Partial Isometries, Duality, and Determinantal Point Processes

Joint work with Tomoyuki SHIRAI (Kyushu Univ.)

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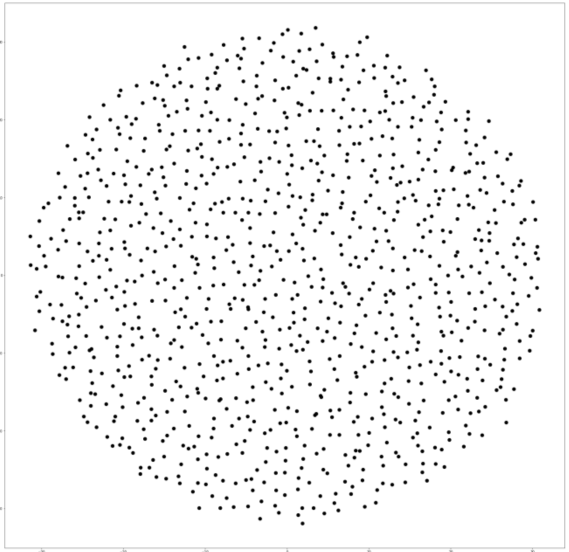
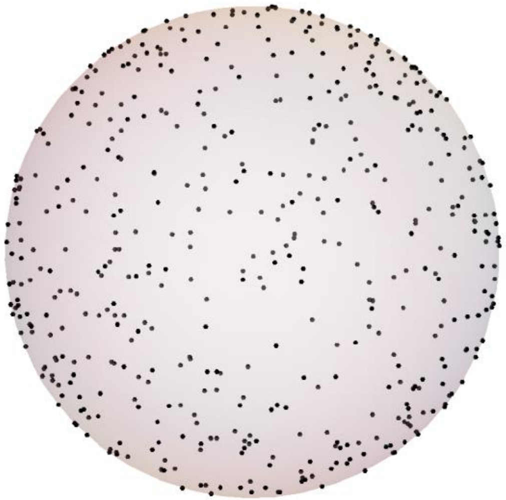




1. Introduction to Determinantal Point Processes (DPPs)

- Let S be a base space, which is locally compact Hausdorff space with countable base, and λ be a Radon measure on S.

- The configuration space over S is given by the set of nonnegative-integer-valued Radon measures;

$$\mathrm{Conf}(S) = \left\{ \xi = \sum_j \delta_{x_j} : x_j \in S,\ \xi(\Lambda) < \infty \text{ for all bounded set } \Lambda \subset S \right\}.$$

  $\mathrm{Conf}(S)$ is equipped with the topological Borel σ-fields with respect to the vague topology; we say $\xi_n, n \in \mathbb{N} := \{1, 2, \dots\}$ converges to ξ in the vague topology, if $\int_S f(x)\xi_n(dx) \to \int_S f(x)\xi(dx)$, $\forall f \in \mathcal{C}_c(S)$, where $\mathcal{C}_c(S)$ is the set of all continuous real-valued functions with compact support.

- A point process on S is a $\mathrm{Conf}(S)$-valued random variable $\Xi = \Xi(\cdot, \omega)$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. If $\Xi(\{x\}) \in \{0, 1\}$ for any point $x \in S$, then the point process is said to be simple.

- **Assume that** $\Lambda_j, j = 1, \ldots, m$**,** $m \in \mathbb{N}$ **are disjoint bounded sets in** S**.**

- **By definition,**

$$\Xi(\Lambda_j) = \textbf{number of points included in } \Lambda_j, j = 1, \ldots, m.$$

- **For** $k_j \in \mathbb{N}_0 := \{0, 1, \ldots\}, j = 1, \ldots, m$ **satisfying** $\sum_{j=1}^m k_j = n \in \mathbb{N}_0$**, we consider the following product of combinatorial numbers,**

$$\prod_{j=1}^{m} \binom{\Xi(\Lambda_j)}{k_j} := \prod_{j=1}^{m} \frac{\Xi(\Lambda_j)!}{k_j!(\Xi(\Lambda_j) - k_j)!}.$$

- **If its expectation is written as**

$$\mathbf{E}\left[\prod_{j=1}^{m} \binom{\Xi(\Lambda_j)}{k_j}\right] = \frac{1}{k_1! \cdots k_m!} \int_{\Lambda_1^{k_1} \times \cdots \times \Lambda_m^{k_m}} \rho^n(x_1, \ldots, x_n) \lambda^{\otimes n}(dx_1 \cdots dx_n),$$

where $\lambda^{\otimes n}$ **denotes the** n**-product measure of** λ**, then** $\rho^n(x_1, \ldots, x_n)$ **is called the** n**-point correlation function with respect to the background measure** λ**.**

- **Determinantal point process (DPP) is defined as follows [Sos00,ST03].**

Definition 1.1 *A simple point process* Ξ *on* (S, λ) *is said to be a determinantal point process (DPP) with correlation kernel* $K : S \times S \to \mathbb{C}$ *if it has correlation functions* $\{\rho^n\}_{n \geq 1}$*, and they are given by*

$$\rho^n(x_1, \dots, x_n) = \det_{1 \leq j,k \leq n} [K(x_j, x_k)] \quad \textit{for every } n \in \mathbb{N}, \textit{ and } x_1, \dots, x_n \in S.$$

The triplet $(\Xi, K, \lambda(dx))$ *denotes the DPP,* $\Xi \in \mathrm{Conf}(S)$*, specified by the correlation kernel* K *with respect to the measure* $\lambda(dx)$*.*

[Sos00] **A. Soshnikov, Determinantal random point fields, Russian Math. Surveys 55 (2000) 923–975.**

[ST03] **T. Shirai and Y. Takahashi, Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point process, J. Funct. Anal. 205 (2003) 414–463.**

- **If the integral operator $\mathcal{K}$ on $L^2(S, \lambda)$ with kernel K is of rank $N \in \mathbb{N}$, then the number of points is N a.s. If $N < \infty$ (resp. $N = \infty$), we call the system a finite DPP (resp. and infinite DPP).**

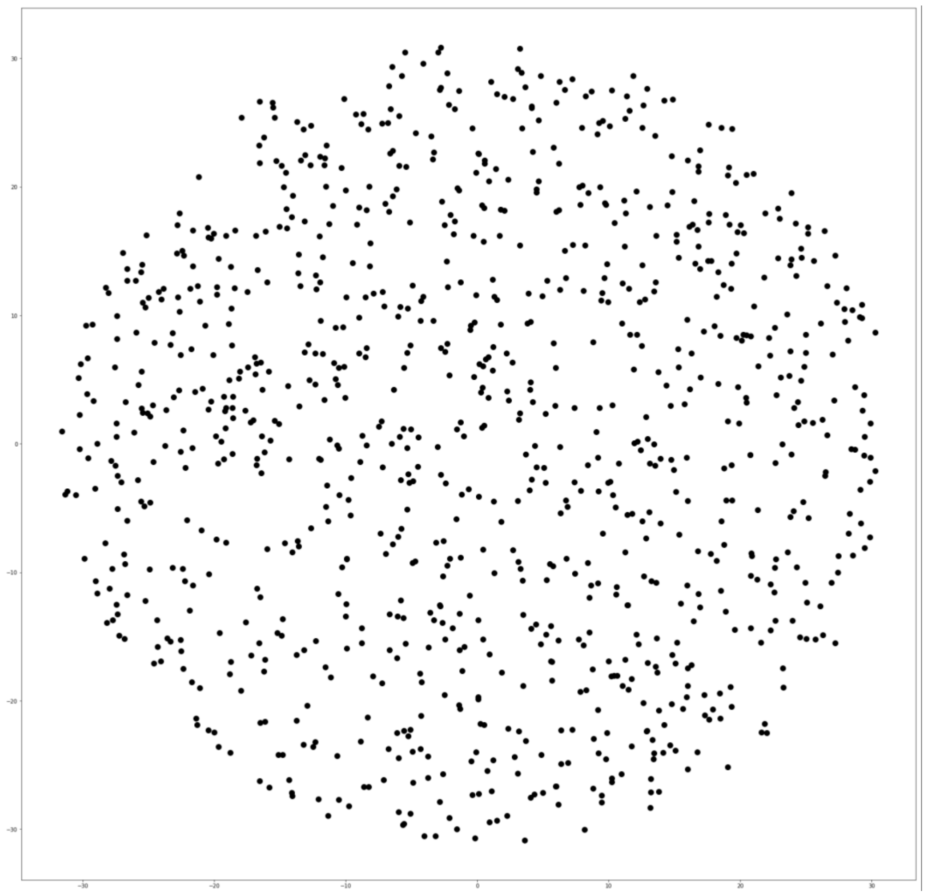
- **The density of points with respect to the background measure $\lambda(dx)$ is given by**

