Stability of heat kernel estimates and parabolic Harnack inequalities for symmetric Dirichlet forms

To the memory of Kazumasa Kuwada.

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Joint work with Z.Q. Chen (Seattle) and J. Wang (Fuzhou)

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1 Introduction

$U \subset \mathbb{R}^d$: Lipschitz domain, $diam(U) = \infty$, $d(\cdot, \cdot)$: inner distance on U. $(\mathcal{E}, W^{1,2}(U))$: Dirichlet form on $L^2(U, dx)$ given by

$$\mathcal{E}(u,v) = \int_U \nabla u(x) \cdot A(x) \nabla v(x)\, dx + \int_U \int_U \frac{(u(x)-u(y))(v(x)-v(y))}{d(x,y)^{d+\alpha}} c(x,y)\, dx\, dy,$$

where $\lambda^{-1} I \leqslant A(x) \leqslant \lambda I$ and $C^{-1} \leqslant c(x,y) = c(y,x) \leqslant C$ (uniformly elliptic).

Diffusion plus jumps.

Question: Investigate the behavior of the corresponding process.

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Mission: i) Detailed short time/long time behavior $\Rightarrow$ through the heat kernel behavior
ii) Regularity of the solution of the heat equation $\Rightarrow$ through Harnack-type inequalities

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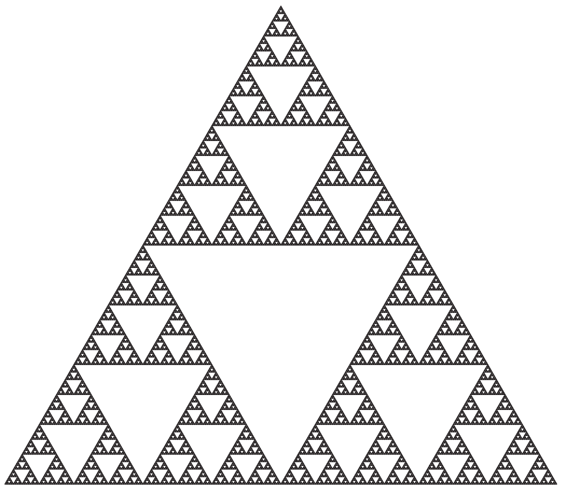
Question: Investigate the behavior of behavior of the corresponding process.

Previous work : Song-Vondracek '07: $\Delta + \Delta^{\alpha/2}$ (i.e. $U = \mathbb{R}^d$, $A(x) \equiv I$, $c(x,y) \equiv 1$), mixture of BM and symmetric α-stable processes: computing the convolution!

Chen-K '10: General diffusions with jumps $U = \mathbb{R}^d$:

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^d} \nabla f(x) \cdot A(x) \nabla f(x)\, dx + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} c(x,y)\, dx\, dy.$$



Our question is more general. Diffusion with jumps on a metric meas. space.

(M, d, μ): Ahlfors d-regular set. Consider a diffusion with jumps whose DF is given by

$$\mathcal{E}(u, v) = \mathcal{E}_{(c)}(u, v) + \int_M \int_M \frac{(u(x) - u(y))(v(x) - v(y))}{d(x, y)^{d+\alpha}} c(x, y)\, \mu(dx)\, \mu(dy),$$

where $\mathcal{E}_{(c)}(\cdot, \cdot)$ is a local regular DF that enjoys sub-Gaussian estimates with the walk dimension $\beta > 2$ and $0 < \alpha < \beta$, (example: Sierpinski gasket).

Question: Investigate the behavior of the corresponding process.

1.1 Framework

Symmetric Dirichlet form

Let (M, d, μ) be a *metric measure space* ($diam(M) = \infty$ for simplicity).

Consider a regular *Dirichlet form* $(\mathcal{E}, \mathcal{F})$ on $L^2(M;\mu)$ as follows:

