On logarithmic Sobolev inequalities

With a result and several questions on the Heisenberg group

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Today's plan

- A brief history of logarithmic Sobolev inequalities.
- The historical proof of Gross for the Gaussian measure.
- Logarithmic Sobolev inequalities on the Heisenberg group.

Bibliography

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Setting

Homogeneous Markov processes

$P_t f(x) = \int f(y) p_t(x, \mathrm{d}y) = \mathbb{E}(f(X_t) | X_0 = x)$.

Reversibility

There exists a probability measure μ such that $\mu(f P_t g) = \mu(g P_t f)$.

Feller property

$|P_t f - f|_{L^2(\mu)} \to 0$ as $t \to 0$.

Properties

Semigroup

$P_{t+s} = P_t P_s$.

Contraction

$|P_t|_{L^p(\mu)\to L^p(\mu)} \leq 1$ for $p \in [1, \infty]$.

Hypercontractivity?

Can we strengthen the contractivity property?
E.g. does it hold $|P_t|_{L^2(\mu)\to L^4(\mu)} \leq 1$ for some $t > 0$?

Generators, carré du champ, and energies

Theorem (Yoshida)

$Lf = \lim_{t\to 0} \frac{P_t f - f}{t}$ *unbounded* $L^2(\mu) \to L^2(\mu)$.
L *has a dense domain* D *in* $L^2(\mu)$, $P_t(D) \subset D$.
$P_t f$ *solves the abstract heat equation* $\partial_t P_t f = L P_t f = P_t L f$.

Carré du champ

$2\Gamma(f,g) = L(fg) - fLg - gLf$ bilinear symmetric positive.

Dirichlet energy

$\mathcal{E}(f,g) = \mu(\Gamma(f,g)) = -\mu(fLg) = -\mu(gLf)$.

Iterated carré du champ

$2\Gamma_2(f,g) = L(\Gamma(f,g)) - \Gamma(f,Lg) - \Gamma(g,Lf)$.

Logarithmic Sobolev inequalities and hypercontractivity

Logarithmic Sobolev inequality

There exists $\rho > 0$ such that for all f

$\mathrm{Ent}_\mu(f^2) := \mu(f^2 \log f^2) - \mu(f^2) \log \mu(f^2) \leq \frac{2}{\rho}\mathcal{E}(f, f)$.

Theorem (Gross 1975; Bakry & Émery 1985)

The invariant measure of a reversible Markov semigroup satisfies a logarithmic Sobolev inequality if and only if it is hypercontractive. More precisely, the Markov semi-group satisfies the logarithmic Sobolev inequality with constant ρ if and only if for all $t \geq 0$:

$|P_t|_{L^2(\mu) \to L^{q(t)}(\mu)} \leq 1$, *with* $q(t) = 1 + \mathrm{e}^{\rho t}$.

The logarithmic Sobolev inequality for the Ornstein-Uhlenbeck semigroup

Dynamic	$\mathrm{d}X_t = \sqrt{2}\mathrm{d}B_t - X_t\mathrm{d}t.$
Invariant measure	$\gamma(\mathrm{d}x) = \mathrm{e}^{-x^2/2}(2\pi)^{-1/2}\mathrm{d}x.$
Semigroup	$P_tf(x) = \int f(\mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}y)\gamma(\mathrm{d}y).$
Generator	$Lf = f'' - xf'.$
Carré du champ	$\Gamma(f,g) = f'g'.$
Iterated carré du champ	$\Gamma_2(f,g) = f'g' + f''g''.$

Theorem (Gross 1975)

This semigroup satisfies a logarithmic Sobolev inequality, hence it is hypercontractive. More precisely, $\mathrm{Ent}_\gamma(f^2) \leq 2\gamma((f')^2)$.

Idea of the proof of Gross

- Prove the logarithmic Sobolev inequality for the Markov dynamic on the two-points space with invariant measure $\nu = \frac{1}{2}(\delta_1 + \delta_{-1})$:

$$\mathrm{Ent}_\nu(f^2) \leq \frac{1}{2}(f(1) - f(-1))^2 = 2\mathcal{E}(f, f).$$

- Show that logarithmic Sobolev inequalities behave well with respect to tensorization:

$$\mathrm{Ent}_{\nu^n}(f^2) \leq \sum_{i=1}^{n} \mathbb{E}_{\nu^n} \mathrm{Ent}_{\nu_i}(f^2) \leq \frac{1}{2} \sum_{i=1}^{n} \mathbb{E}_{\nu^n}(f_i(1) - f_i(-1))^2.$$

- Apply this inequality for ν^n to $f \circ S_n$, with $S_n = \frac{1}{n}\sum_{i=1}^{n} x_i$, pass to the limit $n \to \infty$ and use the central limit theorem.

The logarithmic Sobolev inequality for weighted manifolds

Dynamic	$\mathrm{d}X_t = \sqrt{2}\mathrm{d}B_t - \nabla V(X_t)\mathrm{d}t.$
Invariant measure	$\gamma_V(\mathrm{d}x) = \mathrm{e}^{-V(x)}\mathrm{vol}(\mathrm{d}x).$
Generator	$Lf = \Delta f - \nabla V \cdot \nabla f.$
Carré du champ	$\Gamma(f,g) = \nabla f \cdot \nabla g.$
Iterated carré du champ	$\Gamma_2(f,g) = (\mathrm{Ric} + \nabla^2 V)(\nabla f, \nabla g) + \nabla^2 f \cdot \nabla^2 g.$

Theorem (Bakry & Émery 1985)

The invariant measure of this reversible semigroup satisfies a logarithmic Sobolev inequality if $\mathrm{Ric} + \nabla^2 V \geq K > 0$.