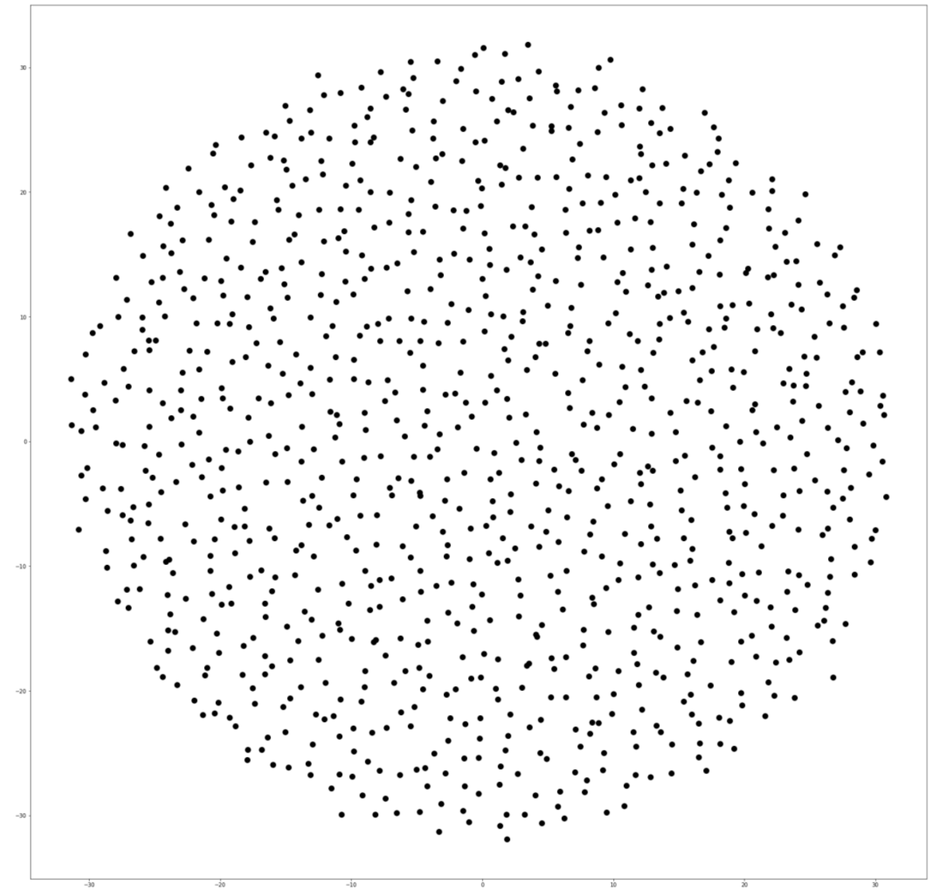
$$\rho(x) := \rho^1(x) = K(x, x).$$

- **The DPP is negatively correlated as shown by**

$$\begin{aligned}\rho^2(x, x') &= \det \begin{bmatrix} K(x,x) & K(x,x') \\ K(x',x) & K(x',x') \end{bmatrix} \\ &= K(x,x)K(x',x') - |K(x,x')|^2 \leq \rho(x)\rho(x'), \quad x, x' \in S,\end{aligned}$$

provided that K is Hermitian.





Poisson point process **an example of DPP (Ginibre DPP)**

(Computer simulation by T. Matsui (Chuo U.))

- **Let $L^2(S,\lambda)$ be an L^2-space.**

- **For operators $\mathcal{A},\mathcal{B}$ on $L^2(S,\lambda)$, we write $\mathcal{A}\geq O$ if $\langle \mathcal{A}f, f\rangle_{L^2(S,\lambda)}\geq 0$ for any $f\in L^2(S,\lambda)$, and $\mathcal{A}\geq\mathcal{B}$ if $\mathcal{A}-\mathcal{B}\geq O$.**

- **For a compact subset $\Lambda\subset S$, the projection from $L^2(S,\lambda)$ to the space of all functions vanishing outside Λ λ-a.e. is denoted by $\mathcal{P}_\Lambda$. $\mathcal{P}_\Lambda$ is the operation of multiplication of the indicator function $\mathbf{1}_\Lambda$ of the set Λ; $\mathbf{1}_\Lambda(x)=1$ if $x\in\Lambda$, and $\mathbf{1}_\Lambda(x)=0$ otherwise.**

- **We say that the bounded Hermitian operator $\mathcal{A}$ on $L^2(S,\lambda)$ is said to be of locally trace class, if the restriction of $\mathcal{A}$ to each compact subset Λ, $\mathcal{A}_\Lambda:=\mathcal{P}_\Lambda\mathcal{A}\mathcal{P}_\Lambda$, is of trace class; $\mathrm{Tr}\,\mathcal{A}_\Lambda<\infty$.**

- **The totality of locally trace class operators on $L^2(S,\lambda)$ is denoted by $\mathcal{I}_{1,\mathrm{loc}}(S,\lambda)$.**

Theorem 1.2 (Sos00,ST03) *Assume that* $\mathcal{K} \in \mathcal{I}_{1,\mathrm{loc}}(S, \lambda)$ *and* $O \leq \mathcal{K} \leq I$. *Then there exists a unique DPP on* S *such that the correlation function is given by*

$$\rho^n(x_1, \dots, x_n) = \det_{1 \leq j,k \leq n} [K(x_j, x_k)], \quad n \in \mathbb{N}, \quad x_1, \dots, x_n \in S.$$

[Sos00] A. Soshnikov, Determinantal random point fields, Russian Math. Surveys 55 (2000) 923–975.

[ST03] T. Shirai and Y. Takahashi, Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point process, J. Funct. Anal. 205 (2003) 414–463.

- **In the present talk, we consider the case that**

$$\mathcal{K}f = f \quad \textbf{for all } f \in (\ker \mathcal{K})^{\perp} \subset L^2(S,\lambda),$$

where $(\ker \mathcal{K})^{\perp}$ **denotes the orthogonal complement of the kernel of** $\mathcal{K}$**.**

- **That is,** $\mathcal{K}$ **is an orthogonal projection.**
- **By definition,**

$$\mathcal{K}f = \begin{cases} f, & \textbf{if } f \in (\ker \mathcal{K})^{\perp}, \\ 0, & \textbf{if } f \in \ker \mathcal{K}. \end{cases}$$

Hence, it is obvious that the condition $O \leq \mathcal{K} \leq I$ **is satisfied.**

- **The purpose of the present talk is to propose a useful method to provide orthogonal projections $\mathcal{K}$ and DPPs whose correlation kernels are given by the Hermitian integral kernels $K(x, x'), x, x' \in S$ of $\mathcal{K}$.**

- **I show that our method gives duality relations between pairs of DPPs.**

- **As examples of DPPs constructed by this method, I discuss the DPPs on higher-dimensional spaces $\mathbb{S}^d, d \in \mathbb{N}$, and their bulk scaling limit.**

- **At the end of my talk, I will show open problems.**

2. Partial Isometry and DPPs

- **First we recall the notion of partial isometries between Hilbert spaces.**
- **Let** $H_\ell, \ell = 1, 2$ **be separable Hilbert spaces with inner products** $\langle \cdot, \cdot \rangle_{H_\ell}$. **For a bounded linear operator** $\mathcal{W} : H_1 \to H_2$, **the adjoint of** $\mathcal{W}$ **is defined as the operator** $\mathcal{W}^* : H_2 \to H_1$, **such that**

$$\langle \mathcal{W} f, g \rangle_{H_2} = \langle f, \mathcal{W}^* g \rangle_{H_1} \quad \text{for all } f \in H_1 \text{ and } g \in H_2.$$