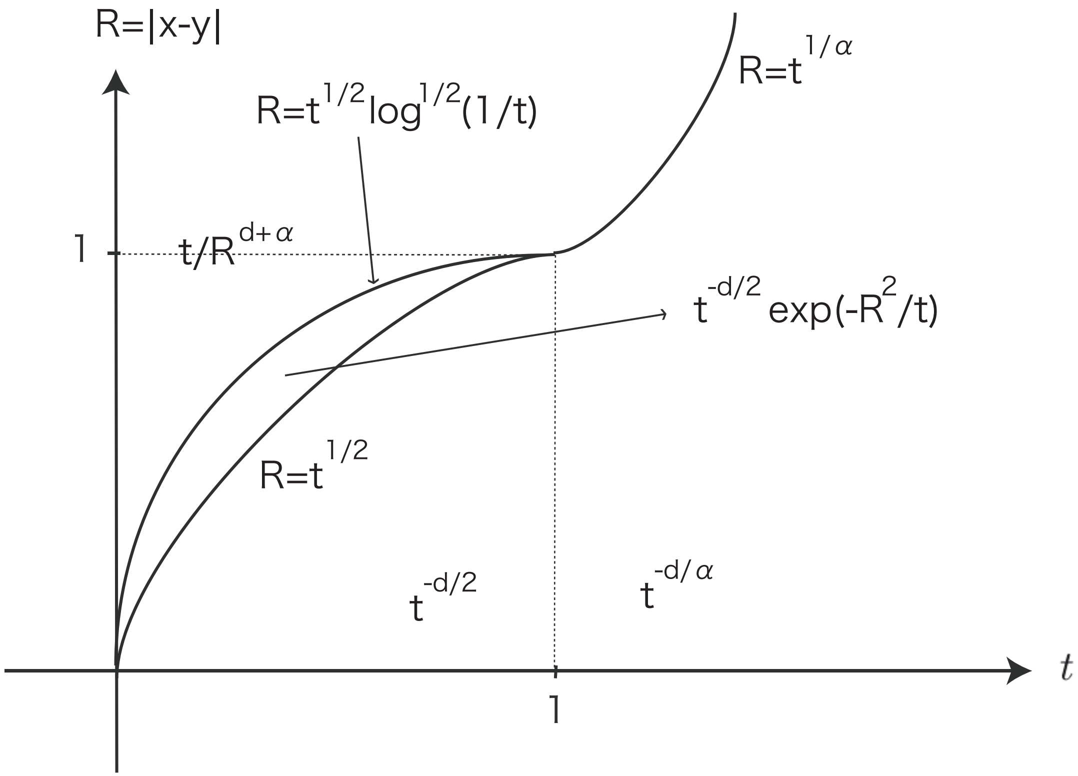
$$\begin{aligned}\mathcal{E}(f,g) =& \mathcal{E}^{(c)}(f,g) + \iint_{M\times M} (f(x)-f(y))(g(x)-g(y))\, J(dx,dy)\\ =:& \, \mathcal{E}^{(c)}(f,g) + \mathcal{E}^{(j)}(f,g),\end{aligned}$$

$(\mathcal{E}^{(c)}, \mathcal{F})$: strongly local part of $(\mathcal{E}, \mathcal{F})$, $J(\cdot,\cdot)$: sym. Radon meas. $M \times M \setminus$ diag.

$$\mathbb{E}^x f(X_t) = P_t f(x) = \int p(t,x,y) f(y)\, \mu(dy), \quad x \in M, f \in L^\infty(M;\mu).$$

1.2 Aim

- Stable characterizations of (upper bounds and) two-sided estimates on heat kernel (HKE) for sym. DFs including both local and non-local terms on MMSs.
- Stable characterizations of parabolic Harnack inequalities (PHI).
- To understand relations between HKE and PHI.



For the case of $\Delta + \Delta^{\alpha/2}$: Convolution!

- Two sided HKE as in the above figure.
- PHI(ϕ) holds with $\phi(r) = r^2 \wedge r^\alpha$, i.e. $\mathrm{PHI}(2)$ for $r \leqslant 1$ and $\mathrm{PHI}(\alpha)$ for $r \geqslant 1$.

In a word, our results say that these are stable under perturbations .

2 Background

Stability of HKE

1. (Local) * Gaussian HKE ($\Leftrightarrow$ VD+PI(2))

Grigor'yan ('91), Saloff-Coste ('92), Sturm ('96), Delmotte ('99).

* Sub-Gaussian HKE $c_1 t^{-\frac{d_f}{d_w}} \exp(-c_2 (d(x,y)^{d_w}/t)^{1/(d_w-1)}$ ($\Leftrightarrow$ VD+PI(d_w)+CSA(d_w))

Barlow-Bass, Barlow-Bass-K, Andres-Barlow, Grigor'yan-Hu-Lau

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2. (Non-local) * $0 < \alpha < 2$ and Ahlfors d-regular: Chen-K('03)

$$p(t,x,y) \asymp \left(t^{-d/\alpha} \wedge \frac{t}{d(x,y)^{d+\alpha}}\right) \Leftrightarrow J(x,y) \asymp \frac{1}{d(x,y)^{d+\alpha}}$$

* General MMS (VD+RVD) Chen-K-Wang (Mem. AMS, to appear)

$HK(\phi_j) \Leftrightarrow J_{\phi_j} + CSJ(\phi_j)$

C.f. Grigor'yan-Hu-Hu ('18); Murugan-Saloff-Coste ('18)

3 Main results

Let (M, d, μ) be a *metric measure space* ($diam(M) = \infty$ for simplicity).

