Consistency and ISDE representation of long range interacting particle systems with jumps

Syota Esaki (Fukuoka University)

joint work with Hideki Tanemura (Keio university)

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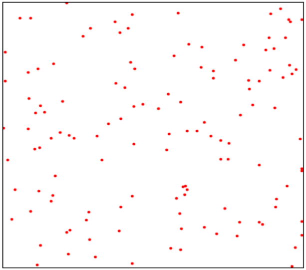
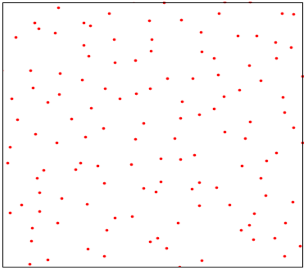
Introduction

Poisson point process
[independent]



Ginibre point process
[2dim Coulomb repulsion]



Examples of "long range" interactions(LR interactions)

- Dyson RPF,
- Ginibre RPF,
- Airy RPF,

These are related to random matrix theory. We remark that for the each associated integral operator K, $\mathrm{Spec}(K)$ contains 1.

We would like to consider jump type dynamics(e.g. α-stable type), whose equilibrium measure is given by the RPFs in the above.

General theory of the construction of dynamics with such LR interactions

- Brownian motion : Osada(1996, 2013)
- jump type : E. (2019 in Tohoku Math. Journal)

Remark 1

These dynamics are constructed on the configuration space. Their studies use the Dirichlet form technique.

Today's our aim

Give infinite dimensional stochastic differential equation(ISDE) representations for the LR interacting jump type systems.

Remark 2

For LR interacting Brownian motion Osada gave ISDE representation.

contents

1. Preliminaries(construct unlabeled process[E.19])
2. Coupling and consistency
3. ISDE representation

Preliminaries

- $S = \mathbb{R}^d$: the state space
- $\mathfrak{M} = \{\xi;\ \xi$ is a non negative integer valued Radon measure. $\}$
  : configuration space

  $\mathfrak{M}$ is a Polish space with vague topology.
- $\mathscr{D}_\circ = \{f : \mathfrak{M} \to \mathbb{R}; f$ is local and smooth$\}$
- $U_r = \{x \in S; |x| \le r\}$.
- $\pi_r(\xi) = \xi(\cdot \cap U_r)$, $\pi_r^c(\xi) = \xi(\cdot \cap U_r^c)$.
- μ : a Random point field on S
- ρ^1 : the 1-correlation function of μ defined by

$$\int_A \rho^1(x)dx = \int_{\mathfrak{M}} \xi(A)d\mu,$$

for any $A \subset \mathcal{B}(S)$.

- For a random point field μ we set

$$\mu_{r,\eta}^{m}(\cdot) = \mu(\pi_r(\xi) \in \cdot | \xi(U_r) = m, \pi_r^c(\xi) = \pi_r^c(\eta))$$

- $\Psi : S \to \mathbb{R} \cup \{\infty\}$ (interaction)
- $\mathcal{H}_r = \sum_{s_i, s_j \in U_r, i<j} \Psi(s_i - s_j)$

Definition 3 (quasi Gibbs meas.)

μ is a Ψ-quasi-Gibbs meas. if $\exists$ $c_{r,\eta}^{m}$ s.t.

$$c_{r,\eta}^{m}{}^{-1} e^{-\mathcal{H}_r} d\Lambda_r^m \leq \mu_{r,\eta}^{m} \leq c_{r,\eta}^{m} e^{-\mathcal{H}_r} d\Lambda_r^m$$

Here $\Lambda_r^m = \Lambda(\cdot | \, \xi(U_r) = m)$ and Λ_r is the Poisson RPF with $\mathbf{1}_{U_r} dx$.

Lemma and Theorem(Osada)

- canonical Gibbs meas. $\Rightarrow$ quasi-Gibbs meas.
- Dyson, Ginibre, Airy random point field are not canonical Gibbs but quasi-Gibbs($\Psi(x) = -\beta \log |x|$).