  A linear operator $\mathcal{W}$ **is called an isometry if**

$$||\mathcal{W} f||_{H_2} = ||f||_{H_1} \quad \text{for all } f \in H_1.$$

- **For** $\mathcal{W}$ **its kernel is denoted as** $\ker \mathcal{W}$ **and the orthogonal complement of** $\ker \mathcal{W}$ **is written as** $(\ker \mathcal{W})^\perp$.
- **A linear operator** $\mathcal{W}$ **is called a partial isometry, if**

$$||\mathcal{W} f||_{H_2} = ||f||_{H_1} \quad \text{for all } f \in (\ker \mathcal{W})^\perp.$$

- **For the partial isometry $\mathcal{W}$, $(\ker \mathcal{W})^{\perp}$ is called the initial space and the range of $\mathcal{W}$, $\mathrm{ran}\mathcal{W}$, is called the final space.**
- **By the definition, $||\mathcal{W}f||^2_{H_2} = \langle \mathcal{W}f, \mathcal{W}f\rangle_{H_2} = \langle f, \mathcal{W}^*\mathcal{W}f\rangle_{H_1}$.**
- **This implies the following.**

Lemma 2.1 *The bounded linear operator $\mathcal{W}$ (resp. $\mathcal{W}^*$) is a partial isometry if and only if $\mathcal{W}^*\mathcal{W}$ (resp. $\mathcal{W}\mathcal{W}^*$) is the identity on $(\ker \mathcal{W})^{\perp}$ (resp. $(ker\mathcal{W}^*)^{\perp}$).*

- **For the partial isometry $\mathcal{W}$, $(\ker \mathcal{W})^{\perp}$ is called the initial space and the range of $\mathcal{W}$, $\mathrm{ran}\mathcal{W}$, is called the final space.**
- **By the definition,** $||\mathcal{W}f||^2_{H_2} = \langle \mathcal{W}f, \mathcal{W}f\rangle_{H_2} = \langle f, \mathcal{W}^*\mathcal{W}f\rangle_{H_1}$.
- **This implies the following.**

Lemma 2.1 *The bounded linear operator $\mathcal{W}$ (resp. $\mathcal{W}^*$) is a partial isometry if and only if $\mathcal{W}^*\mathcal{W}$ (resp. $\mathcal{W}\mathcal{W}^*$) is the identity on $(\ker \mathcal{W})^{\perp}$ (resp. $(\ker\mathcal{W}^*)^{\perp}$).*

- **We put the first assumption.**

> **Assumption 1** **Both $\mathcal{W}$ and $\mathcal{W}^*$ are partial isometries.**

- **Under Assumption 1, the operator $\mathcal{W}^*\mathcal{W}$ (resp. $\mathcal{W}\mathcal{W}^*$) is the projection onto the initial space of $\mathcal{W}$ (resp. the final space of $\mathcal{W}$).**

- **Now we assume that H_1 and H_2 are realized as L^2-spaces, $L^2(S_1, \lambda_1)$ and $L^2(S_2, \lambda_2)$, respectively.**

- **We consider the case in which $\mathcal{W}$ admits an integral kernel $W : S_2 \times S_1 \to \mathbb{C}$ such that**

$$(\mathcal{W}f)(y) = \int_{S_1} W(y, x) f(x) \lambda_1(dx), \quad f \in L^2(S_1, \lambda_1),$$

and then

$$(\mathcal{W}^*g)(x) = \int_{S_2} \overline{W(y, x)} g(y) \lambda_2(dy), \quad g \in L^2(S_2, \lambda_2).$$

- **We put the second assumption.**

Assumption 2 $\mathcal{W}^*\mathcal{W} \in \mathcal{I}_{1,\mathrm{loc}}(S_1, \lambda_1)$ **and** $\mathcal{W}\mathcal{W}^* \in \mathcal{I}_{1,\mathrm{loc}}(S_2, \lambda_2)$**.**

- **We have**

$$(\mathcal{W}^*\mathcal{W}f)(x) = \int_{S_1} K_{S_1}(x, x')f(x')\lambda_1(dx'), \quad f \in L^2(S_1, \lambda_1),$$
$$(\mathcal{W}\mathcal{W}^*g)(y) = \int_{S_2} K_{S_2}(y, y')g(y')\lambda_2(dy'), \quad g \in L^2(S_2, \lambda_2),$$

with the integral kernels,

$$K_{S_1}(x, x') = \int_{S_2} \overline{W(y,x)}W(y,x')\lambda_2(dy) = \langle W(\cdot, x'), W(\cdot, x)\rangle_{L^2(S_2,\lambda_2)},$$
$$K_{S_2}(y, y') = \int_{S_1} W(y,x)\overline{W(y',x)}\lambda_1(dx) = \langle W(y, \cdot), W(y', \cdot)\rangle_{L^2(S_1,\lambda_1)}.$$

- **We see that** $\overline{K_{S_1}(x', x)} = K_{S_1}(x, x')$ **and** $\overline{K_{S_2}(y', y)} = K_{S_2}(y, y')$**.**

- **The main theorem is the following.**

Theorem 2.2 *Under Assumptions 1 and 2, associated with* $\mathcal{W}^*\mathcal{W}$ *and* $\mathcal{W}\mathcal{W}^*$*, there exists a unique pair of DPPs;* $(\Xi_1, K_{S_1}, \lambda_1(dx))$ *on* S_1 *and* $(\Xi_2, K_{S_2}, \lambda_2(dy))$ *on* S_2*. The correlation kernels* $K_{S_\ell}, \ell = 1, 2$ *are Hermitian and given by*

$$K_{S_1}(x, x') = \int_{S_2} \overline{W(y, x)} W(y, x') \lambda_2(dy) = \langle W(\cdot, x'), W(\cdot, x) \rangle_{L^2(S_2, \lambda_2)},$$

$$K_{S_2}(y, y') = \int_{S_1} W(y, x) \overline{W(y', x)} \lambda_1(dx) = \langle W(y, \cdot), W(y', \cdot) \rangle_{L^2(S_1, \lambda_1)}.$$

3. Orthonormal Functions and Correlation Kernels

- **In addition to $L^2(S_\ell, \lambda_\ell)$, $\ell = 1, 2$, we introduce $L^2(\Gamma, \nu)$ as a parameter space for functions in $L^2(S_\ell, \lambda_\ell), \ell = 1, 2$.**

- **Assume that there are two families of measurable functions $\{\psi_1(x, \gamma) : x \in S_1, \gamma \in \Gamma\}$ and $\{\psi_2(y, \gamma) : y \in S_2, \gamma \in \Gamma\}$ such that two bounded operators $\mathcal{U}_\ell : L^2(S_\ell, \lambda_\ell) \to L^2(\Gamma, \nu)$ given by**

$$\widehat{f}(\gamma) = (\mathcal{U}_\ell f)(\gamma) := \int_{S_\ell} \overline{\psi_\ell(x, \gamma)} f(x) \lambda_\ell(dx), \quad \ell = 1, 2,$$

  are well-defined. Then, their adjoints $\mathcal{U}_\ell^* : L^2(\Gamma, \nu) \to L^2(S_\ell, \lambda_\ell), \ell = 1, 2$ are given by

$$(\mathcal{U}_\ell^* F)(\cdot) = \int_\Gamma \psi_\ell(\cdot, \gamma) F(\gamma) \nu(d\gamma).$$

- **Now we define $\mathcal{W} : L^2(S_1, \lambda_1) \to L^2(S_2, \lambda_2)$ by $\mathcal{W} = \mathcal{U}_2^* \mathcal{U}_1$, i.e.,**

$$(\mathcal{W} f)(y) = \int_\Gamma \psi_2(y, \gamma) \widehat{f}(\gamma) \nu(d\gamma).$$

- **We can see the following.**

Lemma 3.1 *If*

$$\mathcal{U}_\ell \mathcal{U}_\ell^* = I_\Gamma \quad \text{for } \ell = 1, 2,$$

then both $\mathcal{W}$ *and* $\mathcal{W}^*$ *are partial isometries.*

Proof **It suffices to show that $\mathcal{W}^*\mathcal{W}$ is an orthogonal projection, or equivalently, it suffices to show $(\mathcal{W}^*\mathcal{W})^2 = \mathcal{W}^*\mathcal{W}$ since $\mathcal{W}^*\mathcal{W}$ is self-adjoint.**
By the assumption, we see that

$$\mathcal{W}^*\mathcal{W} = (\mathcal{U}_2^*\mathcal{U}_1)^*\mathcal{U}_2^*\mathcal{U}_1 = \mathcal{U}_1^*(\mathcal{U}_2\mathcal{U}_2^*)\mathcal{U}_1 = \mathcal{U}_1^*\mathcal{U}_1.$$

