# *Sub-Gaussian heat kernel bounds imply singularity of energy measures*

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**Joint work with Mathav Murugan (UBC)**

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**15:50–16:20**

$\beta = 2$ (**Gaussian**) $\Rightarrow \Gamma(u,u) \ll \mu$.

$\beta > 2$ (**sub-Gaussian**) $\Rightarrow \Gamma(u,u) \perp \mu$**!**

$$p_t(x,y) \asymp c\mu\bigl(B(x,t^{\frac{1}{\beta}})\bigr)^{-1} \exp\Bigl(-\tilde{c}\bigl(d(x,y)^{\beta}/t\bigr)^{\frac{1}{\beta-1}}\Bigr)$$

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▷ $(K, d, \mu, \mathcal{E}, \mathcal{F})$: strongly local reg. symmet. Dir. sp.

$\leftrightarrow (\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in K_\partial})$: $\mu$-sym. diffusion, no killing

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**Fact.** $\widetilde{u}(X_t)-\widetilde{u}(X_0)=M^{[u]}_t+N^{[u]}_t$ & $M^{[u]}$ satisfies

$$\int_K f\,d^{\exists 1}\Gamma(u,u)=\lim_{t\downarrow 0} t^{-1}\mathbb{E}_\mu\left[\int_0^t f(X_s)\,d\langle M^{[u]}\rangle_s\right]$$

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$$=\mathcal{E}(fu,u)-\mathcal{E}(f,u^2)/2 = \text{“}\int_K f\cdot|\nabla u|^2\,d\mu\text{”}.$$

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● **Assume** $\mathbb{P}_x[X_t \in dy] = {}^{\exists}p_t(x,y)\,d\mu(y)$ & ${}^{\exists}\beta > 1$,

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(*cf.* Barlow–Bass '92, '99, Kusuoka–Zhou '92, Barlow–Bass–Kumagai '06)

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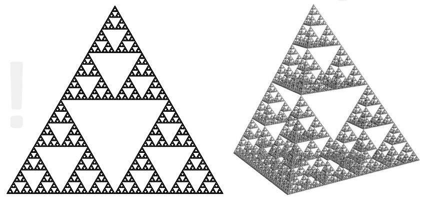

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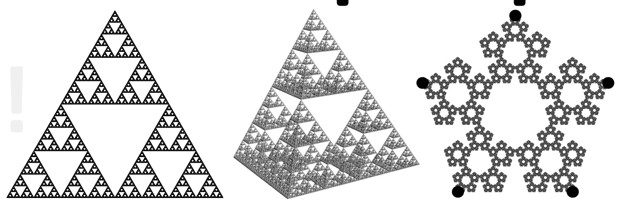

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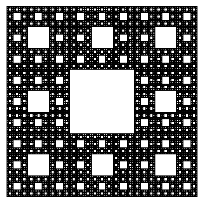
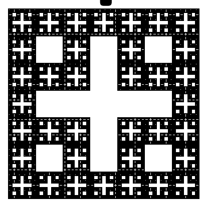
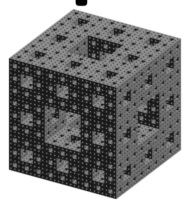

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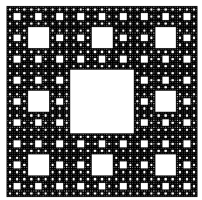
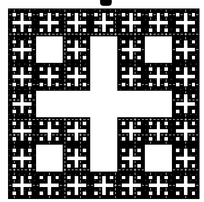
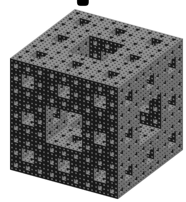

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## 2 General setting & conditions for HK estimates

▷ $\Psi$: homeo. on $[0,\infty)$ with $1<{}^{\exists}\beta_1\le{}^{\exists}\beta_2$, ${}^{\exists}c\ge 1$,

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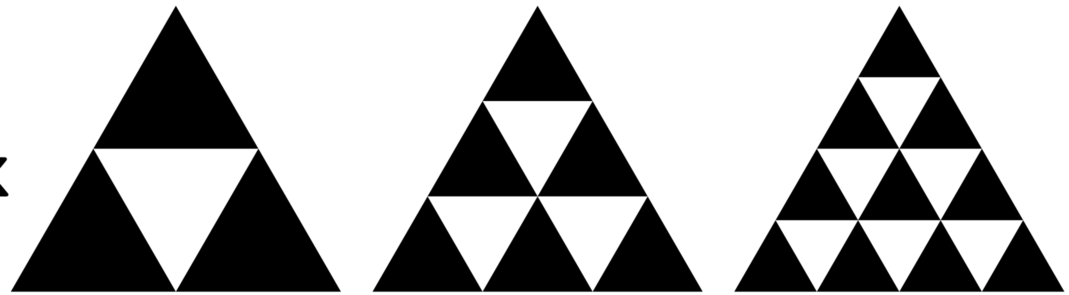

