Time-inconsistent stochastic control and a flow of forward-backward SDEs

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2019.09.03

Introduction

- In recent years, time-inconsistent optimal control problems have attracted an increasing attention.
- "Time-inconsistency" means that the so-called Bellman's principle of optimality does *not* hold. In other words, a restriction of an optimal control for a specific initial pair on a later time interval might not be optimal for that corresponding initial pair.
- We need to reconsider the concept of "optimality" in order to take time-consistency into account.
- In the literatures, the equilibrium control was defined and characterized by a non-standard stochastic equation, namely, a flow of forward-backward SDEs (flow of FBSDEs).
- In this talk, we formulate a notion of equilibrium solutions of flows of FBSDEs in a general framework and show its small-time solvability.

Introduction

$x \in \mathbb{R}^n$, B, Σ, F, G given.
Flow of FBSDE:

$$\begin{cases} dX_s = B(s, X_s, Y_s^s)\, ds + \Sigma(s, X_s, Y_s^s)\, dW_s, \ \ s \in [0, T], \\ dY_s^t = -F(t, X_t, s, X_s, Y_s^s, Y_s^t, Z_s^t)\, ds + Z_s^t\, dW_s, \ \ s \in [t, T], \\ X_0 = x, \ \ Y_T^t = G(t, X_t, X_T), \ \ t \in [0, T]. \end{cases}$$

- (Y_s^t, Z_s^t) is defined on $(t, s) \in \Delta := \{(t, s) | 0 \leq t \leq s \leq T\}$.
- This is a non-standard equation consisting of an SDE for $X = (X_s)_{s\in[0,T]}$ and a continuum of backward SDEs (BSDEs) for $(Y^t, Z^t) = (Y_s^t, Z_s^t)_{s\in[t,T]}$ parametrized by $t \in [0, T]$, that are coupled via the "diagonal term" Y_s^s.
- We want to find a family of adapted processes $(X, \{Y^t, Z^t\}_{t\in[0,T]})$ which satisfies the system in the Itô-sense.

1. Time-inconsistent stochastic control problems

2. Results: Small-time solvability of a flow of FBSDEs

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2. Results: Small-time solvability of a flow of FBSDEs

Notation

$(\Omega, \mathcal{F}, \mathbb{P})$: complete probability space
W: 1-dim. BM
$\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$: $\mathbb{P}$-augmentation of $\mathbb{F}^W$
$\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$

$L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n) := \{\xi | \mathbb{R}^n\text{-valued } \mathcal{F}_t\text{-m'ble r.v., } \mathbb{E}[|\xi|^2] < \infty\}$
$L^2_{\mathbb{F}}(t, T; \mathbb{R}^n) := \{X | \mathbb{R}^n\text{-valued } \mathbb{F}\text{-prog. m'ble proc., } \mathbb{E}[\int_t^T |X_s|^2\, ds] < \infty\}$
$L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n)) := \{X \in L^2_{\mathbb{F}}(t, T; \mathbb{R}^n) | \text{conti.}, \mathbb{E}[\sup_{s\in[t,T]} |X_s|^2] < \infty\}$

The set of control processes:
$\mathcal{U}[t, T] := \{u \in L^2_{\mathbb{F}}(t, T; \mathbb{R}^k); u_t \in U, \ \forall\, t \in [t, T]\}$, where U is a closed subset of $\mathbb{R}^k$.

Stochastic Control Problem

Controlled SDE:

$$\begin{cases} dX_s = b(s, X_s, u_s)\, ds + \sigma(s, X_s)\, dW_s, \ \ s \in [t, T], \\ X_t = x_t. \end{cases}$$

Here, $(t, x_t) \in [0, T) \times L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$ is a given initial condition, $u \in \mathcal{U}[t, T]$ is a *control process*, and $X \equiv X^{t,x_t,u}$ is the corresponding *state process*.