- Consider a regular *Dirichlet form* $(\mathcal{E}, \mathcal{F})$ on $L^2(M; \mu)$ as follows:

$$\begin{aligned}\mathcal{E}(f, g) =& \mathcal{E}^{(c)}(f, g) + \iint_{M\times M} (f(x) - f(y))(g(x) - g(y))\, J(dx, dy) \\ =:& \ \mathcal{E}^{(c)}(f, g) + \mathcal{E}^{(j)}(f, g),\end{aligned}$$

  $(\mathcal{E}^{(c)}, \mathcal{F})$: strongly local part of $(\mathcal{E}, \mathcal{F})$, $J(\cdot, \cdot)$: sym. Radon meas. $M \times M \setminus$ diag.

- Two scaling functions ϕ_c and ϕ_j:

  $\phi_c(r) \leqslant \phi_j(r)$ for all $r \in (0, 1]$, and $\phi_c(r) \geqslant \phi_j(r)$ for all $r \in [1, \infty)$.

- Note: convolution does not make sense on general MMS.

- MMS (M, d, μ). Let $V(x,r) = \mu(B(x,r))$ for all $x \in M$ and $r > 0$.

- VD and RVD

$$c_1\left(\frac{R}{r}\right)^{d_1} \leqslant \frac{V(x,R)}{V(x,r)} \leqslant c_2\left(\frac{R}{r}\right)^{d_2}, \quad x \in M, 0 < r < R. \tag{3.1}$$

- Scaling function ϕ_j: $0 < \exists \beta_{j,1} \leqslant \beta_{j,2}$ s.t.

$$c_3\left(\frac{R}{r}\right)^{\beta_{j,1}} \leqslant \frac{\phi_j(R)}{\phi_j(r)} \leqslant c_4\left(\frac{R}{r}\right)^{\beta_{j,2}}, \quad 0 < r < R. \tag{3.2}$$

- J_{ϕ_j} :

$$J(x,y) \asymp \frac{1}{V(x,d(x,y))\phi_j(d(x,y))}.$$

Heat kernel estimates

- $HK(\phi_c, \phi_j)$:

$$p(t,x,y) \asymp \frac{1}{V(x,\phi_c^{-1}(t))} \wedge \frac{1}{V(x,\phi_j^{-1}(t))} \wedge \left(p^{(c)}(t,x,y) + p^{(j)}(t,x,y)\right),$$

where

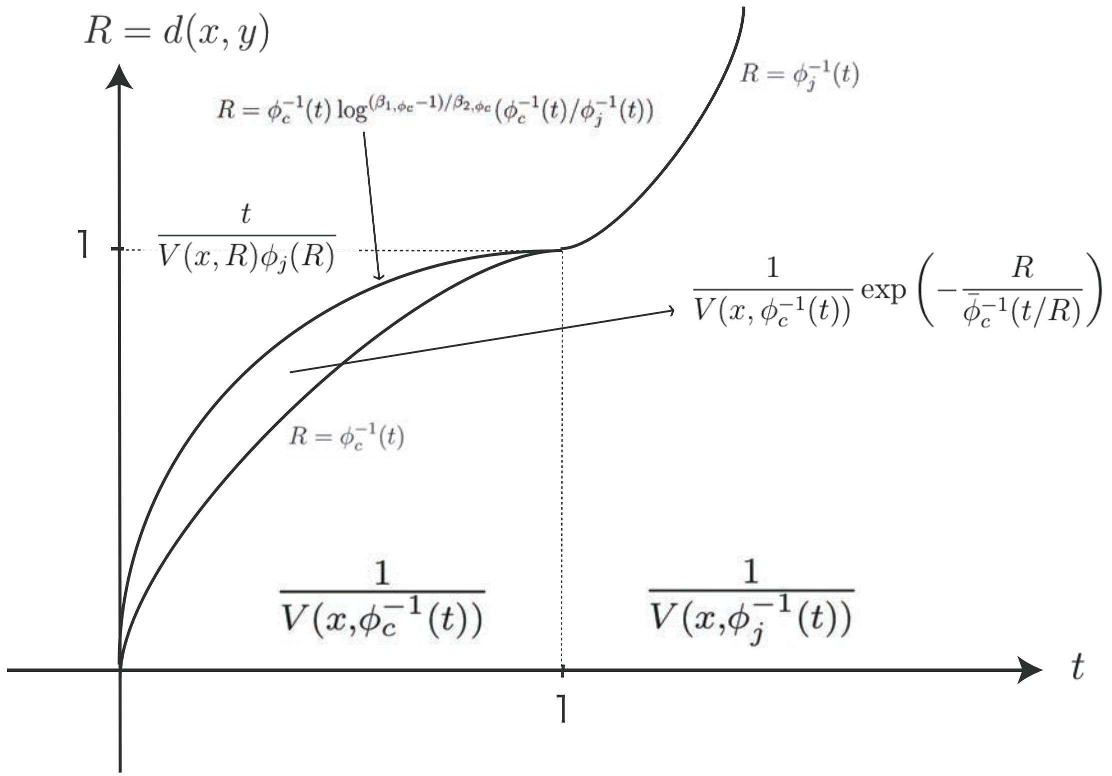
$$p^{(c)}(t,x,y) = \frac{1}{V(x,\phi_c^{-1}(t))} \exp\left(-\sup_{s>0}\left\{\frac{d(x,y)}{s} - \frac{t}{\phi_c(s)}\right\}\right)$$