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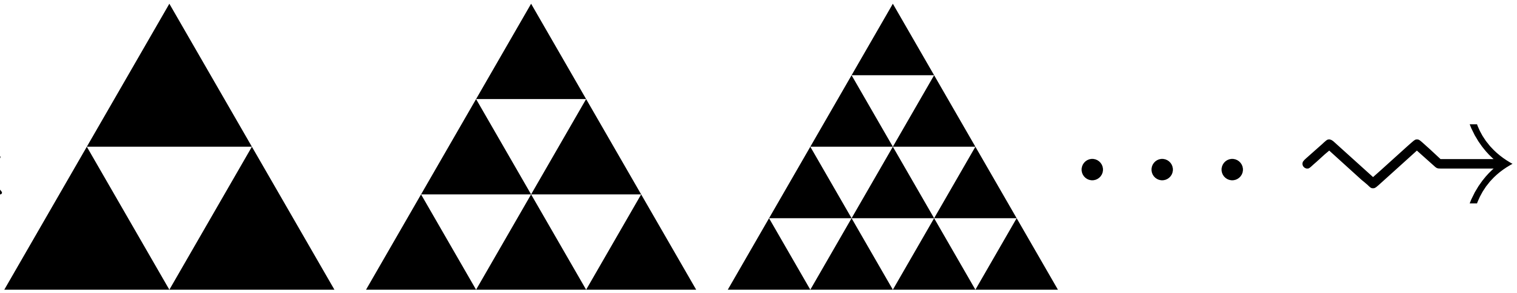

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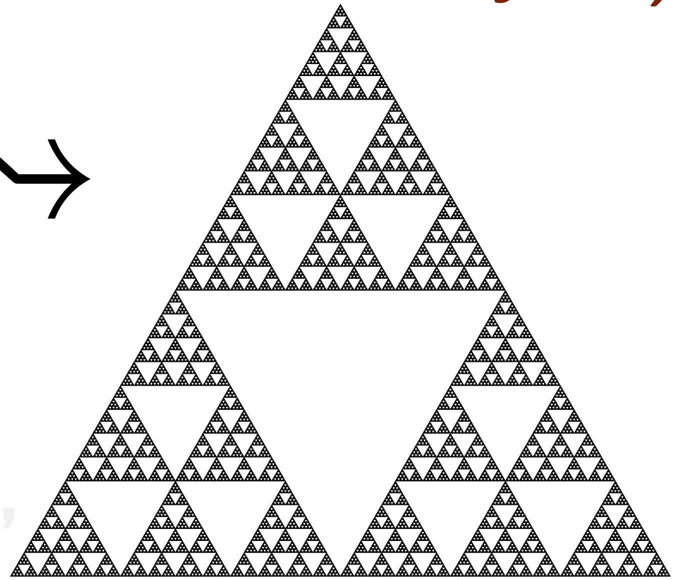

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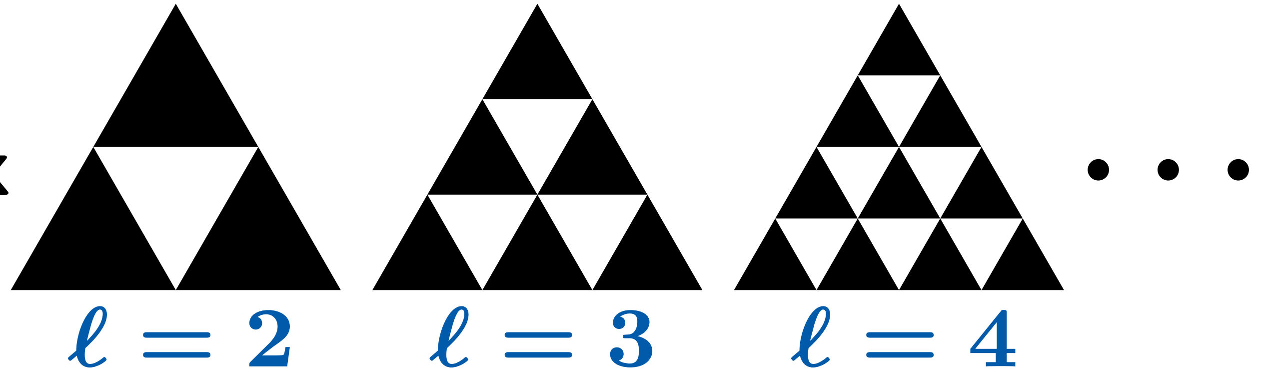