Cost functional:

$$J(t, x_t; u) = \mathbb{E}_t \left[\int_t^T e^{-\delta(s-t)} f(s, X_s, u_s)\, ds + e^{-\delta(T-t)} g(X_T) \right]$$

- Under some mild conditions, for any (t, x_t), and any $u \in \mathcal{U}[t, T]$, $X \equiv X^{t,x_t,u} \in L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n))$ and $J(t, x_t; u) \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R})$ are well-defined.
- We refer to the function $t \mapsto e^{-\delta t}$ as the exponential discounting and $\delta > 0$ as the (constant) discount rate.

Problem: For given $(t, x_t) \in [0, T) \times L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$, find a $\bar{u} \in \mathcal{U}[t, T]$ s.t.

$$J(t, x_t; \bar{u}) = \operatorname{ess\,inf}_{u \in \mathcal{U}[t,T]} J(t, x_t; u) =: V(t, x_t).$$

Stochastic Control Problem

Bellman's principle of optimality:

$$V(t, x_t) = \operatorname{ess\,inf}_{u \in \mathcal{U}[t,\tau]} \mathbb{E}_t \left[\int_t^\tau e^{-\delta(s-t)} f(s, X_s^{t,x_t,u}, u_s)\, ds + e^{-\delta(\tau - t)} V(\tau, X_\tau^{t,x_t,u}) \right],$$

for $(t, x_t) \in [0, T) \times L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$ and $\tau \in (t, T)$.

Let $(\bar{u}, \bar{X})$ be the optimal pair w.r.t. the initial pair (t, x_t).
By the Bellman's principle of optimality, one can show that

$$J(\tau, \bar{X}_\tau; \bar{u}|_{[\tau,T]}) = V(\tau, \bar{X}_\tau) \text{ a.s., } \forall\, \tau \in (t, T).$$

This means that the restriction $\bar{u}|_{[\tau,T]}$ of the optimal control $\bar{u}$ for the initial pair (t, x_t) on a later interval $[\tau, T]$ is an optimal control for the initial pair $(\tau, \bar{X}_\tau)$.

This is called the time-consistency of the problem.

General discounting and time-inconsistency

The empirical studies of human behavior (e.g. Ainslie ('92)) reveals that the assumption of exponential discounting is unrealistic for describing people's time-preferences.

Consider a non-exponential discounting $t \mapsto h(t) \in (0, \infty)$. Then the cost functional is written as

$$J(t, x_t; u) = \mathbb{E}_t \left[\int_t^T h(s-t) f(s, X_s, u_s)\, ds + h(T-t) g(X_T) \right].$$

More generaly, consider the following cost functional:

$$J(t, x_t; u) = \mathbb{E}_t \left[\int_t^T f(t, x_t, s, X_s, u_s)\, ds + g(t, x_t, X_T) \right]. \qquad (1)$$

Since the cost functional (1) depends on the initial pair (t, x_t), the problem is time-inconsistent in general. (i.e. Bellman's principle does *not* hold.)

"Optimality"?

Time-inconsistency

$\rightsquigarrow$ *Today's preference conflicts with tomorrow's preference.*

Choice mechanism (Strotz, '55):

- Precommitment choice
  The initial policy is implemented on the lifetime horizon.
  This approach neglects the time-inconsistency, and the optimal policy is optimal only when viewed at the initial time.
- Naive choice (or myopic choice)
  At each time instant a naive player embarks on the option that currently seems best, namely, this player sticks to the local objective and completely ignores the global interest.
- Sophisticated choice
  The player at different time instants is regarded as different selves, and at any time instant the current self takes account of future selves' decisions.
  Instead of seeking an "optimal control", some kinds of equilibrium solutions are dealt with.

Equilibrium control

Definition

$x_0 \in \mathbb{R}^n$: given. $\hat{u} \in \mathcal{U}[0,T]$ is an equilibrium control (w.r.t. x_0)
$\overset{Def}{\Longleftrightarrow}$ $\forall\, t \in [0,T)$, $\forall\, v \in L^2_{\mathcal{F}_t}(\Omega; U)$,

$$\liminf_{\epsilon \downarrow 0} \frac{J(t, \hat{X}_t; u^{t,\epsilon,v}) - J(t, \hat{X}_t; \hat{u}|_{[t,T]})}{\epsilon} \geq 0, \ \ a.s.,$$

where $\hat{X} \equiv X^{0,x_0,\hat{u}}$ is the state process corresponding to the control $\hat{u}$ (with the initial condition $\hat{X}_0 = x_0$), and $u^{t,\epsilon,v} := v\mathbb{1}_{[t,t+\epsilon)} + \hat{u}\mathbb{1}_{[t+\epsilon,T]} \in \mathcal{U}[t,T]$.

