The strong Feller property of reflected Brownian motions on rough planar domains

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Introduction

- **A Markov operator on a topological space E is said to satisfy the strong Feller property if it maps all bounded measurable functions on E into bounded continuous functions.**
- **Under the strong Feller property, measure theoretic properties (of a process) are strengthened to topological ones.**
- **In this talk, we are concerned with the strong Feller property of reflected Brownian motions (RBMs) on general domains.**

Introduction

Let $D \subset \mathbb{R}^d$ be a domain, and m the Leb. measure on D. We define a Dirichlet form $(\mathcal{E}, H^1(D))$ by

$$H^1(D) := \{f \in L^2(D,m) \mid |\nabla f| \in L^2(D,m)\},$$
$$\mathcal{E}(f,g) := \frac{1}{2}\int_D (\nabla f, \nabla g)\, dm, \quad f,g \in H^1(D).$$

If $(\mathcal{E}, H^1(D))$ is regular on $\overline{D}$, it generates a Hunt process $X = (\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\overline{D}})$ on $\overline{D}$.

We call X a RBM on $\overline{D}$.

Question.

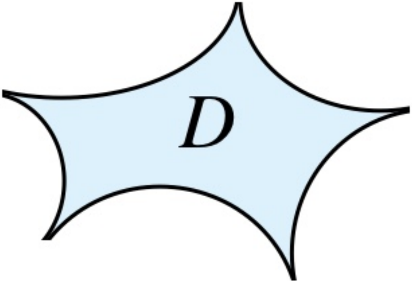
Under what conditions on D (or ∂D), does X have the strong Feller property? (Can we find a version of X with the strong Feller property?)

Known results 1

- Bass and Hsu (1991) consider a RBM X on a bounded Lipschitz domain $D \subset \mathbb{R}^d$. The semigroup $\{P_t^X\}_{t>0}$ of X is strong Feller: $P_t^X(\mathcal{B}_b(\overline{D})) \subset C_b(\overline{D})$ for any $t > 0$.

- Fukushima and Tomisaki (1995, 1996) consider a RBM on a Lipschitz domain $D \subset \mathbb{R}^d$ with cusps. Under a condition on cusps, the resolvent $\{R_\alpha^X\}_{\alpha>0}$ of X satisfies

$$R_\alpha^X(L^1(\overline{D}, m) \cap L^\infty(\overline{D}, m)) \subset C_b(\overline{D}), \quad \alpha > 0.$$



Known results 2

- Gyrya and Saloff-Coste (2011) consider uniform domains.

  (More precisely, they consider RBMs on inner uniform domains. $(\mathcal{E}, H^1(D'))$ on an inner uniform domain D' is not necessarily regular on the topological closure of D'.)

  [Definition of uniform domains (Väisälä)]
  $D \subset \mathbb{R}^d$ is unform domain if there exists $C > 0$ such that for any $x, y \in D$, there is a rectifiable curve γ in D connecting x and y with length$(\gamma) \leq C|x - y|$, and

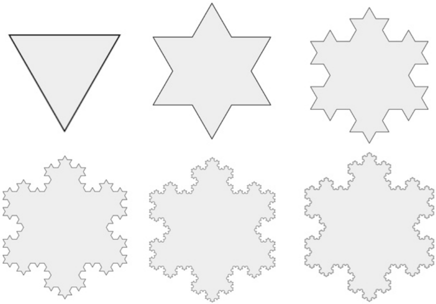
  $$\min\{|x - z|, |z - y|\} \leq C\text{dist}(z, \mathbb{R}^d \setminus D)$$

  for any $z \in \gamma$.

Known results 2

- Gyrya and Saloff-Coste (2011) consider uniform domains.

  (More precisely, they consider RBMs on inner uniform domains. $(\mathcal{E}, H^1(D'))$ on an inner uniform domain D' is not necessarily regular on the topological closure of D'.)
- Bounded Lipschitz domains are uniform domains.
- The interior of the Koch snowflake is an example of uniform domains.



Known results 2

- Gyrya and Saloff-Coste (2011) consider a uniform domain D.