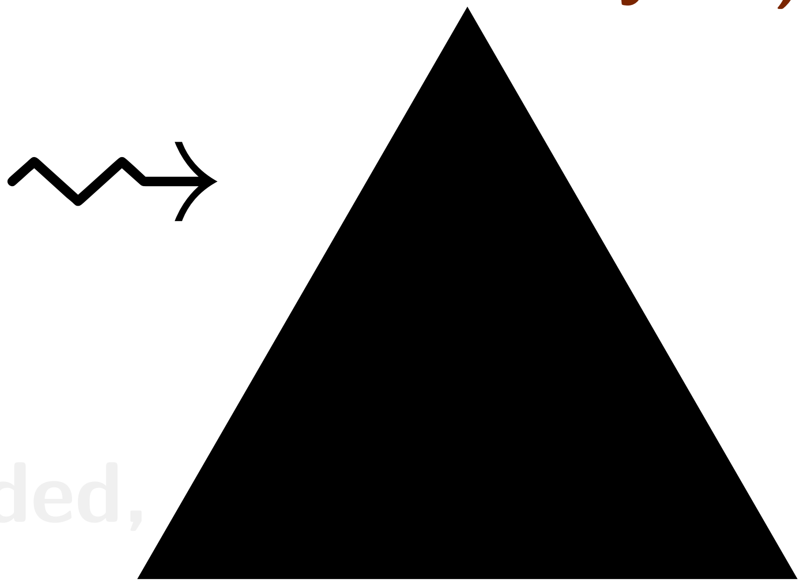

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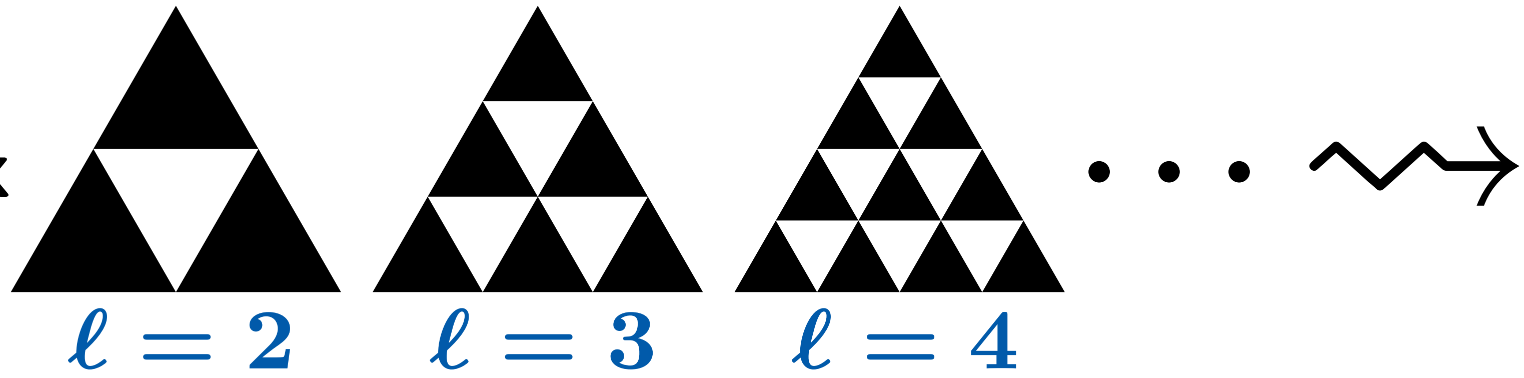