- Consider a small time interval $[t, t+\epsilon)$.
  If the player chooses the control $\hat{u}_s$ for $s \in [0,T] \setminus [t, t+\epsilon)$, then the (asymptotically) optimal choice when viewed at time t is $\hat{u}_t$.
- That is, an equilibrium control $\hat{u}$ can be seen as an Nash equilibrium point for the "game" constructed by infinitely many different selves.

Characterization of the equilibrium control $\hat{u}$

State equation: $dX_s = b(s, X_s, u_s)\, ds + \sigma(s, X_s)\, dW_s,\ s \in [t, T],\ X_t = x_t.$
Cost functional: $J(t, x_t; u) = \mathbb{E}_t \Big[\int_t^T f(t, x_t, s, X_s, u_s)\, ds + g(t, x_t, X_T) \Big].$

Assume (for simplisity) that all coefficients are deterministic, bounded, and sufficiently smooth.

- Let $x_0 \in \mathbb{R}^n$ be given. Fix an arbitrary $\hat{u} \in \mathcal{U}[0, T]$ and denote $\hat{X} \equiv X^{0, x_0, \hat{u}}$.
- For each $t \in [0, T]$, consider the *adjoint equation*

$$\begin{cases} d\hat{Y}_s^t = -H_x(t, \hat{X}_t, s, \hat{X}_s, \hat{u}_s, \hat{Y}_s^t, \hat{Z}_s^t)^\top ds + \hat{Z}_s^t\, dW_s,\ s \in [t, T], \\ \hat{Y}_T^t = g_x(t, \hat{X}_t, \hat{X}_T)^\top, \end{cases} \tag{2}$$

  where the *Hamiltonian* $H(\cdots)$ is defined by

$$H(t, \xi, s, x, u, y, z) := \langle b(s, x, u), y \rangle + \langle \sigma(s, x), z \rangle + f(t, \xi, s, x, u),$$

  for $0 \le t \le s \le T$, $\xi, x, y, z \in \mathbb{R}^n$, $u \in U$.
- Eq.(2) is a backward SDE (BSDE) for $(\hat{Y}^t, \hat{Z}^t)$ parametrized by $t \in [0, T]$. Under our assumptions, there exists a unique adapted solution $(\hat{Y}^t, \hat{Z}^t) = (\hat{Y}_s^t, \hat{Z}_s^t)_{s \in [t, T]} \in L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n) \times L^2_{\mathbb{F}}(t, T; \mathbb{R}^n)$ to BSDE(2) (cf. El Karoui et al. ('97)).
- The processes $\hat{X}$ and $(\hat{Y}^t, \hat{Z}^t)$ $(t \in [0, T])$ are determined by $\hat{u}$.

Characterization of the equilibrium control $\hat{u}$

Proposition (cf. Hu–Jin–Zhou ('12), Yong ('19))

$\hat{u} \in \mathcal{U}[0, T]$ is an equilibrium control if and only if

$$\hat{u}_s \in \operatorname{argmin}_{u \in U} H(s, \hat{X}_s, s, \hat{X}_s, u, \hat{Y}^s_s, \hat{Z}^s_s), \text{ a.e. a.s.} \tag{3}$$

Assume that $\tilde{u}(s, x, y) = \operatorname{argmin}_{u \in U} H(s, x, s, x, u, y, z)$ is well-defined for each $s \in [0, T]$ and $x, y, z \in \mathbb{R}^n$ (which is independent of z), and the function $(s, x, y) \mapsto \tilde{u}(s, x, y)$ is sufficiently regular.