Later on, Bakry showed this is an equivalence.

The Heisenberg group

Lie algebra

$\mathfrak{H} = \mathrm{span}\{X, Y, Z\}$ where $[X, Y] = Z$.

Associated Lie group

$\mathbb{H} = \mathbb{R}^3$ with group law

$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y))$.

$\mathbb{H}$ encodes the increment in $\mathbb{R}^2$ and computes the generated area.

Left-invariant basis of the tangent space

$X = \partial_x - \frac{y}{2}\partial_z, Y = \partial_y + \frac{x}{2}\partial_z, Z = \partial_z$.

Sub-Riemannian structure of the Heisenberg group

Horizontal paths

$h = (x, y, z)\colon [0, 1] \to \mathbb{H}$, $\dot{h} \in \mathrm{span}\{X, Y\}$, $L(h) = \left(\int_0^1 \dot{x}^2 + \dot{y}^2\right)^{1/2}$.

Theorem (Chow)

$\mathbb{H}$ is path connected with horizontal paths.

Carnot-Carathéodory distance

$d(h_0, h_1) = \inf\{L(h) | h \text{ horizontal }, h(0) = h_0, h(1) = h_1\}$.
Topologically $\mathbb{H} = \mathbb{R}^3$ and the Haar measure is the 3-d Lebesgue measure. The metric (Hausdorff) dimension is 4.

Sub-Riemannian operators

$\nabla = \begin{pmatrix} X \\ Y \end{pmatrix}$; $\Delta = X^2 + Y^2$.

The Ornstein-Uhlenbeck semigroup on $\mathbb{H}$

Brownian motion on $\mathbb{H}$	$H_t = (B_t^1, B_t^2, \frac{1}{2}\int(B_t^1 \mathrm{d}B_t^2 - B_t^2 \mathrm{d}B_t^1))$.
Dynamic	$\mathrm{d}X_t = \sqrt{2}\mathrm{d}B_t^1 - X_t \mathrm{d}t$,
	$\mathrm{d}Y_t = \sqrt{2}\mathrm{d}B_t^2 - Y_t \mathrm{d}t$,
	$2\mathrm{d}Z_t = X_t \mathrm{d}Y_t - Y_t \mathrm{d}X_t$.
Invariant measure	$\gamma_{\mathbb{H}} = law(H_1)$.
Generator	$Lf = \Delta f - (x, y) \cdot \nabla f$.
Carré du champ	$\Gamma(f, g) = \nabla f \cdot \nabla g$.
Iterated carré du champ	$\Gamma_2(f, g) =$ we can compute it but we do not use it.

Heuristically on $\mathbb{H}$, Ric $= -\infty$ so we cannot use the result of Bakry & Émery 1985 (see Michel Bonnefont's thesis).

A logarithmic Sobolev inequality on $\mathbb{H}$

Theorem (Bonnefont, Chafaï & Herry 2019+)

$\mathrm{Ent}_{\gamma_{\mathbb{H}}}(f^2) \leq 2\gamma_{\mathbb{H}}(|\nabla f|^2) + \gamma_{\mathbb{H}}((Zf)^2 a)$,

where $a(h) = \mathbb{E}(\int_0^1 (B_t^1)^2 + (B_t^2)^2 \mathrm{d}t | H_1 = h)$.

- This inequality is optimal in the horizontal directions.
- Not known if optimal in the vertical direction.
- The right-hand side contains a vertical term.

Idea of proof

- Essentially the same as the classical proof of Gross 1975.
- By Gross 1975 result and tensorization $\gamma^n_{\mathbb{R}^3}$ satisfies a logarithmic Sobolev inequality.
- Push $\gamma^n_{\mathbb{R}^3}$ forward by $S_n = \frac{1}{n}\sum_{i=1}^n h_i$, where the sum is in $\mathbb{H}$ and pass to the limit in $n \to \infty$.
- The non-commutativity produces the extra term $\gamma_{\mathbb{H}}((Zf)^2 a)$.

Open questions

- Extension to other sub-Riemannian Lie groups?
- Link with some improved contractivity?
- Link with concentration of measure?
- Comparison with other sub-Riemannian inequalities on $\mathbb{H}$? In particular, links with the Prékopa-Leindler inequality for the Haar (=Lebesgue) measure of Balogh, Kristály & Sipos?

Prékopa-Leindler

- Essentially a functional version of $CD(0, N)$.
- On $\mathbb{R}^d$, if $h(tx + (1-t)y) \geq f(x)^t + g(y)^{1-t}$:

$$\int h \geq \left(\int f\right)^t \left(\int g\right)^{1-t}.$$

- Same on CD spaces but with distortion coefficients.
- Can be used to derive the logarithmic Sobolev inequality / Bakry-Emery for log-concave measure.

Open questions 2

- Balogh et al. have a version of Prékopa-Leindler on $\mathbb{H}$. Can we derive our (or an other) logarithmic Sobolev inequality for $\gamma_{\mathbb{H}}$ from it?
- More generally, I am interested, in what can be said about the law of the heat kernel on manifold from Prékopa-Leindler (in general, the heat kernel is not log-concave). If you know some references, please let me know!

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