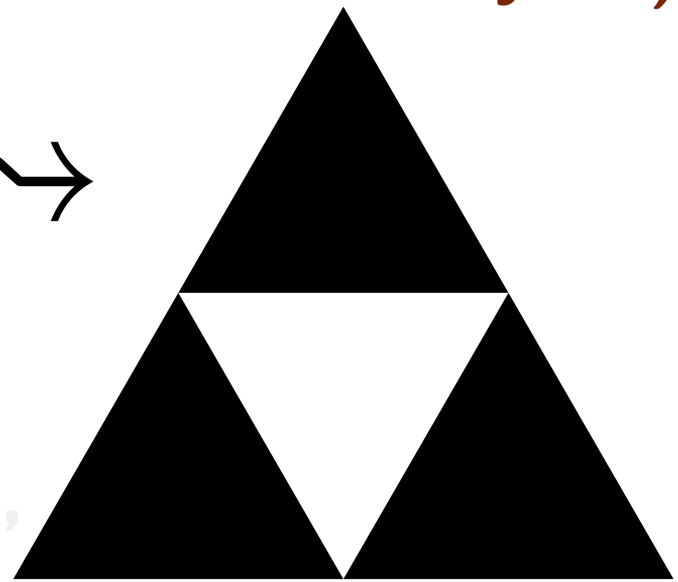

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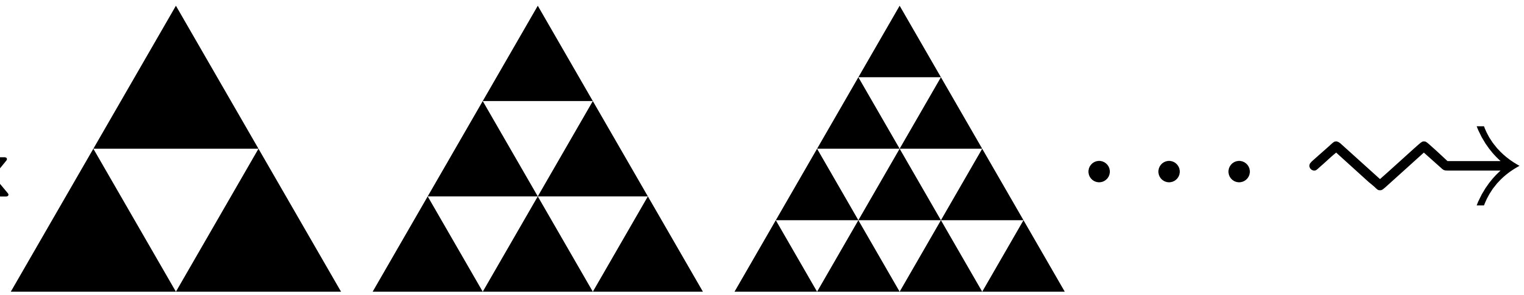
Mix 

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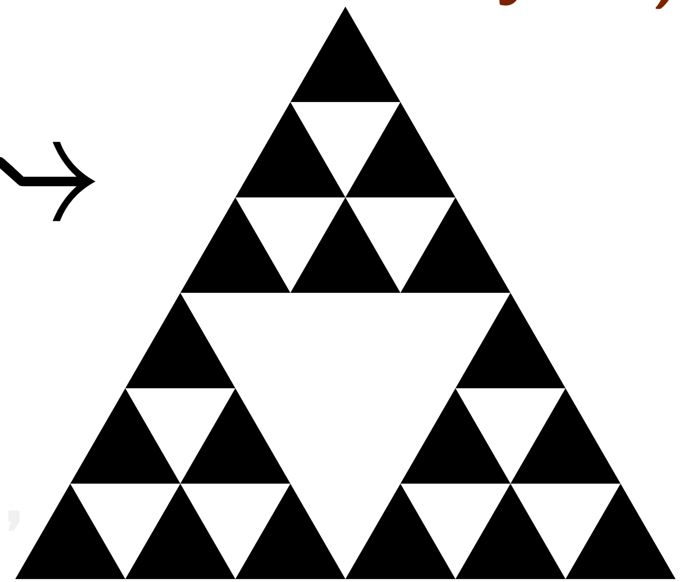

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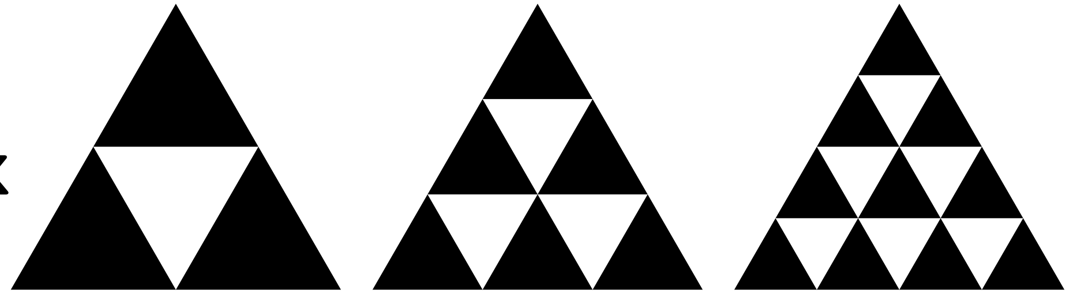

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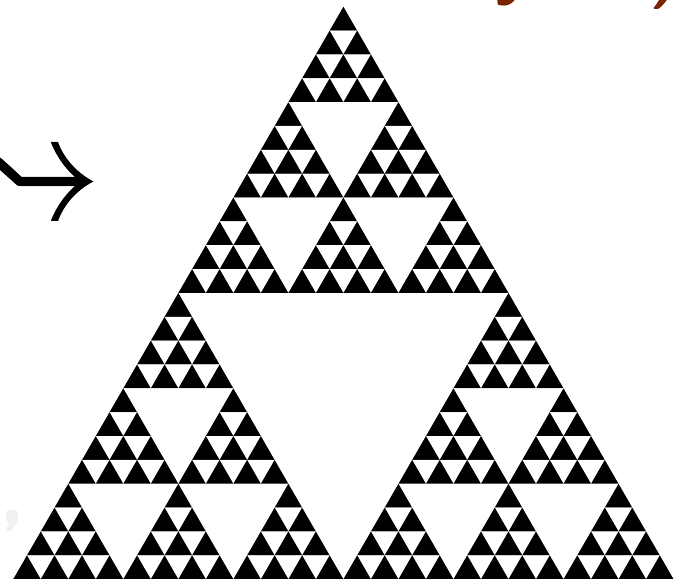