Hence, $(\mathcal{W}^*\mathcal{W})^2 = \mathcal{U}_1^*\mathcal{U}_1\mathcal{U}_1^*\mathcal{U}_1 = \mathcal{U}_1^*\mathcal{U}_1 = \mathcal{W}^*\mathcal{W}$.
By symmetry, the assertion for $\mathcal{W}^*$ also follows. ∎

- **We note from the proof that $\mathcal{W}^*\mathcal{W} = \mathcal{U}_1^*\mathcal{U}_1$ and $\mathcal{W}\mathcal{W}^* = \mathcal{U}_2^*\mathcal{U}_2$ so that $\mathcal{U}_\ell, \ell = 1, 2$ are partial isometries.**

Assumption 3 **We assume that $\mathcal{U}_\ell \mathcal{U}_\ell^* = I_\Gamma$ for $\ell = 1, 2$.**

Assumption 3 can be rephrased as the following orthonormality relations:

$$\langle \psi_\ell(\cdot, \gamma), \psi_\ell(\cdot, \gamma') \rangle_{L^2(S_\ell, \lambda_\ell)} \nu(d\gamma) = \delta(\gamma - \gamma') d\gamma, \quad \gamma, \gamma' \in \Gamma, \quad \ell = 1, 2.$$

We will use these relations below.

The following is immediately obtained as a corollary of Theorem 2.2.

Corollary 3.2 *Let $\mathcal{W} = \mathcal{U}_2^*\mathcal{U}_1$ as in the above. We assume Assumption 3 in addition to Assumption 2. Then, there exist a unique pair of DPPs; $(\Xi_1, K_{S_1}, \lambda_1(dx))$ on S_1 and $(\Xi_2, K_{S_2}, \lambda_2(dy))$ on S_2. Here the correlation kernels $K_{S_\ell}, \ell = 1, 2$ are given by*

$$K_{S_1}(x, x') = \int_\Gamma \psi_1(x, \gamma) \overline{\psi_1(x', \gamma)} \nu(d\gamma) = \langle \psi_1(x, \cdot), \psi_1(x', \cdot) \rangle_{L^2(\Gamma, \nu)},$$
$$K_{S_2}(y, y') = \int_\Gamma \psi_2(y, \gamma) \overline{\psi_2(y', \gamma)} \nu(d\gamma) = \langle \psi_2(y, \cdot), \psi_2(y', \cdot) \rangle_{L^2(\Gamma, \nu)}.$$

- Now we consider a simplified version of the preceding setting.
- Let $\Gamma \subseteq S_2$ and $\nu = \lambda_2|_\Gamma$.
- We define $\mathcal{U}_2 : L^2(S_2, \lambda_2) \to L^2(\Gamma, \nu)$ as the restriction onto Γ and then its adjoint $\mathcal{U}_2^*$ is given by $(\mathcal{U}_2^* F)(y) = F(y)$ for $y \in \Gamma$, and by 0 for $y \in S_2 \setminus \Gamma$.
- It is obvious that $\mathcal{U}_2 \mathcal{U}_2^* = I_\Gamma$ and hence $\mathcal{U}_2$ is a partial isometry.
- For $\Gamma \subseteq S_2$, we assume that there is a family of measurable functions $\{\psi_1(x, y) : x \in S_1, y \in \Gamma\}$ such that a bounded operator $\mathcal{U}_1 : L^2(S_1, \lambda_1) \to L^2(\Gamma, \nu)$ given by $(\mathcal{U}_1 f)(\gamma) := \int_{S_1} \overline{\psi_1(x, \gamma)} f(x) \lambda_1(dx)$, $(\gamma \in \Gamma)$ is well-defined.

Assumption 3' We assume that $\mathcal{U}_1 \mathcal{U}_1^* = I_\Gamma$. This can be rephrased as

$$\langle \psi_1(\cdot, y), \psi_1(\cdot, y') \rangle_{L^2(S_1, \lambda_1)} \lambda_2(dy) = \delta(y - y')dy, \quad y, y' \in \Gamma.$$

- Now we define $\mathcal{W} : L^2(S_1, \lambda_1) \to L^2(S_2, \lambda_2)$ by $\mathcal{W} = \mathcal{U}_2^* \mathcal{U}_1$ as before.
- It follows from Assumption 3' that $\mathcal{W}$ is a partial isometry. Corollary 3.2 is reduced to the following.

Corollary 3.3 *Let $\mathcal{W} = \mathcal{U}_2^* \mathcal{U}_1$ as in the above. We assume Assumption 3' in addition to Assumption 2. Then there exists a unique DPP, (Ξ, K, λ_1) on S_1 with the correlation kernel*

$$K_{S_1}(x, x') = \int_\Gamma \psi_1(x, y) \overline{\psi_1(x', y)} \lambda_2(dy) = \langle \tilde{\psi}_1(x, \cdot), \tilde{\psi}_1(x', \cdot) \rangle_{L^2(\Gamma, \lambda_2)}.$$

4. Duality

4.1 Duality relations

- **For $f \in \mathcal{C}_c(S)$, the Laplace transform of the probability measure $\mathbf{P}$ for a point process Ξ is defined as**

$$\Psi[f] = \mathbf{E}\left[\exp\left(\int_S f(x)\Xi(dx)\right)\right].$$

- **For the DPP, $(\Xi, K, \lambda(dx))$, this is given by the Fredholm determinant on $L^2(S, \lambda)$,**

$$\operatorname*{Det}_{L^2(S,\lambda)}[I-(1-e^f)\mathcal{K}] := 1 + \sum_{n \in \mathbb{N}} \frac{(-1)^n}{n!} \int_{S^n} \det_{1 \le j,k \le n}[K(x_j, x_k)] \prod_{\ell=1}^{n} (1-e^{f(x_\ell)}) \lambda^{\otimes n}(d\boldsymbol{x}).$$

Lemma 4.1 *Between two DPPs, $(\Xi_1, K_{S_1}, \lambda_1(dx))$ on S_1 and $(\Xi_2, K_{S_2}, \lambda_2(dy))$ on S_2, given by Theorem 2.2, the following equality holds with an arbitrary parameter $\alpha \in \mathbb{C}$,*

$$\operatorname*{Det}_{L^2(S_1,\lambda_1)}[I + \alpha\mathcal{K}_{S_1}] = \operatorname*{Det}_{L^2(S_2,\lambda_2)}[I + \alpha\mathcal{K}_{S_2}].$$

- **For $\Lambda_\ell \subset S_\ell, \ell = 1, 2$, let**

$$\widetilde{\mathcal{W}} := \mathcal{P}_{\Lambda_2} \mathcal{W} \mathcal{P}_{\Lambda_1}, \quad \mathcal{K}_{S_1}^{(\Lambda_2)} := \mathcal{W}^* \mathcal{P}_{\Lambda_2} \mathcal{W}, \quad \mathcal{K}_{S_2}^{(\Lambda_1)} := \mathcal{W} \mathcal{P}_{\Lambda_1} \mathcal{W}^*.$$

- **They admit the following integral kernels,**

$$\begin{aligned} \widetilde{W}(y,x) &= \mathbf{1}_{\Lambda_2}(y) W(y,x) \mathbf{1}_{\Lambda_1}(x), \\ K_{S_1}^{(\Lambda_2)}(x,x') &= \int_{\Lambda_2} \overline{W(y,x)} W(y,x') \lambda_2(dy), \\ K_{S_2}^{(\Lambda_1)}(y,y') &= \int_{\Lambda_1} W(y,x) \overline{W(y',x)} \lambda_1(dx). \end{aligned}$$