  $\overline{D}$ is regarded as a metric space endowed with the intrinsic distance ρ defined by $(\mathcal{E}, H^1(D))$. They prove **(VD)** and **(PI)** for $(\overline{D}, \rho, \mathcal{E}, m)$.

  As a result, X has a jointly continuous heat kernel $p_t^X(x,y)$. $p_t^X(x,y)$ satisfies the two-sided Gaussian HKE: ${}^\exists c_1, c_2 \in (0,\infty)$ such that

  $$p_t^X(x,y) \asymp c_1 m(B_\rho(x,\sqrt{t}))^{-1} \exp(-c_2 \rho(x,y)^2/t)$$

  for any $t > 0, x, y \in \overline{D}$.
  There exists $\alpha \in (0,1]$ such that the map

  $$\overline{D} \ni x \longmapsto p_t^X(x,y) \in \mathbb{R}$$

  is α-Hölder continuous for any $y \in \overline{D}$ and $t > 0$.

Summary of main results

- We prove the semigroup strong Feller property for RBMs on a class of bounded planar domains.

- The class consists of Jordan domains which are images of the unit disk $\mathbb{D}$ under Hölder continuous conformal maps.

  The class is a subclass of Hölder domains

  The class $\ni$ a non-inner uniform domain

  The class $\supset$ {bdd simply connected planar uniform domains}

  $\ni$ the interior of the Koch snowflake.

- On the bdd simply cnnctd planar uniform domains, the HKs of RBMs are Hölder continuous.
  In this case, we give lower bounds for the Hölder exponents by using a geometric quantity.

Conformal invariance of planar RBM

Let $D \subset \mathbb{R}^2 \cong \mathbb{C}$ be a Jordan domain.

There exists a conformal map $\phi : \mathbb{D} \to D$, which is extended to a homeo. $\bar{\mathbb{D}} \to \overline{D}$.

Let $Y = (\{Y_t\}_{t\geq 0}, \{P_y^Y\}_{y \in \bar{\mathbb{D}}})$ be a RBM on $\bar{\mathbb{D}}$.
Define $X = (\{X_t\}_{t\geq 0}, \{P_x^X\}_{x \in \overline{D}})$ by

$$P_x^X := P_{\phi^{-1}(x)}^Y, \quad x \in \overline{D},$$
$$X_t := \phi(Y_{A_t^{-1}}), \quad t \geq 0,$$

where

$$A_t := \int_0^t |\phi'(Y_s)|^2 \mathbf{1}_{\mathbb{D}}(Y_s)\, ds \nearrow \infty \text{ as } t \to \infty.$$

Then, the Dirichlet form of X is identified with $(\mathcal{E}, H^1(D))$ and is regular on $\overline{D}$.

Denote by $\{R_\alpha^X\}_{\alpha>0}$ the resolvent of X. Then, we have

Main results

Theorem 1. (M.)

Suppose that $\phi : \mathbb{D} \to D$ is κ-Hölder continuous.
Then, ${}^\forall\alpha > 0$, ${}^\forall\varepsilon \in (0, \kappa)$, ${}^\exists C = C_{\alpha,\varepsilon,\kappa} > 0$ s.t.

$$
\begin{aligned}
&|R^X_\alpha f(x) - R^X_\alpha f(y)| \\
&\quad \leq C\|f\|_\infty |\phi^{-1}(x) - \phi^{-1}(y)|^{\{(1-\varepsilon)(\kappa-\varepsilon)\}\wedge(1/2)}
\end{aligned}
$$

for ${}^\forall x, y \in \overline{D}$ and ${}^\forall f \in \mathcal{B}_b(\overline{D})$.

If $D \subset \mathbb{R}^2$ is a bdd simply cnnctd uniform domain, it is known that $\phi^{-1} : \bar{\mathbb{D}} \to \overline{D}$ is also Hölder continuous.

If ϕ^{-1} is λ-Hölder continuous, $R^X_\alpha f$ is

$$\lambda \times [\{(1-\varepsilon)(\kappa-\varepsilon)\} \wedge (1/2)]\text{-Hölder continuous.}$$

$\lambda \times [\{(1-\varepsilon)(\kappa-\varepsilon)\} \wedge (1/2)] \fallingdotseq \lambda \times (\kappa \wedge (1/2))$.