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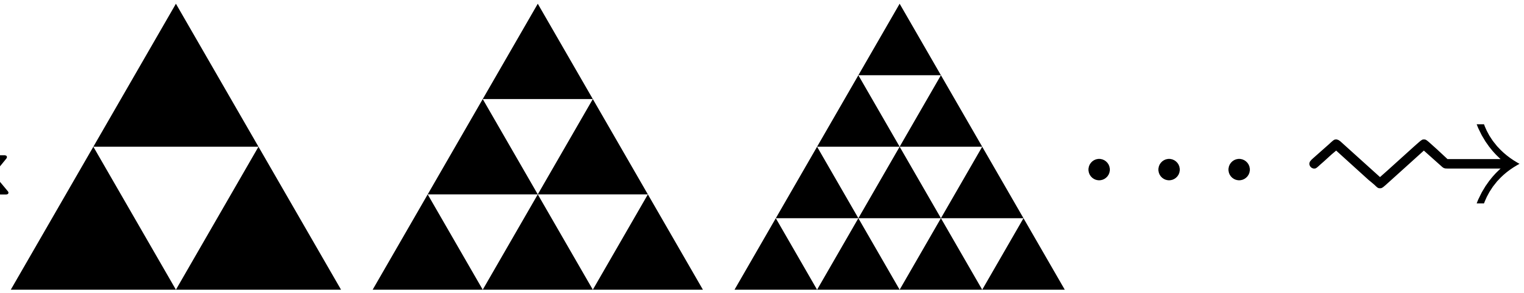

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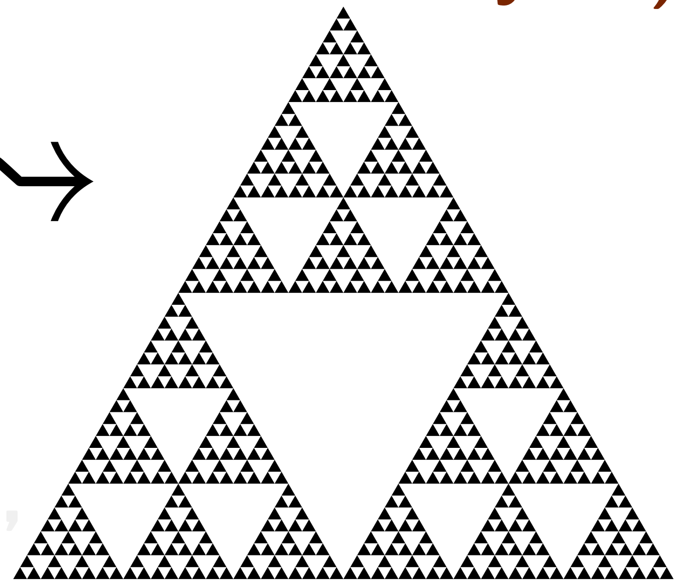

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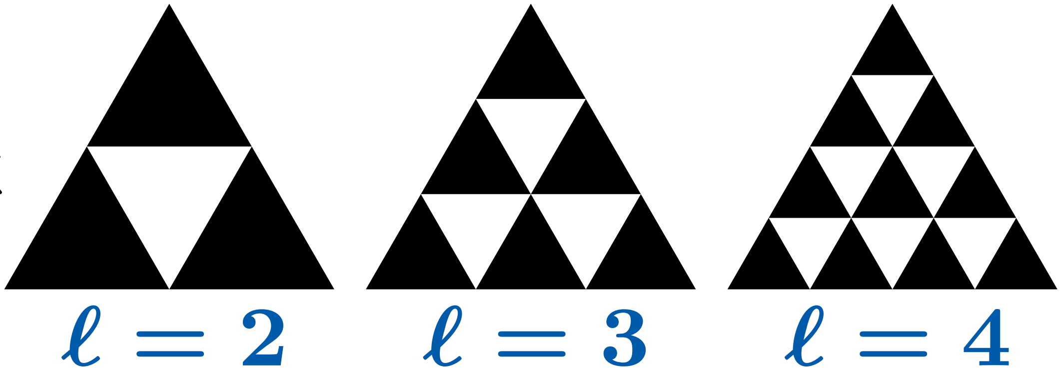
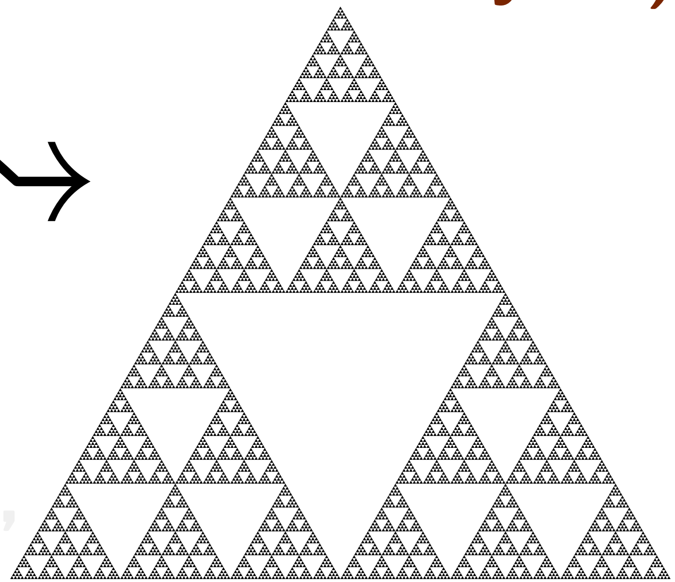