- **Using Lemma 4.1, the following is proved.**

Proposition 4.2 *Let* $(\Xi_1^{(\Lambda_2)}, K_{S_1}^{(\Lambda_2)}, \lambda_1(dx))$ *and* $(\Xi_2^{(\Lambda_1)}, K_{S_2}^{(\Lambda_1)}, \lambda_2(dy))$ *be DPPs associated with the kernels* $K_{S_1}^{(\Lambda_2)}$ *and* $K_{S_2}^{(\Lambda_1)}$ *given as above, respectively. Then*

$$\mathbf{P}(\Xi_1^{(\Lambda_2)}(\Lambda_1) = m) = \mathbf{P}(\Xi_2^{(\Lambda_1)}(\Lambda_2) = m), \quad \forall m \in \mathbb{N}_0.$$

- **For $\Lambda_\ell \subset S_\ell, \ell = 1, 2$, let**

$$\widetilde{\mathcal{W}} := \mathcal{P}_{\Lambda_2} \mathcal{W} \mathcal{P}_{\Lambda_1}, \quad \mathcal{K}_{S_1}^{(\Lambda_2)} := \mathcal{W}^* \mathcal{P}_{\Lambda_2} \mathcal{W}, \quad \mathcal{K}_{S_2}^{(\Lambda_1)} := \mathcal{W} \mathcal{P}_{\Lambda_1} \mathcal{W}^*.$$

- **They admit the following integral kernels,**

$$\begin{aligned}
\widetilde{W}(y, x) &= \mathbf{1}_{\Lambda_2}(y) W(y, x) \mathbf{1}_{\Lambda_1}(x), \\
K_{S_1}^{(\Lambda_2)}(x, x') &= \int_{\Lambda_2} \overline{W(y, x)} W(y, x') \lambda_2(dy), \\
K_{S_2}^{(\Lambda_1)}(y, y') &= \int_{\Lambda_1} W(y, x) \overline{W(y', x)} \lambda_1(dx).
\end{aligned}$$

- **Using Lemma 4.1, the following is proved.**

Proposition 4.2 *Let* $(\Xi_1^{(\Lambda_2)}, K_{S_1}^{(\Lambda_2)}, \lambda_1(dx))$ *and* $(\Xi_2^{(\Lambda_1)}, K_{S_2}^{(\Lambda_1)}, \lambda_2(dy))$ *be DPPs associated with the kernels* $K_{S_1}^{(\Lambda_2)}$ *and* $K_{S_2}^{(\Lambda_1)}$ *given as above, respectively. Then*

$$\mathbf{P}(\Xi_1^{(\Lambda_2)}(\Lambda_1) = m) = \mathbf{P}(\Xi_2^{(\Lambda_1)}(\Lambda_2) = m), \quad \forall m \in \mathbb{N}_0.$$

4.2 Example

- **We consider an application of Corollary 3.3 (the simplified version).**
- **Let** $S_1 = \mathbb{C}$ **and** $S_2 = \mathbb{N}_0$ **with** $\lambda_1(dx) = \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx)$.
- **Consider the normal complex Gaussian measure,** $\lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx) = \frac{1}{\pi} e^{-|x|^2} dx,\ x \in \mathbb{C}$.
- **We put**

$$\varphi_n(x) = \frac{x^n}{\sqrt{n!}}, \quad n \in \mathbb{N}_0.$$

Note that $\{\varphi_n(x)\}_{n \in \mathbb{N}_0}$ **forms a complete orthonormal system of the Bargmann–Fock space, which is the space of square-integrable analytic functions on** $\mathbb{C}$ **with respect to** $\lambda_{\mathrm{N}(0,1;\mathbb{C})}$**;**

$$\langle \varphi_n, \varphi_m \rangle_{L^2(\mathbb{C}, \lambda_{\mathrm{N}(0,1;\mathbb{C})})} = \delta_{nm},\ n, m \in \mathbb{N}_0.$$

- **If we assume that** $\Gamma = S_2 = \mathbb{N}_0$ **and apply Corollary 3.3, we obtain the DPP on** $\mathbb{C}$ **in which the correlation kernel with respect to** $\lambda_{\mathrm{N}(0,1:\mathbb{C})}$ **is given by**

$$K_{\mathrm{BF}}(x, x') = \sum_{n \in \mathbb{N}_0} \varphi_n(x) \overline{\varphi_n(x')} = \sum_{n=0}^{\infty} \frac{(x\overline{x'})^n}{n!} = e^{x\overline{x'}}, \quad x, x' \in \mathbb{C}.$$

- **This is the reproducing kernel in the Bargmann–Fock space and obtained DPP is identified with the Ginibre ensemble (in the bulk scaling limit)** $(\Xi, K_{\mathrm{Ginibre}}, \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx))$.

- Now we show an application of the duality relation.
- Let Λ_1 be a disk (*i.e.*, two-dimensional ball) $\mathbb{B}_r^2$ with radius $r \in (0, \infty)$ centered at the origin in $S_1 = \mathbb{C} \simeq \mathbb{R}^2$ and $\Lambda_2 = S_2 = \mathbb{N}_0$.
- We obtain

$$
\begin{aligned}
K_{\mathbb{C}}^{\mathbb{N}_0}(x, x') &= \sum_{n=0}^{\infty} \varphi_n(x)\overline{\varphi_n(x')} = e^{x\overline{x'}} \\
&= K_{\text{Ginibre}}(x, x'), \quad x, x' \in \mathbb{C}, \\
K_{\mathbb{N}_0}^{\mathbb{B}_r^2}(n, n') &= \int_{\mathbb{B}_r^2} \overline{\varphi_n(x)}\varphi_{n'}(x)\lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx) = \frac{1}{\pi\sqrt{n!n'!}} \int_0^r ds\, e^{-s^2} s^{n+n'+1} \int_0^{2\pi} d\theta\, e^{i\theta(n'-n)} \\
&= 2\delta_{nn'} \int_0^r \frac{s^{2n+1}e^{-s^2}}{n!} ds = \delta_{nn'} \int_0^{r^2} \frac{u^n e^{-u}}{n!} du, \quad n, n' \in \mathbb{N}_0.
\end{aligned}
$$

- Define

$$
\lambda_n(r) := \int_0^{r^2} \frac{u^n e^{-u}}{n!} du = \sum_{k=n+1}^{\infty} \frac{r^{2k} e^{-r^2}}{k!}, \quad n \in \mathbb{N}_0, \quad r \in (0, \infty).
$$

where the second equality is due to Eq.(4.1) in [Shi15].

[Shi15] T. Shirai, Ginibre-type point processes and their asymptotic behavior, J. Math. Soc. Jpn. 67 (2015) 763–787.

- **Now we show an application of the duality relation.**
- **Let Λ_1 be a disk (*i.e.*, two-dimensional ball) $\mathbb{B}_r^2$ with radius $r \in (0, \infty)$ centered at the origin in $S_1 = \mathbb{C} \simeq \mathbb{R}^2$ and $\Lambda_2 = S_2 = \mathbb{N}_0$.**
- **We obtain**

$$K_{\mathbb{C}}^{\mathbb{N}_0}(x, x') = \sum_{n=0}^{\infty} \varphi_n(x)\overline{\varphi_n(x')} = e^{x\overline{x'}}$$
$$= K_{\text{Ginibre}}(x, x'), \quad x, x' \in \mathbb{C},$$
$$K_{\mathbb{N}_0}^{\mathbb{B}_r^2}(n, n') = \int_{\mathbb{B}_r^2} \overline{\varphi_n(x)}\varphi_{n'}(x)\lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx) = \frac{1}{\pi\sqrt{n!n'!}} \int_0^r ds\, e^{-s^2} s^{n+n'+1} \int_0^{2\pi} d\theta\, e^{i\theta(n'-n)}$$
$$= 2\delta_{nn'} \int_0^r \frac{s^{2n+1}e^{-s^2}}{n!} ds = \delta_{nn'} \int_0^{r^2} \frac{u^n e^{-u}}{n!} du, \quad n, n' \in \mathbb{N}_0.$$

- **Define**

$$\lambda_n(r) := \int_0^{r^2} \frac{u^n e^{-u}}{n!} du = \sum_{k=n+1}^{\infty} \frac{r^{2k} e^{-r^2}}{k!}, \quad n \in \mathbb{N}_0, \quad r \in (0, \infty).$$

where the second equality is due to Eq.(4.1) in [Shi15].

[Shi15] T. Shirai, Ginibre-type point processes and their asymptotic behavior, J. Math. Soc. Jpn. 67 (2015) 763–787.