We define our bilinear form $(\mathfrak{E}, \mathfrak{D}_\infty)$. For $f, g \in \mathscr{D}_\circ$ we set $\mathbb{D}[f,g] : \mathfrak{M} \to \mathbb{R}$ by the following.

$$\mathbb{D}[f,g](\xi) = \frac{1}{2}\sum_i \int_S (f(\xi^{s_i,y}) - f(\xi))(g(\xi^{s_i,y}) - g(\xi))p(|s_i - y|)dy,$$

where $s_i \in S$, $\xi = \sum_i \delta_{s_i}$, $\xi^{s_i,y_i} = \xi + \delta_{y_i} - \delta_{s_i}$, p is a nonnegative measurable function on S satisfying
(p.1) $p(z) = O(|z|^{-(d+\alpha)})$ as $|z| \to \infty$ for some $0 < \alpha < 2$.
(p.2) $p(z) = O(|z|^{-(d+\gamma)})$ as $|z| \to +0$ for some $0 < \gamma < 2$.

$$\mathfrak{E}(f,g) = \int_{\mathfrak{M}} \mathbb{D}[f,g](\xi)d\mu, \quad f, g \in \mathfrak{D}_\infty,$$

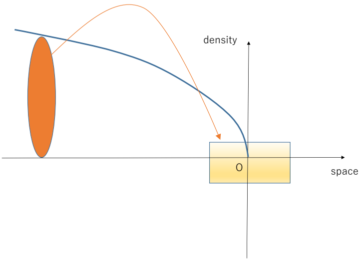
$$\mathfrak{D}_\infty = \{f \in \mathscr{D}_\circ \cap L^2(\mathfrak{M}, \mu); \mathfrak{E}(f,f) < \infty\}.$$

Assumptions

(A.1) μ is a Ψ-quasi Gibbs meas. with suitable Ψ.

(A.2) $\rho^1(x) = O\left(|x|^\kappa\right)$ as $|x| \to \infty$ for some $0 \leq \kappa < \alpha$.

(+ some technical assumptions)



(p.1) and (A.2) $\Rightarrow$

$$\int_S \rho^1(y)p(|y-x|)dy < C\rho^1(y)$$

Proposition 4

(A.1) $\Rightarrow (\mathfrak{E}, \mathfrak{D}_\infty)$ *is closable on* $L^2(\mathfrak{M}, \mu)$.

$\to (\mathfrak{E}, \mathfrak{D})$: the closure of $((\mathfrak{E}, \mathfrak{D}_\infty), L^2(\mathfrak{M}, \mu))$.

Theorem 1 (E.(2019 in Tohoku Mathematical Journal))

Assume (p.1)–(p.2), (A.1)–(A.2) (+α). Then $(\mathfrak{E}, \mathfrak{D})$ *is a quasi-regular Dirichlet form on* $L^2(\mathfrak{M}, \mu)$. *Hence there exists a Hunt process* $(\Xi(t), \{\mathbb{P}_\xi\}_{\xi \in \mathfrak{M}})$ *associated with* $((\mathfrak{E}, \mathfrak{D}), L^2(\mathfrak{M}, \mu))$. *Moreover* $\Xi(t)$ *is reversible with invariant measure* μ.

Examples of interacting α-stable systems

μ : Dyson random point field (on $\mathbb{R}$), Ginibre random point field (on $\mathbb{C}$)
$\rightarrow \rho^1 \equiv \mathrm{const.}$
$\rightarrow$ We can construct interacting symmetric α-stable processes for $0 < \alpha < 2$.

μ : Airy random point field (on $\mathbb{R}$)
$\rightarrow \rho^1(x) = O(|x|^{1/2})$ as $x \rightarrow -\infty$.
$\rightarrow$ We can construct interacting symmetric α-stable processes for $\frac{1}{2} < \alpha < 2$.

We would like to give ISDE representations.

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HOWEVER, it is NOT trivial!!

Reason 1:
Let $\Xi(t) = \sum_{j=1}^{\infty} \delta_{X_j(t)}$.
The coordinate of each particle $X_j(t)$ is NOT a "GOOD" functional of Ξ.
That is, the coordinate function can NOT be in the domain of the Dirichlet form associated with the unlabeled dynamics.

Reason 2:
The labeled space $(\mathbb{R}^d)^{\mathbb{N}}$ is too large to apply the Dirichlet form theory directly.