Combining the state equation (SDE), adjoint equation (BSDE), and the *equilibrium condition* (3), we obtain the following *Hamiltonian system*:

$$\begin{cases} dX_s = b(s, X_s, \tilde{u}(s, X_s, Y^s_s))\, ds + \sigma(s, X_s)\, dW_s, \ s \in [0, T], \\ dY^t_s = -H_x(t, X_t, s, X_s, \tilde{u}(s, X_s, Y^s_s), Y^t_s, Z^t_s)^\top ds + Z^t_s\, dW_s, \ s \in [t, T], \\ X_0 = x_0, \ Y^t_T = g_x(t, X_t, X_T)^\top, \ t \in [0, T]. \end{cases} \tag{4}$$

If there exists a "solution" of Eq.(4), then the equilibrium control $\hat{u}$ is characterized by $\hat{u}_s = \tilde{u}(s, X_s, Y^s_s), \ s \in [0, T]$.

Remarks on the Hamiltonian system (4)

Hamiltonian system (4):

$$\begin{cases} dX_s = b(s, X_s, \tilde{u}(s, X_s, Y_s^s))\, ds + \sigma(s, X_s)\, dW_s,\ \ s \in [0, T], \\ dY_s^t = -H_x(t, X_t, s, X_s, \tilde{u}(s, X_s, Y_s^s), Y_s^t, Z_s^t)^\top\, ds + Z_s^t\, dW_s,\ \ s \in [t, T], \\ X_0 = x_0,\ \ Y_T^t = g_x(t, X_t, X_T)^\top,\ \ t \in [0, T]. \end{cases}$$

- (Y_s^t, Z_s^t) is defined on $(t, s) \in \Delta := \{(t, s) | 0 \le t \le s \le T\}$.
- For each $t \in [0, T]$, $(Y^t, Z^t) = (Y_s^t, Z_s^t)_{s \in [t,T]}$ is the adapted solution of the corresponding BSDE (parametrized by $t \in [0, T]$).
- Eq.(4) is a non-standard equation consisting of a (forward) SDE for X and a continuum of BSDEs for (Y^t, Z^t), that are coupled via the "*diagonal term*" Y_s^s.
- We call Eq.(4) a flow of forward-backward SDEs (flow of FBSDEs, or FFBSDE).

Flow of FBSDE (FFBSDE)

$x \in \mathbb{R}^n$, B, Σ, F, G given.
FFBSDE:

$$\begin{cases} dX_s = B(s, X_s, Y_s^s)\, ds + \Sigma(s, X_s, Y_s^s)\, dW_s, \ \ s \in [0, T], \\ dY_s^t = -F(t, X_t, s, X_s, Y_s^s, Y_s^t, Z_s^t)\, ds + Z_s^t\, dW_s, \ \ s \in [t, T], \\ X_0 = x, \ \ Y_T^t = G(t, X_t, X_T), \ \ t \in [0, T]. \end{cases} \tag{5}$$

Definition

$(X, \{(Y^t, Z^t)\}_{t \in [0,T]})$ *is an equilibrium solution of FFBSDE(5)*
$\overset{Def}{\Longleftrightarrow}$ *For each $t \in [0, T]$, they satisfy Eq.(5) in the Itô-sence, where:*

- $X \in L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathbb{R}^n))$,
- $(Y^t, Z^t) \in L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^m)) \times L^2_{\mathbb{F}}(t, T; \mathbb{R}^m)$, $\forall\, t \in [0, T]$,
- $(Y_t^t)_{t \in [0,T]}$ *is progressively measurable.*