and

$$p^{(j)}(t,x,y) = \frac{1}{V(x,\phi_j^{-1}(t))} \wedge \frac{t}{V(x,d(x,y))\phi_j(d(x,y))}.$$

- $HK_-(\phi_c, \phi_j)$: the lower bound is replaced by

$$\left(\frac{1}{V(x,\phi_c^{-1}(t))} \wedge \frac{1}{V(x,\phi_j^{-1}(t))} \wedge p^{(j)}(t,x,y)\right).$$



- $HK(\phi_c, \phi_j)$:

$$p(t,x,y) \asymp \frac{1}{V(x,\phi_c^{-1}(t))} \wedge \frac{1}{V(x,\phi_j^{-1}(t))} \wedge \left(p^{(c)}(t,x,y) + p^{(j)}(t,x,y)\right).$$

Main result: heat kernel estimates

Theorem 3.1 (Chen-K-Wang,'19) *Suppose that* (M, d, μ) *satisfies* VD *and* RVD*, and that* ϕ_c, ϕ_j *satisfy* (3.2)*. Then the following are equivalent:*

(1) $HK_-(\phi_c, \phi_j)$.

(2) $PI(\phi)$, J_{ϕ_j} *and* $CS(\phi)$,

where

$$\phi(r) := \phi_c(r) \wedge \phi_j(r) = \begin{cases} \phi_c(r), & r \in (0, 1], \\ \phi_j(r), & r \in [1, \infty). \end{cases}$$

If (M, d, μ) *is connected and geodesic, then they are equivalent to:*

(4) $HK(\phi_c, \phi_j)$.

- $CS(\phi)$: $\exists C_0 \in (0,1], C_1, C_2 > 0$ s.t. $0 < \forall r \leqslant R$, a.e. $x_0 \in M$ and any $f \in \mathcal{F}$,

  $\exists$ a cut-off function $\varphi \in \mathcal{F}_b$ for $B(x_0, R) \subset B(x_0, R + r)$ s.t.

$$\begin{aligned}&\int_{B(x_0,R+(1+C_0)r)} f^2\, d\Gamma(\varphi,\varphi)\\ \leqslant & C_1\left(\int_{B(x_0,R+r)} \varphi^2\, d\Gamma_c(f,f) + \int_{B(x_0,R+r)\times B(x_0,R+(1+C_0)r)} \varphi^2(x)(f(x)-f(y))^2\, J(dx,dy)\right)\\ &+\frac{C_2}{\phi(r)}\int_{B(x_0,R+(1+C_0)r)} f^2\, d\mu.\end{aligned}$$

Remark: Under $J_{\phi_j,\leqslant}$, $CS(\phi)$ always holds if $\beta_{j,2} < 2$ in J_{ϕ_j}.