Main results

If $D \subset \mathbb{R}^2$ is a bdd simply connected uniform domain, X has a continuous heat kernel $p_t^X(x,y)$. Moreover,

$$\sup_{x,y\in\overline{D}} p_t^X(x,y) < \infty \quad \text{for any } t > 0.$$

Using a result of Bass–Kassmann–Kumagai (2010), we have

Corollary. (M.)

Let D be a bdd simply cnnctd planar uniform domain. (Assume that $\phi : \overline{\mathbb{D}} \to \overline{D}$ is κ-Hölder conti. and $\phi^{-1} : \overline{D} \to \overline{\mathbb{D}}$ is λ-Hölder conti.) Then, ${}^\forall\varepsilon \in (0,\kappa), {}^\forall x \in \overline{D}$ and ${}^\forall t > 0$,

$$\overline{D} \ni y \longmapsto p_t^X(x,y)$$

is $\lambda \times [\{(1-\varepsilon)(\kappa-\varepsilon)\} \wedge (1/2)]$-Hölder continuous.

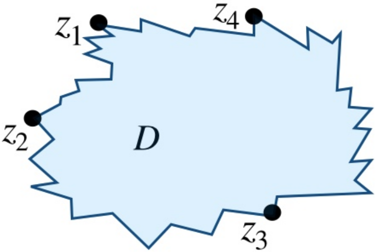
Estimates for κ and λ

Theorem. (Näkki–Palka (1980))

$\lambda > 1/2$ and $\kappa \geq (2\arcsin^2 k)/(\pi(\pi - \arcsin k))$.

$$k = \inf \frac{|z_1 - z_3||z_2 - z_4|}{|z_1 - z_2||z_3 - z_4| + |z_1 - z_4||z_2 - z_3|} \in (0, 1],$$

where the infimum is extended over the quadruples z_1, z_2, z_3, z_4 of finite points of Jordan arc ∂D with the property that z_1 and z_3 separate z_2 and z_4.



For the Koch snowflake, the value k is......?

Outline of proof (mirror couplings of RBMs)

- **Atar and Burdzy (2004) construct mirror couplings of RBMs on a class of Euclidean domains.**

- **Let ν be the inward unit normal vector on $\partial\mathbb{D}$. The mirror coupling of RBMs (Y, Z) on $\bar{\mathbb{D}}$ is described as**

$$Y_t = y + B_t + \int_0^t \nu(Y_s)\, dL_s^Y,$$
$$Z_t = z + W_t + \int_0^t \nu(Z_s)\, dL_s^Z,$$
$$W_t = B_t - 2\int_0^t \frac{Y_t - Z_t}{|Y_t - Z_t|^2}(Y_s - Z_s, dB_s).$$
$$t < T_{\mathsf{cpl}} := \inf\{t > 0 \mid X_t = Y_t\},$$

- **W_t is the mirror image of the Brownian motion B_t w.r.t. the hyperplane between Y_t and Z_t.**

Outline of proof

- Recall $X = (\{X_t\}_{t\ge 0}, \{P_x^X\}_{x\in\overline{D}})$ is described as

$$P_x^X := P_{\phi^{-1}(x)}^Y, \quad x \in \overline{D},$$
$$X_t := \phi(Y_{A_t^{-1}}), \quad t \ge 0,$$

  where Y is a RBM on $\bar{\mathbb{D}}$, and $A_t = \int_0^t |\phi'(Y_s)|^2\,ds$.