Discretized flow

Let $\mathcal{P}_{[0,T]}$ be the set of finite partitions $\Pi = \{t_k | k = 0, \dots, N\}$ of $[0, T]$ $(0 = t_0 < t_1 < \cdots < t_N = T)$.
For each $\Pi \in \mathcal{P}_{[0,T]}$, consider the *discretized flow* (FFBSDE$^\Pi$):

$$\begin{cases} dX_s^\Pi = B(s, X_s^\Pi, \mathcal{Y}_s^\Pi)\, ds + \Sigma(s, X_s^\Pi, \mathcal{Y}_s^\Pi)\, dW_s, \;\; s \in [0, T], \\ dY_s^{\Pi,k} = -F(t_{k-1}, X_{t_{k-1}}^\Pi, s, X_s^\Pi, \mathcal{Y}_s^\Pi, Y_s^{\Pi,k}, Z_s^{\Pi,k})\, ds \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad + Z_s^{\Pi,k}\, dW_s, \;\; s \in [t_{k-1}, T], \\ X_0^\Pi = x, \;\; Y_T^{\Pi,k} = G(t_{k-1}, X_{t_{k-1}}^\Pi, X_T^\Pi), \;\; k = 1, \dots, N, \\ \mathcal{Y}_s^\Pi = \sum_{j=1}^N Y_s^{\Pi,j} \mathbb{1}_{[t_{j-1}, t_j)}(s), \;\; s \in [0, T]. \end{cases} \tag{6}$$

Definition

$(X^\Pi, \{(Y^{\Pi,k}, Z^{\Pi,k})\}_{k=1,\dots,N})$ is a Π-equilibrium solution of FFBSDE$^\Pi$ (6)
$\overset{Def}{\Longleftrightarrow}$ *For each $k = 1, \dots, N$, they satisfy Eq.(6) in the Itô-sence, where:*

- $X^\Pi \in L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathbb{R}^n))$,
- $(Y^{\Pi,k}, Z^{\Pi,k}) \in L^2_{\mathbb{F}}(\Omega; C([t_{k-1}, T]; \mathbb{R}^m)) \times L^2_{\mathbb{F}}(t_{k-1}, T; \mathbb{R}^m)$, $\forall\, k = 1, \dots, N$.

Assumptions

(1) $B, \Sigma \colon \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$,
$F \colon \Omega \times \Delta \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, and
$G \colon \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ are measurable.
Moreover, $B(\cdot, x, \eta), \Sigma(\cdot, x, \eta), F(t, \xi, \cdot, x, \eta, y, z)$ are $\mathbb{F}$-prog. m'ble, and $G(t, \xi, x)$ is $\mathcal{F}_T$-m'ble.

(2) $R := \mathbb{E}[\int_0^T (|B|^2 + |\Sigma|^2)(s, 0, 0)\, ds] +$
$\sup_{t \in [0,T]} \mathbb{E}[\int_t^T |F(t, 0, s, 0, 0, 0, 0)|^2\, ds + |G(t, 0, 0)|^2] < \infty$.

(3) B, Σ, F, G are L-Lipschitz conti. w.r.t. ξ, x, η, y, z. $(L > 0)$

(4) $\exists\, \rho \colon [0, \infty) \to [0, \infty)$, conti., non-decreasing, $\rho(0) = 0$, s.t.
$|F(t, \xi, s, x, \eta, y, z) - F(t', \xi, s, x, \eta, y, z)| + |G(t, \xi, x) - G(t', \xi, x)|$
$\leq \rho(|t - t'|)(1 + |\xi| + |x| + |\eta| + |y| + |z|)$,
$\forall\, s \in [0, T],\ t, t' \in [0, s],\ \xi, x \in \mathbb{R}^n, \eta, y, z \in \mathbb{R}^m$, a.s.

Main results

Theorem 1 (Small-time solvability of FFBSDE$^\Pi$)

Assume Conditions (1)-(3) (i.e. measurability, integrability, L-Lip. continuity) hold. Then $\exists\, \delta_1 = \delta_1(L) > 0$ *s.t.* $\forall\, T \le \delta_1,\ \forall\, \Pi \in \mathcal{P}_{[0,T]}$,
$\exists!$ Π*-equilibrium sol.* $(X^\Pi, \{(Y^{\Pi,k}, Z^{\Pi,k})\}_{k=1,\dots,N})$ *of FFBSDE*$^\Pi$.