PHI(ϕ)

Let $Q := (0, 4T) \times B(x_0, 2R)$. For $Q \subset M$, $u(t,x) : M \to \mathbb{R}_+$ is caloric on Q, if

$$\frac{\partial u}{\partial t}(t,x) = \mathcal{L}u(t,x), \qquad \forall t \in Q.$$

We say $PHI(\phi)$ (**parabolic Harnack inequality**) holds, if $\exists C_1 > 0$ s.t. $\forall u = u(t,x)$ caloric and $\geqslant 0$ on M with $T = \phi(R)$, then

$$\sup_{Q_-} u \leqslant C_1 \inf_{Q_+} u.$$

PHR: Important consequence of PHI.

Proposition 3.2 *Assume $PHI(\phi)$. Then, $\forall u$ bounded and caloric in $Q(x_0, \phi(r), r)$,*

$$|u(t', x') - u(t'', x'')| \leqslant C \left(\frac{\phi^{-1}(|t' - t''|) + d(x', x'')}{r} \right)^{\gamma} \sup_{Q(x_0, \phi(r), r)} u$$

for $dt \times \mu$-a.e. $(t', x'), (t'', x'') \in Q(x_0, \delta\phi(r), \delta r)$.

- The De Giorgi-Nash-Moser theory in PDE.

Main result: parabolic Harnack inequalities

Theorem 3.3 (Chen-K-Wang,'19) *Suppose that* (M, d, μ) *satisfies* VD *and* RVD*, and that* ϕ_c, ϕ_j *satisfy* (3.2). *Then*

$$PHI(\phi) \Leftrightarrow PI(\phi) + J_{\phi,\leqslant} + CS(\phi) + UJS \Leftrightarrow PHR(\phi) + J_{\phi,\leqslant} + E_\phi + UJS$$

$$\Leftrightarrow EHR + J_{\phi,\leqslant} + E_\phi + UJS$$

In particular, $\quad HK_-(\phi_c, \phi_j) \Longleftrightarrow PHI(\phi) + J_{\phi_j}.$

- E_ϕ: $\exists c_1 > 1$ s.t. $\forall r > 0$, μ-a.a. $x \in M$,

$$c_1^{-1}\phi(r) \leqslant \mathbb{E}^x[\tau_{B(x,r)}] \leqslant c_1\phi(r),$$

where define the exit time $\tau_A = \inf\{t > 0 : X_t \in A^c\}$ for $A \subset M$.

- UJS (Barlow-Bass-K ('09)): for a.e. $x, y \in M$,

$$J(x, y) \leqslant \frac{c}{V(x, r)} \int_{B(x,r)} J(z, y)\,\mu(dz), \quad r \leqslant \frac{1}{2} d(x, y).$$

Example 3.4 **($PHI(\phi)$ alone does not imply J_{ϕ_j})** *Let $M=\mathbb{R}^d$, and*

$$J(x,y)\asymp\begin{cases}\frac{1}{|x-y|^{d+\alpha}} & |x-y|\leqslant 1;\\ \frac{1}{|x-y|^{d+\beta}} & |x-y|\geqslant 1,\end{cases}$$

where $\alpha,\beta\in(0,2)$. — Diffusion part is just Brownian motion.

Then, $PHI(\phi)$ holds with $\phi(r)=r^\beta\wedge r^2$ for all $r>0$. (Note that ϕ does not depend on the choice of $\alpha\in(0,2)$.) Since $PHI(\phi)$ holds regardless of the choice of $\alpha\in(0,2)$, $PHI(\phi)$ alone does not imply the bound of the jumping kernel.

4 HKE for general symmetric pure jump Dirichlet forms

We also have studied stability of HKE and PHI for general symmetric pure jump Dirichlet forms. Let us illustrate this by an example.

Suppose (M, d, μ) has a nice diffusion having $HK(d_w)$ with $d_w \geqslant 2$.

Let $0 < \alpha < d_w$ and $\gamma > d_w$. Consider a regular DF on $(\mathcal{E}, \mathcal{F})$:

$$\mathcal{E}(u,u) = \int_{M\times M} (u(x) - u(y))^2 J(x,y)\mu(dx)\mu(dy),$$

where

$$J(x,y) \asymp \begin{cases} \frac{1}{V(x,d(x,y))d(x,y)^{\alpha}} & d(x,y) \leqslant 1, \\ \frac{1}{V(x,d(x,y))d(x,y)^{\gamma}} & d(x,y) \geqslant 1. \end{cases}$$