- Using the mirror coupling of RBMs (Y, Z), we have

$$\begin{aligned}
&|R_\alpha^X f(\phi(y)) - R_\alpha^X f(\phi(z))| \\
&\le 2E_{yz}\left[\left(\alpha\int_0^{T_{\mathsf{cpl}}} |\phi'(Y_s)|^2\,ds\right)\wedge 1\right] \\
&\quad + 2E_{yz}\left[\left(\alpha\int_0^{T_{\mathsf{cpl}}} |\phi'(Z_s)|^2\,ds\right)\wedge 1\right], \quad y, z \in \bar{\mathbb{D}}.
\end{aligned}$$

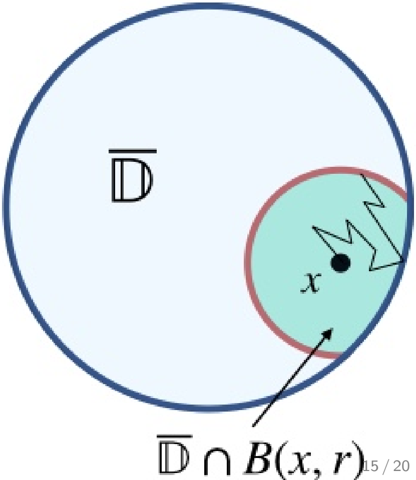
Outline of proof

$\phi : \bar{\mathbb{D}} \to \overline{D}$ is κ-Höl. continuous.

It can be shown that

- $P_{x,y}(T_{\mathsf{cpl}} > t)$
  $\leq \int_0^{|x-y|/2} \frac{2e^{-u^2}}{(2\pi t)^{1/2}}\, du,$
- $P_x^Y(\tau_{\bar{\mathbb{D}}\cap B(x,r)} \leq t)$
  $\lesssim \exp(-r^2/128t),\ r \geq 0,\ t > 0,$
- $E_x^Y\left[\int_0^{\tau_{\bar{\mathbb{D}}\cap B(x,r)}} |\phi'(Y_s)|^2\, ds\right]$
  $\lesssim -r^{2\kappa}\log r, \quad r \in (0, 1/32]$
  $\leq (1/\varepsilon) \times r^{2\kappa-\varepsilon}, \quad \varepsilon \in (0, 2\kappa).$

$\tau_{\bar{\mathbb{D}}\cap B(x,r)}$
$= \inf\{t > 0 \mid Y_t \notin \bar{\mathbb{D}} \cap B(x,r)\}.$



Remark

Theorem 1. (M.)

Suppose that $\phi : \mathbb{D} \to D$ is κ-Hölder continuous.
Then, $^{\forall}\alpha > 0$, $^{\forall}\varepsilon \in (0, \kappa)$, $^{\exists}C = C_{\alpha,\varepsilon,\kappa} > 0$ s.t.

$$|R_\alpha^X f(x) - R_\alpha^X f(y)| \\ \leq C\|f\|_\infty |\phi^{-1}(x) - \phi^{-1}(y)|^{\{(1-\varepsilon)(\kappa-\varepsilon)\}\wedge(1/2)}$$

for $^{\forall}x, y \in \overline{D}$ and $^{\forall}f \in \mathcal{B}_b(\overline{D})$. In particular, the resolvent of X is strong Feller since $\phi : \overline{\mathbb{D}} \to \overline{D}$ is a homeo.

- The semigroup P_t^X of X is strong Feller?
- If P_t^X is ultracontractive, there is no problem. In the setting of Thm 1, it is hard to verify that P_t^X has a ultracontractivity.
- There exists a non-inner uniform Jordan domain $D \subset \mathbb{R}^2$ satysfying the condition in Thm 1. The domain is also not a $W^{1,2}$-extension domain.

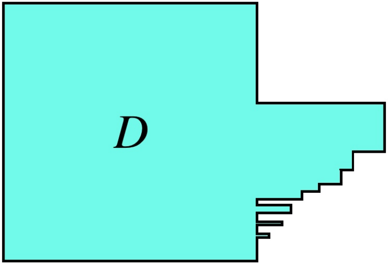
Example by Becker and Pommerenke (1982)

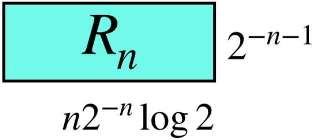
Define a Jordan domain D (which is not inner uniform) by

$$D = \{(u,v) \in \mathbb{R}^2 \mid |u| < 1,\ |v| < 1\} \cup \bigcup_{n=1}^{\infty} R_n,$$

$$R_n = \{(u,v) \in \mathbb{R}^2 \mid 0 \le u - 1 \le \frac{n \log 2}{2^n}, |v - (1/n)| \le 2^{-n}\}.$$