$\cdots \rightsquigarrow$



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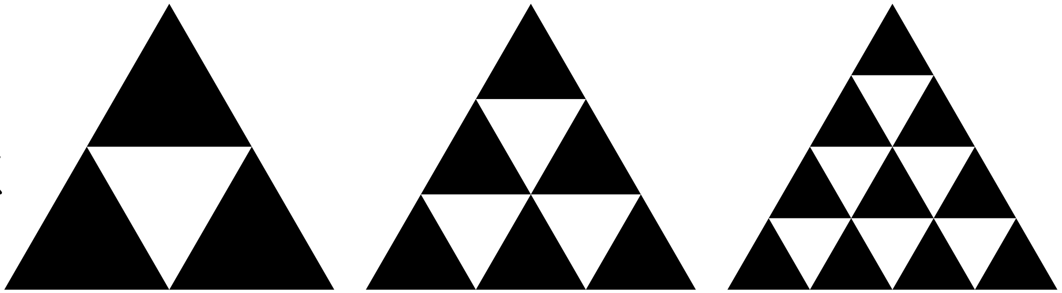


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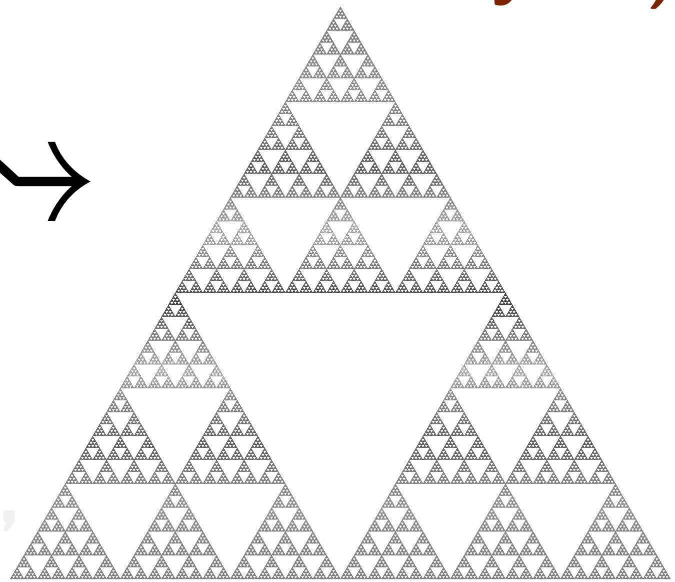

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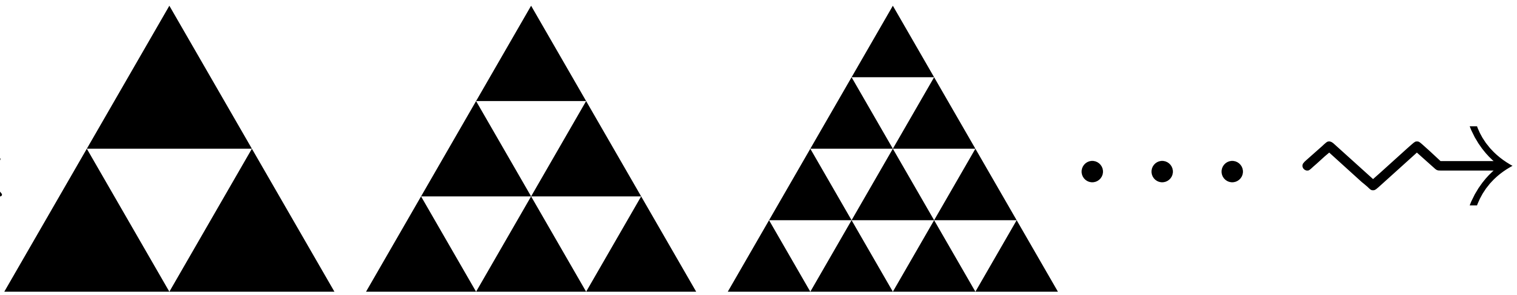