Hence, to construct ISDE representation we consider infinite k-labeled dynamics being consistent with unlabeled dynamics, where k is the number of labeled particle.

Here, we explain the case $k = 1$. For other k we can consider similarly.

- μ_s : reduced Palm meas. of μ conditioned at s

$$\mu_s(\cdot) = \mu\left(\cdot - \delta_s \middle| \xi(s) \geq 1\right)$$

- μ^1 : 1-Campbell mesure on $\mathfrak{M} \times S$ defined by

$$\mu^1(A \times B) = \int_B \mu_s(A)\rho^1(s)ds.$$

- $\rho_s^1(x)$: 1-correlation function of μ_s

We define another bilinear form similar with the previous one.
For $f, g \in \mathfrak{D}_\circ \otimes C_0^\infty(S)$

$$\mathbb{D}^1[f,g](\xi,x) := \mathbb{D}[f(\cdot,x),g(\cdot,x)](\xi) + \nabla^{[1]}[f(\xi,\cdot),g(\xi,\cdot)](x).$$

Here for $\phi, \psi \in C_0^\infty(S)$ we define a square field $\nabla^{[1]}$ by the following.

$$\nabla^{[1]}[\phi,\psi](x) = \frac{1}{2}\int_S (\phi(y)-\phi(x))(\psi(y)-\psi(x))p(|x-y|)dy.$$

$$\mathfrak{E}^1(f,g) = \int_{\mathfrak{M}\times S} \mathbb{D}^1[f,g](\xi,\boldsymbol{x})\mu^1(d\xi dx),$$

$$\mathfrak{D}^1_\infty = \{f \in \mathfrak{D}_\circ \otimes C_0^\infty(S) \cap L^2(\mathfrak{M}\times S,\mu^1); \mathfrak{E}^1(f,f) < \infty\}.$$

additional assumptions

(A.2.1) $\rho_s^1(x) = O(|x|^\kappa)$ as $|x| \to \infty$ for all $s \in S$.
(A.3.1) μ_x and μ_y are mutually absolute continuous for a.e. x and y.
(+ some technical assumptions)

Proposition 5

(A.1), (A.3.1) $\Rightarrow (\mathfrak{E}^1, \mathfrak{D}^1_\infty)$ *is closable on* $L^2(\mathfrak{M} \times S, \mu^1)$.

$\rightarrow (\mathfrak{E}^1, \mathfrak{D}^1)$: the closure of $((\mathfrak{E}^1, \mathfrak{D}^1_\infty), L^2(\mathfrak{M} \times S, \mu^1))$.

Theorem 2 (E.-Tanemura)

Assume (p.1)–(p.2), (A.1), (A.2.1)–(A.3.1) $(+\alpha)$. *Then* $(\mathfrak{E}^1, \mathfrak{D}^1)$ *is a quasi-regular Dirichlet form on* $L^2(\mathfrak{M} \times S, \mu^1)$. *Hence there exists a Hunt process* $((\Xi^\diamond(t), X(t)), \{\mathbb{P}^{\mu^1}_{(\xi,x)}\}_{(\xi,x)\in\mathfrak{M}\times S})$ *associated with* $((\mathfrak{E}^1, \mathfrak{D}^1), L^2(\mathfrak{M} \times S, \mu^1))$.

To prove the consistency we need to introduce some natural assumptions.
(NCL) There is no collision among particles.
(NEX) Any tagged particle never explodes.
(NBJ) Only a finite number of particles visit a given bounded set during any finite time interval.

additional assumption

(A.4) (NCL), (NEX) and (NBJ) hold.

Remark 6

We have sufficient conditions for (NCL), (NEX) and (NBJ). These are proved by the Dirichlet form technique and the analysis of Markov processes.

Fix a label ℓ. Let $\ell(\Xi(t)) = (X_j(t))_{j=1}^{\infty}$, where $\Xi(t) = \sum_{j=1}^{\infty} \delta_{X_j(t)}$ be the unlabeled process associated with $(\mathfrak{E}, \mathfrak{D})$.