- **That is, if we write the Gamma distribution with parameters (a, b) as $\Gamma(a, b)$ and the Poisson distribution with parameter c as $\mathrm{Po}(c)$,**

$$\lambda_n(r) := \mathbf{P}(S_n \le r^2) = \mathbf{P}(Y_{r^2} \ge n+1),$$

provided $S_n \sim \Gamma(n+1, 1)$ and $Y_{r^2} \sim \mathrm{Po}(r^2)$.

- **Then DPP $(\Xi_2, K_{\mathbb{N}_0}^{\mathbb{B}_r^2})$ is the product measure $\bigotimes_{n\in\mathbb{N}_0} \mu_{\lambda_n(r)}^{\mathrm{Bernoulli}}$ under the natural identification between $\{0,1\}^{\mathbb{N}_0}$ and the power set of $\mathbb{N}_0$, where $\mu_p^{\mathrm{Bernoulli}}$ denotes the Bernoulli measure of probability $p \in [0, 1]$.**

- **Proposition 4.2 gives the duality relation**

$$\mathbf{P}(\Xi_{\mathrm{Ginibre}}(\mathbb{B}_r^2) = m) = \mathbf{P}(\Xi_2(\mathbb{N}_0) = m), \quad \forall m \in \mathbb{N}_0,$$

where Ξ_{Ginibre} denotes the Ginibre DPP.

- **If we introduce a series of random variables $X_n^{(r)} \in \{0, 1\}, n \in \mathbb{N}_0$, which are mutually independent and $X_n^{(r)} \sim \mu_{\lambda_n(r)}^{\mathrm{Bernoulli}}$, $n \in \mathbb{N}_0$, then the above implies the equivalence in probability law**

$$\Xi_{\mathrm{Ginibre}}(\mathbb{B}_r^2) \overset{\text{(law)}}{=} \Xi_2(\mathbb{N}_0) \overset{\text{(law)}}{=} \sum_{n\in\mathbb{N}_0} X_n^{(r)}, \quad r \in (0, \infty).$$

5. DPPs on *d*-Dimensional Spheres
5.1 Harmonic Ensembles

- **We consider a unit sphere in $\mathbb{R}^{d+1}$ denoted by $\mathbb{S}^d$, in which we use the polar coordinates for $x = (x^{(1)}, \dots, x^{(d+1)}) \in \mathbb{S}^d$,**

$$
\begin{aligned}
x^{(1)} &= \sin\theta_d \cdots \sin\theta_2 \sin\theta_1, \\
x^{(a)} &= \sin\theta_d \cdots \sin\theta_a \cos\theta_{a-1}, \quad a = 2, \dots, d, \\
x^{(d+1)} &= \cos\theta_d, \quad \textbf{with } \theta_1 \in [0, 2\pi), \quad \theta_a \in [0, \pi], \quad a = 2, \dots, d.
\end{aligned}
$$

- **The standard measure on $\mathbb{S}^d$ is given by the Lebesgue area measure expressed as**

$$
d\sigma_d(x) = \sin^{d-1}\theta_d \sin^{d-2}\theta_{d-1} \cdots \sin\theta_2 d\theta_1 \cdots d\theta_d, \quad x \in \mathbb{S}^d.
$$

The total measure of $\mathbb{S}^d$ is calculated as

$$
\omega_d = \sigma_d(\mathbb{S}^d) = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}.
$$

- **We write the space of harmonic polynomials of degree $k \in \mathbb{N}_0$, $\mathcal{H}_k$, restricted on $\mathbb{S}^d$ as**

$$\mathcal{Y}_{(d,k)} = \left\{ h\Big|_{\mathbb{S}^d} : h \in \mathcal{H}_k \right\}, \quad k \in \mathbb{N}_0.$$

We can see that

$$D(d,k) = \dim \mathcal{Y}_{(d,k)} = \frac{(d+2k-1)(d+k-2)!}{(d-1)!k!} = \frac{2}{(d-1)!}k^{d-1} + o(k^{d-1}).$$

- **Consider an orthonormal basis $\{Y_j^{(d,k)}\}_{j=1}^{D(d,k)}$ of $\mathcal{Y}_{(d,k)}$ with respect to $d\sigma_d$;**

$$\langle Y_n^{(d,k)}, Y_m^{(d,k)} \rangle_{L^2(\mathbb{S}^d, d\sigma_d)} = \int_{\mathbb{S}^d} Y_n^{(d,k)}(x)\overline{Y_m^{(d,k)}(x)}d\sigma_d(x) = \delta_{nm}, \quad n,m \in \mathbb{N}_0.$$

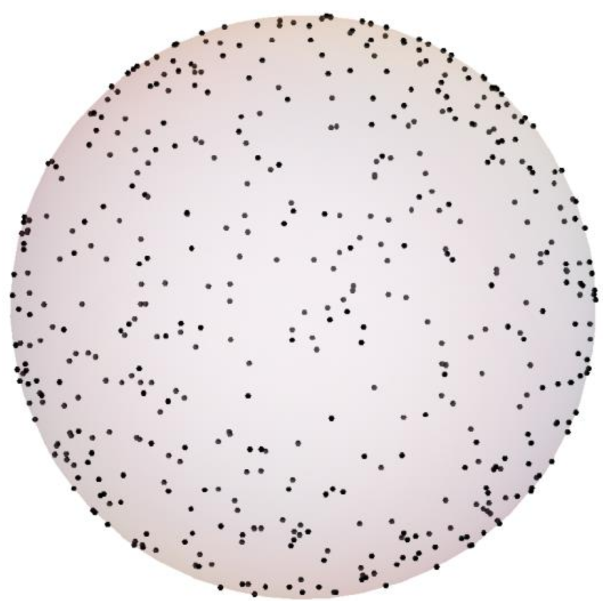
- **Then, if we put** $$K^{\mathcal{Y}_{(d,k)}}(x,x') = \sum_{j=1}^{D(d,k)} Y_j^{(d,k)}(x)\overline{Y_j^{(d,k)}(x')}, \quad x' \in \mathbb{S}^d,$$

then $\{K^{\mathcal{Y}_{(d,k)}}(x,x')\}_{x,x' \in \mathbb{S}^d}$ give the reproducing kernel in $\mathcal{Y}^{(d,k)}$ in the sense that

$$Y(x') = \int_{\mathbb{S}^d} Y(x)\overline{K^{\mathcal{Y}_{(d,k)}}(x,x')}d\sigma_d(x), \quad \forall Y \in \mathcal{Y}_{(d,k)}.$$

- **Fix $d \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then, if we consider the case that $S_1 = \mathbb{S}^d$, $S_2 = \mathbb{N}$ with $\lambda_1(dx) = d\sigma_d(x)$, $L^2(\Gamma, \nu) = \ell^2(\{1, \dots, D(d,k)\}) \subset S_2$, and $\psi_1(x,n) = Y_n^{(d,k)}(x)$.**
- **Then Corollary 3.3 determines a unique DPP on $\mathbb{S}^d$, in which the correlation kernel is given by $K^{\mathcal{Y}_{(d,k)}}(x,x'), x, x' \in \mathbb{S}^d$.**
- **It is obvious that the obtained DPP is rotationally invariant on $\mathbb{S}^d$, since the kernel $K^{\mathcal{Y}_{(d,k)}}(x,x')$ depend only on the inner product $x \cdot x'$. The density of points is uniform on $\mathbb{S}^d$ and is given with respect to $\sigma_d(dx)$ by**

$$\begin{aligned}\rho^{\mathcal{Y}_{(d,k)}} &= K^{\mathcal{Y}_{(d,k)}}(x,x) \\ &= \frac{D(d,k)}{\omega_d} = \frac{2k^{d-1}}{(d-1)!\omega_d} + \mathrm{o}(k^{d-1}).\end{aligned}$$



- Next we consider the DPP on $\mathbb{S}^d$ for fixed $d \in \mathbb{N}$ and $L \in \mathbb{N}$ such that the correlation kernel is given by the following finite sum,

$$K^{(N(d,L))}_{\mathrm{harmonic}(\mathbb{S}^d)}(x, x') = \sum_{k=0}^{L-1} K^{\mathcal{Y}_{(d,k)}}(x, x'),$$

where the total number of points on $\mathbb{S}^d$ is given by

$$N(d, L) = \sum_{k=0}^{L-1} D(d, k) = \frac{2L + d - 2}{d} \binom{d + L - 2}{L - 1} = \frac{2}{d!} L^d + \mathrm{o}(L^d).$$

- The DPP $(\Xi, K^{(N)}_{\mathrm{harmonic}(\mathbb{S}^d)}, d\sigma_d(x))$ is rotationally invariant in $\mathbb{S}^d$ and is called the harmonic ensemble in $\mathbb{S}^d$ with N points by Beltrán *et al.*

  [BMOC16] C. Beltrán, J. Marzo and J. Ortega-Cerdà, Energy and discrepancy of rotationally invariant determinantal point processes in high dimensional spheres, Journal of Complexity 37 (2016) 76–109.