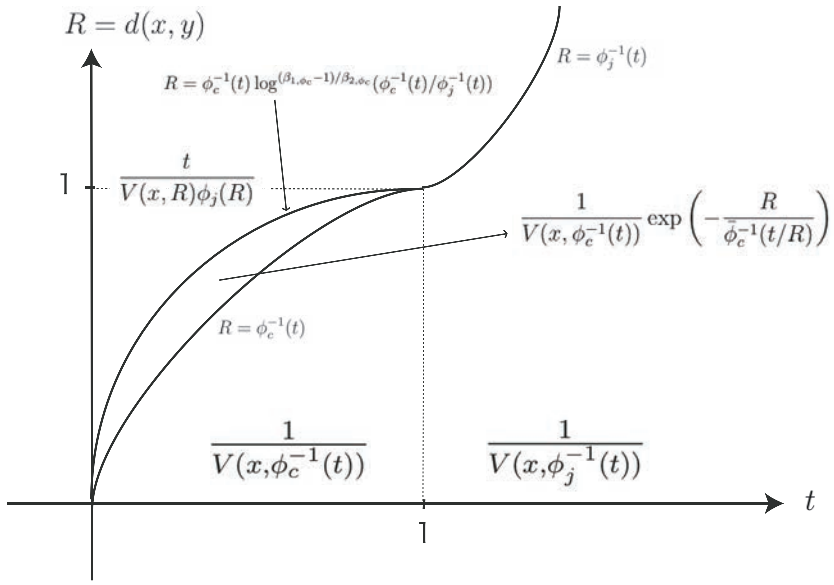
The heat kernel of $(\mathcal{E},\mathcal{F})$ satisfies

$$p(t,x,y) \asymp \begin{cases} p^{(j)}(t,x,y), & t \leqslant 1 \\ \frac{1}{V(x,t^{1/d_w})} \wedge \left(p^{(j)}(t,x,y) + p^{(c)}(t,x,y)\right), & t \geqslant 1, \end{cases}$$

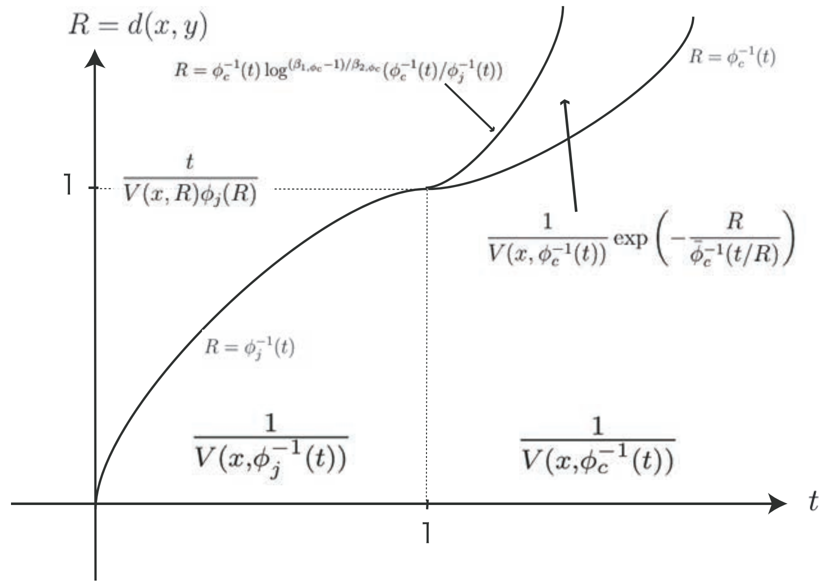
- C.f. Diffusions with jumps:

$$p(t,x,y) \asymp \frac{1}{V(x,\phi_c^{-1}(t))} \wedge \frac{1}{V(x,\phi_j^{-1}(t))} \wedge \left(p^{(c)}(t,x,y) + p^{(j)}(t,x,y)\right).$$

- J. Bae, J. Kang, P. Kim and J. Lee, arXiv:1904.10189



Diffusion+Jumps



Pure jumps with lighter poly. tails

We have more general stability theory for pure jump processes with

$$J(x,y) \asymp \frac{1}{V(x,d(x,y))\phi_j(d(x,y))}.$$

— For example, when $d_w = 2$ and $\phi_j(r) = r^\alpha \vee r^2$ with $\alpha \in (0,2)$, we can take

$$\phi_c(r) := r^2 \mathbf{1}_{\{0 \leqslant r \leqslant 1\}} + \frac{r^2}{\log(1+r)} \mathbf{1}_{\{r>1\}}, \quad \phi(r) = r^\alpha \vee \frac{r^2}{\log(1+r)}.$$

Thank you!