Theorem 3 (E. -Tanemura)

Assume (p.1)–(p.2), (A.1)–(A.2), (A.2.1)–(A.3.1) and (A.4) $(+\alpha)$. Then the process $(\Xi^{\diamond}(t), X(t))$ associated with $((\mathfrak{E}^1, \mathfrak{D}^1), L^2(\mathfrak{M} \times S, \mu^1))$ is consistent with $\Xi(t)$. Here we say $(\Xi^{\diamond}(t), X(t))$ is consistent with $\Xi(t)$ if the following condition holds:

$$\{(\Xi^{\diamond}(t), X(t))\}_t = \left\{ \left(\sum_{j=2}^{\infty} \delta_{X_k(t)}, X_1(t) \right) \right\}_t \quad \textit{in law}$$

$\to$ We call $(\Xi^{\diamond}(t), X(t))$ 1-labeled dynamics.

Sketch of proof of Theorem 3

To show Theorem 3, we consider a finite-region scheme. Let

- $\sigma_r^0 := \inf\{t > 0; \Xi(t)(U_r) \neq \Xi(t-)(U_r)\}$,
- $\sigma_r^1 = \inf\{t > 0; X_t \notin U_r$ or $\Xi^\diamond(t-)(U_r) \neq \Xi^\diamond(t-)(U_r)\}$,
- $\{\Xi^{\sigma_r^0}(t)\}_t := \{\Xi(t \wedge \sigma_r^0)\}_t$.
- $\{(X^{\sigma_r^1}(t), \Xi^{\diamond,\sigma_r^1}(t))\}_t := \{(X(t \wedge \sigma_r^1), \Xi^\diamond(t \wedge \sigma_r^1))\}$.

By considering some parts of $\mathbb{P}^\mu$ and $\mathbb{P}^{\mu^1}$ and some arguments we can show the following lemma.

Lemma 4

Assume (p.1)–(p.2), (A.1)–(A.2), (A.2.1)–(A.3.1) and (A.4) (+α). Then we have for any $n \in \mathbb{N}$

$$(X^{\sigma_r^1}(t), \Xi^{\diamond,\sigma_r^1}(t)) = (\Xi^{\sigma_r^0}(t)) \quad \textit{in law.}$$

- $r(i) = r$ if i is odd, and $r(i) = r + 1$ if i is even.
- For $i \geq 2$ let

$$\bar{\sigma}_i^0 := \inf\{t > \bar{\sigma}_{i-1}^0; \Xi(t)(U_{r(i)}) \neq \Xi(t-)(U_{r(i)})\},$$
$$\bar{\sigma}_i^1 := \inf\{t > \bar{\sigma}_{i-1}^1; X(t) \notin U_{r(i)} \text{ or } \Xi^{\diamond}(t)(U_{r(i)}) \neq \Xi^{\diamond}(t-)(U_{r(i)})\},$$

  where we set $\bar{\sigma}_1^a = \sigma_r^a \ (a = 0, 1)$.
- $\bar{\sigma}_\infty^a = \lim_{i\to\infty} \bar{\sigma}_i^a$ (on suitable case).

We would like to show $\bar{\sigma}^a_\infty = \infty$ for $a = 0, 1$.

Proof:
We assume $\bar{\sigma}^0_\infty < \infty$.
(NBJ) $\Rightarrow \exists N = N(\omega) \in \mathbb{N}$ such that
$\sharp\{X^j_t \in U_r$ for some $t \in [0, \bar{\sigma}^0_\infty]\} = N$.
$\Rightarrow 1 \le \exists J_1, J_2 \le N$ such that

- X^{J_1} cross ∂U_r infinitely many times up to $\bar{\sigma}^0_\infty$.
- X^{J_2} cross ∂U_{r+1} infinitely many times up to $\bar{\sigma}^0_\infty$.

On the other hand, $\forall \varepsilon > 0$, X^{J_1} and X^{J_2} have finite number of jumps with ranges larger than ε.