- If we introduce the Jacobi polynomials defined as

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha + 1)_n}{n!} F\left(-n, n + \alpha + \beta + 1; \alpha + 1; \frac{1 - x}{2}\right),$$

the above kernel is written as follows,

$$K^{(N(d,L))}_{\mathrm{harmonic}(\mathbb{S}^d)}(x, x') = \frac{N(d, L)}{\omega_d} \frac{P^{(d/2,(d-2)/2)}_{L-1}(x \cdot x')}{P^{(d/2,(d-2)/2)}_{L-1}(1)}.$$

- **In particular, when $d = 1$, $N(1, L) = 2L - 1$ and**

$$K^{(N(1,L))}_{\text{harmonic}(\mathbb{S}^1)}(x, x')d\sigma_1(x) = \frac{1}{2\pi} F\left(\frac{1-(2L-1)}{2}, \frac{1+(2L-1)}{2}; \frac{3}{2}; \sin^2 \frac{\theta - \theta'}{2}\right) d\theta$$
$$= \frac{\sin\{N(\theta - \theta')/2\}}{\sin\{(\theta - \theta')/2\}} \frac{d\theta}{2\pi}.$$

This verifies the identification of the 1-sphere case of the present DPP with the Circular Unitary Ensemble (CUE) studied in random matrix theory.

- **On the other hand, when $d = 2$, $N(2, L) = L^2$ and**

$$K^{(N(2,L))}_{\text{harmonic}(\mathbb{S}^2)}(x, x') = \frac{L^2}{4\pi} F\left(-L+1, L+1; 2; \frac{1 - x \cdot x'}{2}\right)$$
$$= \frac{N}{4\pi} F\left(-\sqrt{N}+1, \sqrt{N}+1; 2; \frac{||x - x'||^2_{\mathbb{R}^3}}{4}\right).$$

- **This DPP on $\mathbb{S}^2$ is different from the spherical ensemble studied by Caillol (1981) and Krishnapur (2009).**

5.2 Bulk scaling limit

- **Now we consider the vicinity of the north pole** $e_{d+1} = (0, \dots, 0, 1)$ **on** $\mathbb{S}^d$ **and put** $\theta_d = r/L$, $r \in [0, \infty)$. **Then the polar coordinates behave as**

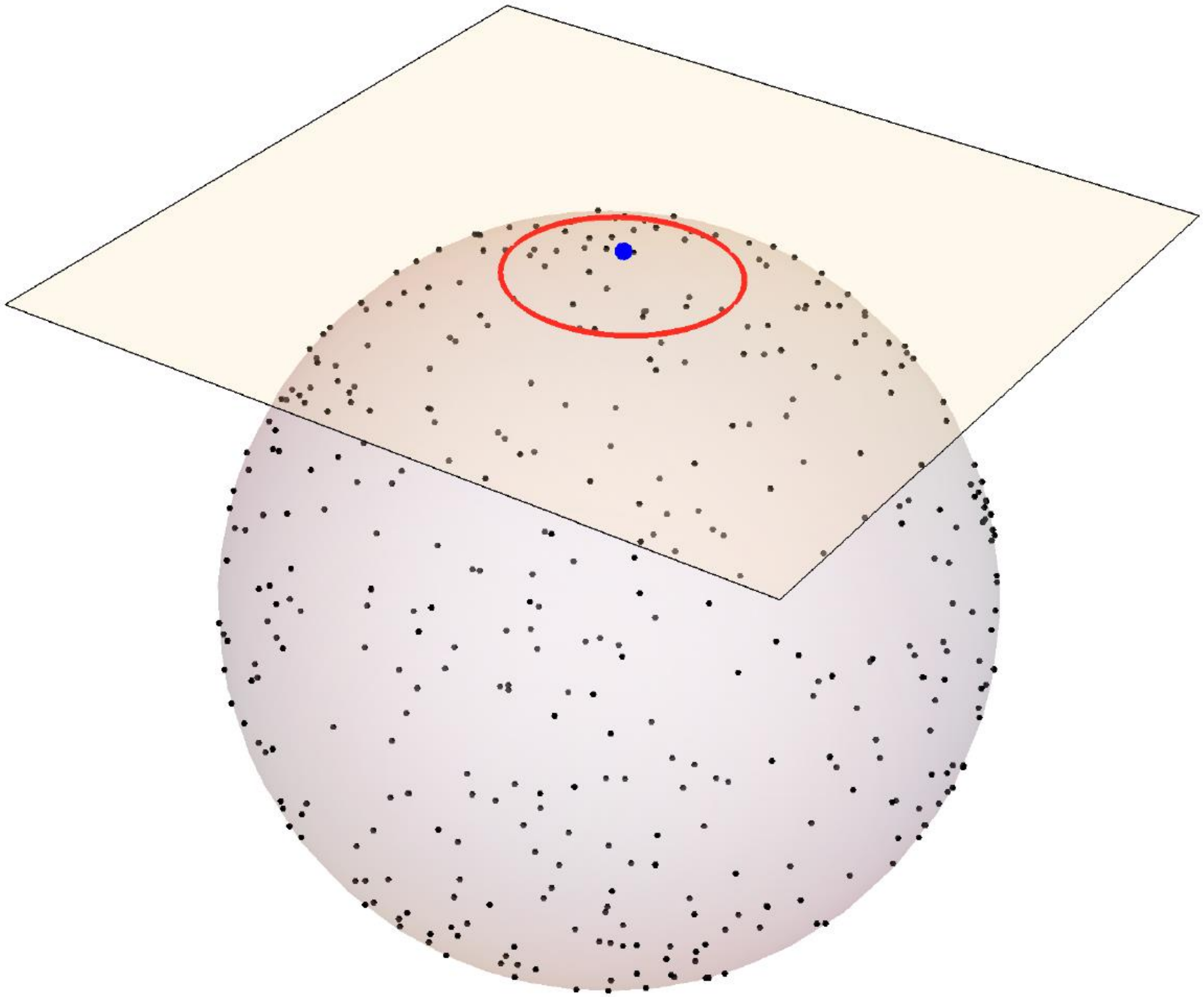
$$
\begin{aligned}
x^{(1)} &\simeq \frac{r}{L} \sin\theta_{d-1} \cdots \sin\theta_2 \sin\theta_1 =: \frac{1}{L}\widetilde{x}^{(1)}, \\
x^{(a)} &\simeq \frac{r}{L} \sin\theta_{d-1} \cdots \sin\theta_k \cos\theta_{a-1} =: \frac{1}{L}\widetilde{x}^{(a)}, \qquad a = 2, \dots, d, \\
x^{(d+1)} &\simeq 1 - \frac{1}{2}\left(\frac{r}{L}\right)^2 .
\end{aligned}
$$

- **In this case, for** $x, x' \in \mathbb{S}^d$, $x \cdot x' = \sum_{a=1}^{d+1} x^{(a)} x'^{(a)} = 1 - \frac{1}{2L^2} ||\widetilde{x} - \widetilde{x}'||^2_{\mathbb{R}^d} + \mathrm{o}\left(\frac{1}{L^2}\right)$, **as** $L \to \infty$, **where** $\widetilde{x}, \widetilde{x}' \in \mathbb{R}^d$ **and** $|| \cdot ||_{\mathbb{R}^d}$ **denotes the Euclidean norm in** $\mathbb{R}^d$. **Hence we can conclude that**

$$
x \cdot x' = \cos\left(\frac{r}{L}\right) + \mathrm{o}\left(\frac{1}{L^2}\right), \quad \textbf{with } r := ||\widetilde{x} - \widetilde{x}'||_{\mathbb{R}^d}, \qquad \textbf{as } L \to \infty.
$$