Theorem 2 (Small-time solvability of FFBSDE and approximation)

Assume Conditions (1)-(4) (i.e. ρ*-continuity w.r.t. the t-variable) hold. Then* $\exists\, \delta_2 = \delta_2(L) \le \delta_1,\ \exists\, C = C(L) > 0$ *s.t.*

(A) $\forall\, T \le \delta_2$, $\exists!$ *equilibrium sol.* $(X, \{(Y^t, Z^t)\}_{t\in[0,T]})$ *of FFBSDE.*

(B) $\forall\, T \le \delta_2,\ \forall\, \Pi \in \mathcal{P}_{[0,T]}$,

$$\mathbb{E}\left[\sup_{s\in[0,T]} |X^\Pi_s - X_s|^2 + \int_0^T |\mathcal{Y}^\Pi_s - Y^s_s|^2\, ds\right] \le C(R + |x|^2)(\rho(\|\Pi\|)^2 + \|\Pi\|),$$

where $\mathcal{Y}^\Pi_s = \sum_{j=1}^N Y^{\Pi,j}_s \mathbb{1}_{[t_{j-1},t_j)}(s),\ s \in [0, T]$, *and* $\|\Pi\| := \max_k |t_k - t_{k-1}|$.

Remarks on the main results

- FFBSDE$^{\Pi}$ can be seen as a system of *finitely many FBSDEs*, and we can construct the Π-equilibrium solution $(X^{\Pi}, \{(Y^{\Pi,k}, Z^{\Pi,k})\}_{k=1,\dots,N})$ *backward inductively* w.r.t. $k = 1, \dots, N$.
- Then by showing some estimates for the Π-equilibrium solutions that hold *uniformly in* $\Pi \in \mathcal{P}_{[0,T]}$, we can show that, when $\|\Pi\| \downarrow 0$, the corresponding Π-equilibrium solutions are Cauchy in an appropriate Banach space.
- We can show that the limit is the unique equilibrium solution of the original FFBSDE.
- Our result says that, under natural assumptions, FFBSDE is well-posed *when the time interval* $[0, T]$ *is small*. The *global solvability* is a challenging problem *even for the discretized version; FFBSDE*$^{\Pi}$.
- However, we can obtain the global solution *if a (non-standard) PDE system has a classical solution*. (See the next slide.)

Relationship between FFBSDE and (non-standard) PDE

Let B, Σ, F, G be deterministic and smooth. $T > 0$ is arbitrary.

(Non-local) PDE system:

$$\begin{cases} \theta_s^{t,\xi}(s,x) + \theta_x^{t,\xi}(s,x) B(s,x,\theta^{s,x}(s,x)) \\ \quad + \frac{1}{2} \Sigma(s,x,\theta^{s,x}(s,x))^\top \theta_{xx}^{t,\xi}(s,x) \Sigma(s,x,\theta^{s,x}(s,x)) \\ \quad + F(t,\xi,s,x,\theta^{s,x}(s,x),\theta^{t,\xi}(s,x),\theta_x^{t,\xi}(s,x)\Sigma(s,x,\theta^{s,x}(s,x))) = 0, \\ \qquad\qquad\qquad\qquad (t,s) \in \Delta,\ \xi, x \in \mathbb{R}^n, \\ \theta^{t,\xi}(T,x) = G(t,\xi,x),\ t \in [0,T],\ \xi, x \in \mathbb{R}^n. \end{cases} \tag{7}$$

Assume that Eq.(7) has a classical solution $\theta^{t,\xi}(s,x)$, and that the following SDE has a unique strong solution:

$$\begin{cases} dX_s = B(s,X_s,\theta^{s,X_s}(s,X_s))\,ds + \Sigma(s,X_s,\theta^{s,X_s}(s,X_s))\,dW_s,\ s \in [0,T], \\ X_0 = x. \end{cases}$$

Then,

$$\begin{cases} X_s,\ s \in [0,T], \\ Y_s^t = \theta^{t,X_t}(s,X_s),\ Z_s^t = \theta_x^{t,X_t}(s,X_s)\Sigma(s,X_s,\theta^{s,X_s}(s,X_s)),\ (t,s) \in \Delta, \end{cases}$$

is an equilibrium solution of FFBSDE(5).