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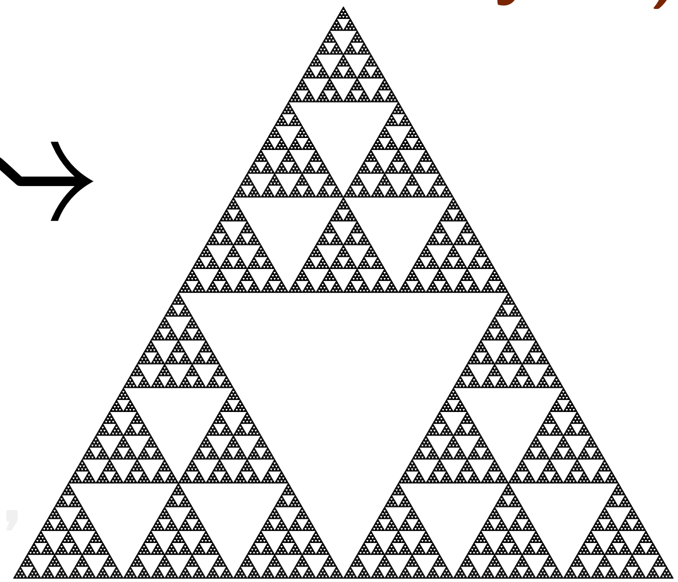

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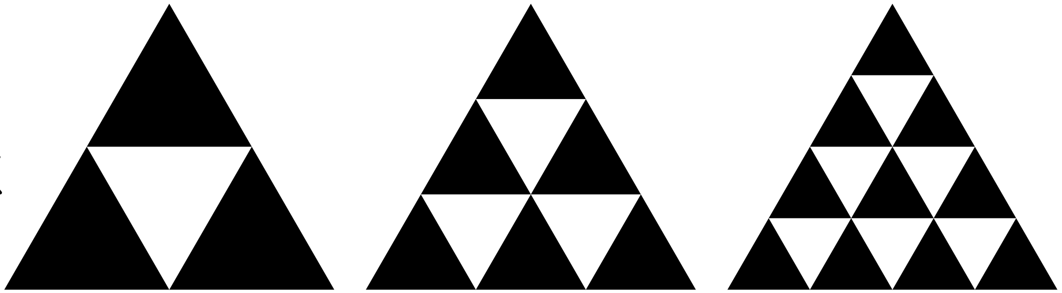
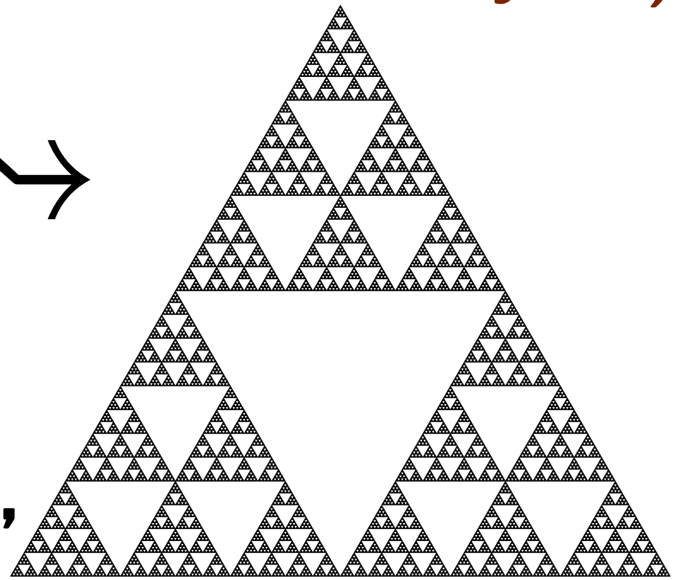
Mix  $\ell=2$ $\ell=3$ $\ell=4$ $\cdots \rightsquigarrow$ 

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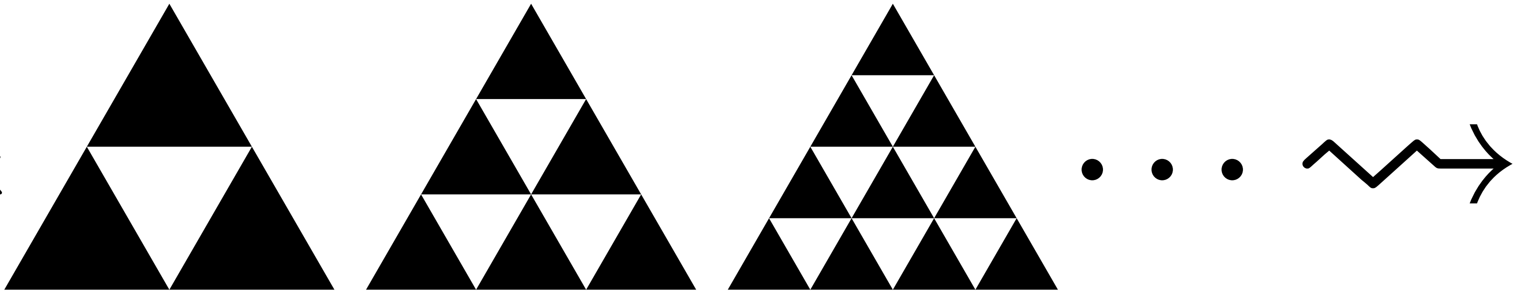