- **In this limit, the measure on** $\mathbb{S}^d$ **behaves as**

$$
d\sigma_d(x) \simeq \frac{1}{L^d} r^{d-1} \sin^{d-3}\theta_{d-2} \cdots \sin\theta_2 \, dr d\theta_1 \cdots d\theta_{d-1} = \frac{1}{L^d} d\widetilde{x}, \quad \widetilde{x} \in \mathbb{R}^d.
$$



The following limit is proved for the correlation kernel $K_{\mathrm{harmonic}(\mathbb{S}^d)}^{(N(d,L))}$

Lemma 5.1 *When* $x \cdot x' = \cos\left(\frac{r}{L}\right) + \mathrm{o}\left(\frac{1}{L^2}\right)$, *with* $r := ||\widetilde{x} - \widetilde{x}'||_{\mathbb{R}^d}$, *as* $L \to \infty$ *holds, the limit*

$$k^{(d)}(r) = \lim_{L\to\infty} \frac{1}{L^d} K_{\mathrm{harmonic}(\mathbb{S}^d)}^{(N(d,L))}(x, x')$$

exists and have the following expressions,

$$k^{(d)}(r) = \frac{J_{d/2}(r)}{(2\pi r)^{d/2}}, = \frac{1}{(2\pi)^{d/2} r^{(d-2)/2}} \int_0^1 s^{d/2} J_{(d-2)/2}(rs) ds,$$

where $J_\nu(z)$ *is the Bessel function of the first kind with index* ν.

- **This result implies that for each $d \in \mathbb{N}$ we obtain an infinite-dimensional DPP on $\mathbb{R}^d$ such that it is uniform and isotropic on $\mathbb{R}^d$ and the correlation kernel is given by**

$$K^{(d)}(x, x') = k^{(d)}(||x - x'||_{\mathbb{R}^d}), \quad x, x' \in \mathbb{R}^d.$$

- **We can give the following alternative expression for $K^{(d)}$.**

Lemma 5.2 *For $d \in \mathbb{N}$, the correlation kernel $K^{(d)}$ is written as*

$$K^{(d)}(x, x') = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathbf{1}_{\mathbb{B}^d}(y) e^{i(x-x')\cdot y} dy = \frac{1}{(2\pi)^d} \int_{\mathbb{B}^d} e^{i(x-x')\cdot y} dy,$$

where $\mathbb{B}^d$ denotes the unit ball centered at the origin; $\mathbb{B}^d := \{y \in \mathbb{R}^d : |y| \leq 1\}$.

- **This kernel is obtained as the correlation kernel K_{S_1} given by Corollary 3.3, if we consider the case such that**

$$S_1 = S_2 = \mathbb{R}^d, \ \lambda_1(dx) = dx, \ \lambda_2(dy) = \nu(dy) = dy, \ \psi_1(x, y) = e^{ix\cdot y}, \ \Gamma = \mathbb{B}^d \subsetneq \mathbb{R}^d.$$

- **The kernels $K^{(d)}$ on $\mathbb{R}^d, d \geq 1$ derived as the bulk scaling limit of $K^{(N)}_{\mathrm{harmonic}(\mathbb{S}^d)}$ have been studied by Zelditch.**
- **Zelditch regarded them as the Szegö kernels for the reduced Euclidean motion group.**
- **Here we call the DPPs associated with the correlation kernels in this form the Euclidean family of DPPs on $\mathbb{R}^d, d \in \mathbb{N}$.**

Definition 6.1 *The Euclidean family of DPP on $\mathbb{R}^d, d \in \mathbb{N}$ is defined by $\left(\Xi, K^{(d)}_{\mathrm{Euclidean}}, dx\right)$ with the correlation kernel*

$$
\begin{aligned}
K^{(d)}_{\mathrm{Euclid}}(x, x') &= \frac{1}{(2\pi)^{d/2}} \frac{J_{d/2}(||x - x'||_{\mathbb{R}^d})}{||x - x'||^{d/2}_{\mathbb{R}^d}} \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathbf{1}_{\mathbb{B}^d}(y) e^{i(x-x')\cdot y} dy, \quad x, x' \in \mathbb{R}^d.
\end{aligned}
$$

[Zel55] **S. Zelditch, From random polynomials to symplectic geometry, in Proceedings of ICMP 2000,** arXiv:math-ph/0010012

- **We introduce the following operation.**

(Dilatation) **For** $c > 0$**, we set** $c \circ \Xi := \sum_j \delta_{cx_j}$

$$c \circ K(x, x') := K\left(\frac{x}{c}, \frac{x'}{c}\right), \quad x, x' \in cS,$$

and $c \circ \lambda(dx) := \lambda(dx/c)$**. We define** $c \circ (\Xi, K, \lambda(dx)) := (c \circ \Xi, c \circ K, c \circ \lambda(dx))$**.**

The result reported in the previous section is summarized as follows.

Proposition 6.2 *The following is established for $d \in \mathbb{N}$,*

$$\left(\frac{d!}{2}\right)^{1/d} N^{1/d} \circ \left(\Xi, K^{(N)}_{\text{harmonic}(\mathbb{S}^d)}, d\sigma_d(x)\right) \overset{N\to\infty}{\Longrightarrow} \left(\Xi, K^{(d)}_{\text{Euclid}}, dx\right).$$

- **For lower dimensions, the correlation kernels and the densities are given as follows,**

$$K^{(1)}_{\text{Euclid}}(x, x') = \frac{\sin(x - x')}{\pi(x - x')} = K_{\text{sinc}}(x, x') \quad \textbf{with} \quad \rho^{(1)}_{\text{Euclid}} = \frac{1}{\pi},$$

$$K^{(2)}_{\text{Euclid}}(x, x') = \frac{J_1(||x - x'||_{\mathbb{R}^2})}{2\pi||x - x'||_{\mathbb{R}^2}} \quad \textbf{with} \quad \rho^{(2)}_{\text{Euclid}} = \frac{1}{4\pi},$$

$$K^{(3)}_{\text{Euclid}}(x, x') = \frac{1}{2\pi^2||x - x'||^2_{\mathbb{R}^3}} \left(\frac{\sin||x - x'||_{\mathbb{R}^3}}{||x - x'||_{\mathbb{R}^3}} - \cos||x - x'||_{\mathbb{R}^3}\right) \quad \textbf{with} \quad \rho^{(3)}_{\text{Euclid}} = \frac{1}{6\pi^2}.$$

- **This family of DPPs includes the DPP with the sinc kernel K_{sinc} as the lowest dimensional case with $d = 1$.**

- **Note that, if d is odd,**

$$k^{(d)}(r) = \left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{(d-1)/2} \frac{\sin r}{\pi r}.$$

6. Concluding Remarks

- With $L^2(S, \lambda)$ and $L^2(\Gamma, \nu)$, we can consider the system of biorthonormal functions, which consists of a pair of distinct families of measurable functions $\{\psi(x, \gamma) : x \in S, \gamma \in \Gamma\}$ and $\{\varphi(x, \gamma) : x \in S, \gamma \in \Gamma\}$ satisfying the biorthonormality relations

$$\langle \psi(\cdot, \gamma), \varphi(\cdot, \gamma') \rangle_{L^2(S,\lambda)} \nu(d\gamma) = \delta(\gamma - \gamma') d\gamma, \quad \gamma, \gamma' \in \Gamma.$$

- If the integral kernel defined by

$$K^{\mathrm{bi}}(x, x') = \int_\Gamma \psi(x, \gamma) \overline{\varphi(x', \gamma)} \nu(d\gamma), \quad x, x' \in S,$$

  is of finite rank, we can construct a finite DPP on S whose correlation kernel is given by K^{bi} following a standard method of random matrix theory.

- By the above biorthonormality, it is easy to verify that K^{bi} is a projection kernel, but it is not necessarily an orthogonal projection. This observation means that such a DPP is not constructed by the method reported in this talk. Generalization of the present framework in order to cover such DPPs associated with biorthonormal systems is required.

- Moreover, the dynamical extensions of DPPs called determinantal processes shall be studied in the context of the present talk.

Thank you very much for your attention.

- **M. Katori, T. Shirai,**
  Partial isometries, duality, and determinantal point processes,
  arXiv: math.PR/1903.04945.