{eq}}$

$$H_N(\boldsymbol{x}) = N^2 I(\mu_{\mathrm{eq}}) + 2N\sum_{i=1}^N \xi(x_i) - \underbrace{\int\int \log|x-y|\mathrm{d}f_N^{\boldsymbol{x}}(x)\mathrm{d}f_N^{\boldsymbol{x}}(y)}_{F_N(\boldsymbol{x})}$$

with $\xi = -\log * \mu_{\mathrm{eq}} + \frac{1}{4}|\cdot|^2 + c$ effective potential

- electric interpretation: $\mathbb{R} \hat{=} \mathbb{R} \times \{0\} \subset \mathbb{R}^2$, $\log|\cdot|$ Coulomb potential

$$\phi_N^{\boldsymbol{x}}(x) = -\log * \left(\Sigma_i \delta_{x_i} - N\mu_{\mathrm{eq}}\right)(x)\,, \quad -\Delta\phi_N^{\boldsymbol{x}} = 2\pi\left(\Sigma_i \delta_{x_i} - N\mu_{\mathrm{eq}}\right)$$



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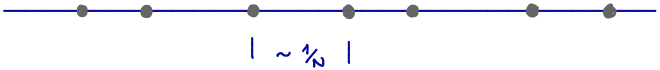

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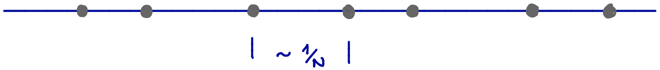

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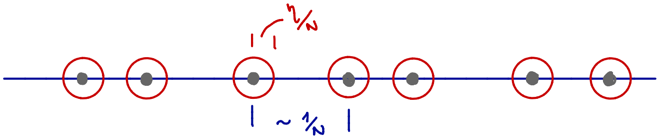
regularization by smearing out charges: $\delta_x^\eta = \text{Unif}(\partial B_\eta(x))$



renormalized interaction energy

- for a point configuration $\mathcal{C}$ on $\mathbb{R}$ set

$$\mathcal{W}(\mathcal{C}) = \inf_E \qquad \frac{1}{2\pi}\int \qquad |E \quad|^2 + \log\eta$$

$$-\mathrm{div}E = 2\pi\big(\mathcal{C} - \mathrm{d}x\big) \;,$$

- for a point process P on $\mathbb{R}$ (with intensity 1) set

$$\mathcal{W}(P) = \mathbb{E}_P\big[\mathcal{W}(\mathcal{C})\big]$$

- **Remark:** if $\mathcal{W}(\mathcal{C}) < \infty$ we have

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where Π^1 is Poisson point process on $\mathbb{R}$ with intensity measure $\mathrm{d}x$

renormalized free energy

- for a stationary point process P (with intensity 1), $\beta \geq 0$ set

$$\mathcal{F}_\beta(P) = \mathcal{E}(P) + \beta \mathcal{W}(P)$$

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- $\mathcal{F}_\beta$ is l.s.c., with compact sublevels, hence compact set of minimizers
- [LEBLÉ–SERFATY '17] sine$_\beta$ minimizes $\mathcal{F}_\beta$

Theorem ([E.–HUESMANN–LEBLÉ 18+])

Sine$_\beta$ is the unique minimizer of $\mathcal{F}_\beta$.

Problem: $P \mapsto \mathcal{F}_\beta(P)$ is affine w.r.t. linear interpolation

Idea: use displacement interpolation from optimal transport

Variational characterization of sine$_\beta$

renormalized free energy

- for a stationary point process P (with intensity 1), $\beta \geq 0$ set

$$\mathcal{F}_\beta(P) = \mathcal{E}(P) + \beta \mathcal{W}(P)$$

- $\mathcal{F}_\beta$ is l.s.c., with compact sublevels, hence compact set of minimizers
- [LEBLÉ–SERFATY '17] sine$_\beta$ minimizes $\mathcal{F}_\beta$

Theorem ([E.–HUESMANN–LEBLÉ 18+])

Sine$_\beta$ is the unique minimizer of $\mathcal{F}_\beta$.

Problem: $P \mapsto \mathcal{F}_\beta(P)$ is affine w.r.t. linear interpolation

Idea: use displacement interpolation from optimal transport

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Recap on optimal transport

- **Monge problem**: $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^n)$,
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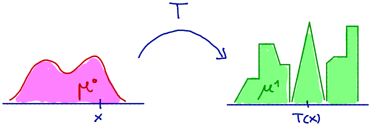

- **Brenier's theorem:** If μ_0 absolutely continuous w.r.t. Lebesgue there exists an optimal map T and $T = \nabla\phi$ for some $\phi : \mathbb{R}^n \to \mathbb{R}^n$ convex

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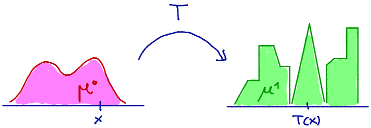

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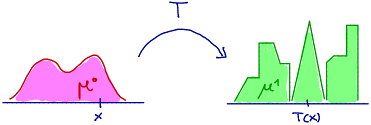

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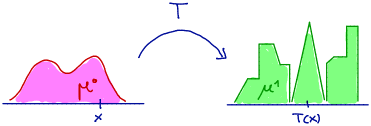