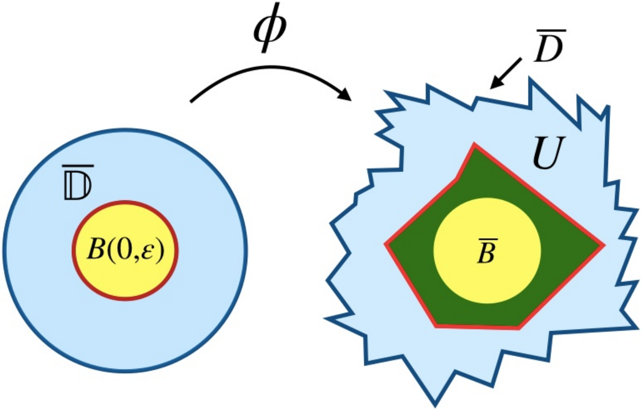
For $n \ge 5$, $R_n \cap R_{n+1} = \emptyset$.





A Refinement of Theorem 1.

- D is a domain with the condition in Theorem 1.
- Let U be an open subset of $\overline{D}$ such that $U \subset \overline{D} \setminus \overline{B}$, where $\overline{B} \subset D$ is a closed disk such that $\underline{\phi(B(0,\varepsilon)) \subset \overline{B}}$
  $\mathcal{L}_U$: the Laplacian on U with the Dirichlet bdry. cond. on red line and the Neumann bdry. cond. on blue line.



A Refinement of Theorem 1.

- $G_{\bar{\mathbb{D}}\setminus\phi^{-1}(\overline{B})}(x,y) \leq 2\log(1+\varepsilon^{-1}) - 2\log|x-y|$
  (Burdzy–Chen–Marshall (2006)).
- $\mathcal{L}_U$ has discrete spectrum, and ${}^{\exists}C_1, C_2 \in (0,\infty)$ s.t.

  (the first eigenvalue of $-\mathcal{L}_U$) $\geq \dfrac{C_1\{2\log(1+\varepsilon^{-1})+C_2\}^{-1}}{m(U)\log\left(2+m(U)^{-1}\right)}$

  for any $\varepsilon \in (0,1)$ and any closed disk $\overline{B} \subset D$ such that $\varphi(B(\varepsilon)) \subset B$, and any open subset U of $\overline{D}$ such that $U \subset \overline{D} \setminus B$

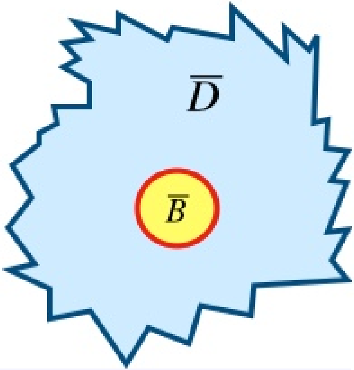
Lemma.

The semigroup of the part process $X^{\overline{D}\setminus\overline{B}}$ of X on $\overline{D} \setminus \overline{B}$ has a ultracontractivity.

A Refinement of Theorem 1.

Denote by $\{P_t^{\overline{D}\setminus\overline{B}}\}_{t>0}$ the semigroup of the part process $X^{\overline{D}\setminus\overline{B}}$ of X on $\overline{D}\setminus\overline{B}$.

- $X^{\overline{D}\setminus\overline{B}}$ is smgrp strong Feller.
- $\lim_{x\to z\in\partial B} P_t^{\overline{D}\setminus\overline{B}} f(x) = 0$ for any $t>0$ and any $f\in\mathcal{B}_b(\overline{D}\setminus\overline{B})$.



By shrinking the radius of $\overline{B}$, we have

Theorem 2. (M.)

Suppose that $\phi:\mathbb{D}\to D$ is Hölder continuous.
Then, the semigroup $\{P_t^X\}_{t>0}$ of X is strong Feller: $P_t^X(\mathcal{B}_b(\overline{D}))\subset C_b(\overline{D})$.