$\Rightarrow$ by a limit argument $X^{J_1} \in \partial U_r$ and $X^{J_2} \in \partial U_{r+1}$ at $\bar{\sigma}^0_\infty$.
$\Rightarrow$ This contradicts the locally boundedness of ρ^n(, which is included in assumptions).
$\Rightarrow \bar{\sigma}^0_\infty = \infty$

SDE representation

We set

$$c(x, y, \xi) = \frac{1}{2} p(x, y) \left(1 + \frac{\rho^1(y)}{\rho^1(x)} \frac{d\mu_y}{d\mu_x}(\xi)\right) \ : \text{ jump rate}$$

where $\frac{d\mu_y}{d\mu_x}$ is the Radon-Nikodym derivative.

By

- the consistent process
- Fukushima decomposition
- some limit arguments

we can give the ISDE representation in the next slide:

Theorem 5 (E. -Tanemura)

Suppose that the assumptions in Theorem 3 holds. $\exists \mathfrak{M}_0 \subset \mathfrak{M}$ *such that* $\mu(\mathfrak{M}_0) = 1$ *and that* $\forall \boldsymbol{x} \in \mathfrak{u}^{-1}(\mathfrak{M}_0)$, *there exists a solution* $\boldsymbol{X}(t) = (X_j(t))$ *satisfying*

$$X_j(t) = X_j(0) + \int_{[0,t)\times S\times[0,\infty)} N_j(dsdudr) \\ \times u\mathbf{1}\left(0 \le r \le c(\Xi^{(j)}(s-), X_j(s-), X_j(s-)+u)\right),$$

$$(X_j(0))_{j\in\mathbb{N}} = \boldsymbol{x}, \quad \boldsymbol{X}(t) \in \mathfrak{u}^{-1}(\mathfrak{M}_0) \quad \textit{for all } t,$$

where $\mathfrak{u} : S^{\mathbb{N}} \to \mathfrak{M}$ *s.t.* $\mathfrak{u}((s_j)) = \sum_j \delta_{s_j}$, $\boldsymbol{N} = (N_j)_{j\in\mathbb{N}}$ *are independent Poisson random point fields with intensity* $dsdudr$ *and* $\Xi^{(j)}(s-) = \Xi(s-) - \delta_{X_j(s-)}$.

- Dyson interacting system (on $\mathbb{R}$)

$$c(\xi, x; y) = p(|y-x|)\left(1 + \lim_{r\to\infty} \prod_{|s_i|<r} \frac{|y-s_i|^\beta}{|x-s_i|^\beta}\right), \quad \beta = 1, 2, 4.$$

- Airy interacting system (on $\mathbb{R}$)

$$c(\xi, x; y) = p(|y-x|) \\ \times \left(1 + \lim_{r\to\infty} \left\{ \exp\left(\beta(x-y)\int_{|u|<r} \frac{\widehat{\rho}(x)}{-u} du\right) \prod_{|s_i|<r} \frac{|y-s_i|^\beta}{|x-s_i|^\beta} \right\}\right),$$

$\beta = 1, 2, 4$, where $\widehat{\rho}(x) = \frac{\mathbf{1}_{(-\infty,0)}(x)}{\pi}\sqrt{-x}$.
$\xi = \sum_{i\in\mathbb{N}} \delta_{s_i}$.

- Bessel interacting system (on $\mathbb{R}$)

$$c(\xi, x; y) = p(|y-x|) \left(1 + \frac{y^\alpha}{x^\alpha} \lim_{r\to\infty} \prod_{|s_i|<r} \frac{|y-s_i|^2}{|x-s_i|^2} \right), \quad \alpha \geq 1.$$

- Ginibre interacting system (on $\mathbb{C}$)

$$c(\xi, x; y) = p(|y-x|) \left(1 + \frac{e^{-|y|^2}}{e^{-|x|^2}} \lim_{r\to\infty} \prod_{|s_i|<r} \frac{|y-s_i|^2}{|x-s_i|^2} \right).$$

$\xi = \sum_{i\in\mathbb{N}} \delta_{s_i}$.

Future problems

We also have the result of the pathwise uniqueness of the ISDE(cf. our arXiv).

Our Results

- ISDE representation
- Pathwise uniqueness

$\Downarrow$

- Dynamical universality(c.f. Osada-Kawamoto)
- Tagged particle problem
  - CLT(c.f. Kipnis-Varadhan, Jara)
  - subdiffusive(c.f. Osada: Ginibre BM$\Rightarrow$ subdiffusive)

Thank you for your attention.