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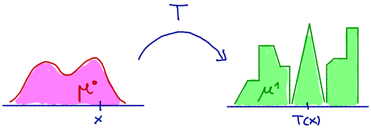

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Ideas of the proof for uniqueness

Let P^0, P^1 stationary point processes minimizing $\mathcal{F}_\beta$

Step 1: Large box approximation

- let $P^i|_{\Lambda_R}$ restriction of P^i to $\Lambda_R = [-R, R]$, $i = 0, 1$
- screening procedure yields pp's P^i_R on Λ_R s.t.

$$(1) \text{a.s. } 2R \text{ points in } \Lambda_R$$

$$(2) \frac{1}{R} \mathrm{Ent}\left[P^i_R | \Pi^1_{\Lambda_R}\right] \leq \mathcal{E}(P^i) + \varepsilon$$

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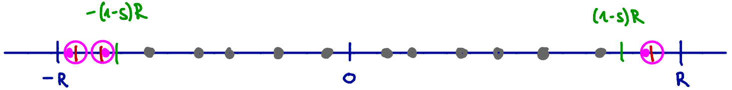

Step 2: displacement interpolation

- identify P_R^i with $\mu^i \in \mathcal{P}(\mathbb{R}^{2R})$ via labeling
- displacement interpolant $\mu^h := \left(\frac{1}{2}\mathrm{id} + \frac{1}{2}\nabla\phi\right)_{\#}\mu^0$
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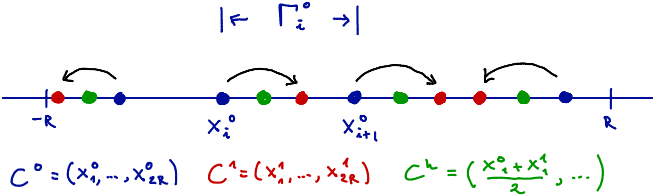
- convexity of entropy

$$\begin{aligned}\mathrm{Ent}\left[P_R^h|\Pi^1_{\Lambda_R}\right] &= \mathrm{Ent}\left[\mu^h|\mathrm{Leb}\right] + c_R\\ (\mathrm{McCann}) &\le \frac{1}{2}\mathrm{Ent}\left[\mu^0|\mathrm{Leb}\right] + \frac{1}{2}\mathrm{Ent}\left[\mu^1|\mathrm{Leb}\right] + c_R\\ &\le \frac{R}{2}\cdot\left(\mathcal{E}(P^0) + \mathcal{E}(P^1)\right) + \varepsilon\end{aligned}$$



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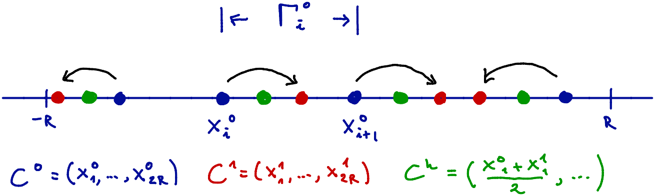

Ideas of the proof for uniqueness

- establish strict convexity of interaction energy with gain or order R
- use estimate

$$-\log\left(\frac{x+y}{2}\right) \le -\frac{1}{2}\left(\log x + \log y\right) - \frac{1}{8}\frac{(x-y)^2}{x^2+y^2}$$

- estimate interaction of interpolated points

$$\begin{aligned}\mathcal{W}_{\text{int}}(\mathcal{C}^h) &= \sum_{i<j} -\log\left[\frac{x_i^0+x_i^1}{2} - \frac{x_j^0+x_j^1}{2}\right] \\ &\le \frac{1}{2}\sum_{i<j} -\log\left[x_i^0 - x_j^0\right] - \log\left[x_i^1 - x_j^1\right] - \frac{1}{8}\sum_i \frac{|\Gamma_i^0 - \Gamma_i^1|^2}{|\Gamma_i^0|^2 + |\Gamma_i^1|^2} \\ &= \frac{1}{2}\mathcal{W}_{\text{int}}(\mathcal{C}^0) + \frac{1}{2}\mathcal{W}_{\text{int}}(\mathcal{C}^1) - \text{Gain}_R(\mathcal{C}^0, \mathcal{C}^1)\end{aligned}$$

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Step 3: build global competitor

- partition $\mathbb{R}$ in boxes of size $2R$, place iid copies of P_R^h, randomize origin
  $\Rightarrow$ obtain stationary point process P^h on $\mathbb{R}$

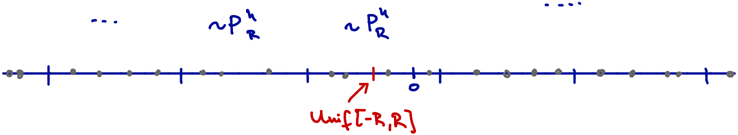
- check that copies on different boxes do not interact too much

$$\mathcal{W}(P^h) \leq \frac{1}{R}\mathcal{W}_{\mathrm{int}}(P_R^h) + \mathrm{error}$$

- conclude contradiction

$$\mathcal{F}_\beta(P^h) \leq \frac{1}{2}\mathcal{F}_\beta(P^0) + \frac{1}{2}\mathcal{F}_\beta(P^1) + \mathrm{error} - g < \mathcal{F}_{\min}$$

□



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- Can one gain strict displacement convexity along this interpolation?
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