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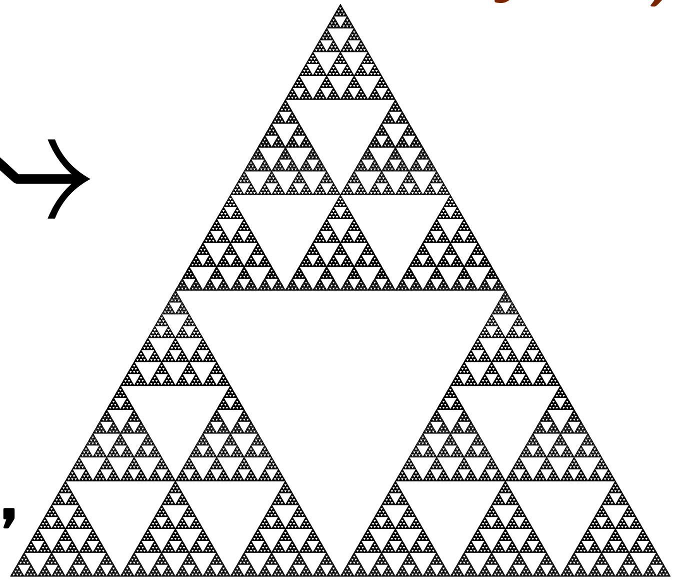

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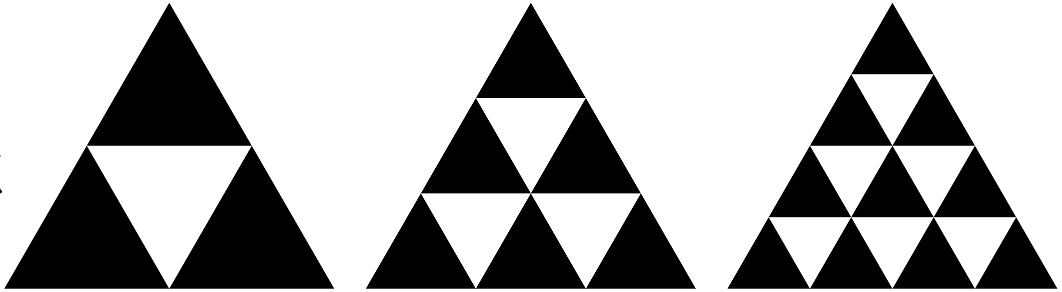

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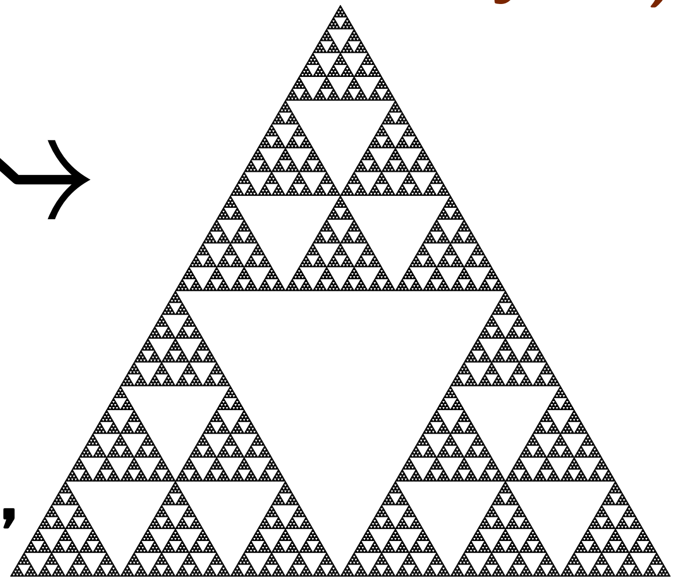

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● Suffices to show $\Gamma(u,u)|_V\perp\mu|_V$ when $u|_V$ is harmonic:

**Lem.** Let $u\in\mathcal{F}\cap\mathcal{C}_c(K)$, $u\geq 0$, $F_n:=u^{-1}(2^{-n}\mathbb{Z})$ and

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**Prop (Reverse PI).** If $h\in\mathcal{F}\cap L^\infty$ & $h|_{B(x,2r)}$ is harmonic,

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