Loewner chains and evolution families on parallel slit half-planes

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Kyoto University (JSPS Research Fellow)

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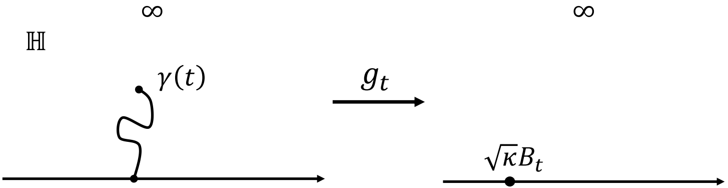
Usually, SLE is defined via the ODE

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \xi(t)}, \quad g_0(z) = z.$$

- $\mathbb{H} \xrightarrow{\text{replace}} D = \mathbb{H} \setminus \bigcup_{j=1}^{N} C_j$ (parallel slit half-plane)
- $\gamma(0, t] \xrightarrow{\text{replace}} F_t$ (unbounded $\mathbb{H}$-hull)



For the conformal mapping $g_t \colon D \setminus F_t \to D_t$ $(t \geq 0)$,

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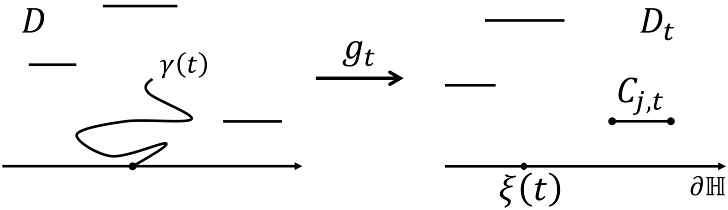

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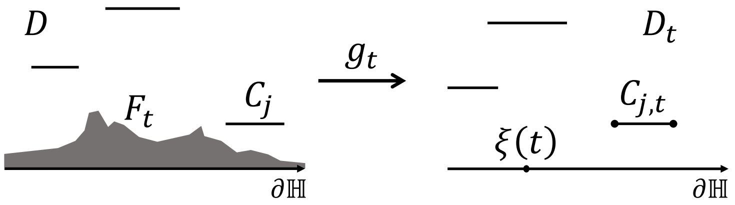

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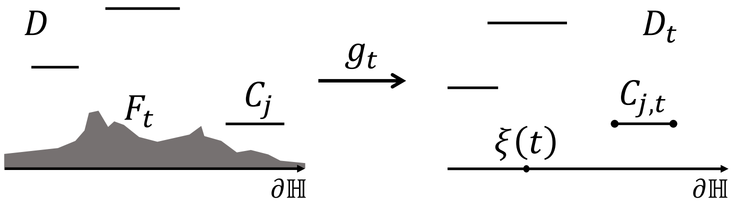

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Plan of my talk

1. **Motivations from probability theory**
   - Multiple SLE and its “hydrodynamic limit”
   - DLA and related models
2. **Motivations from complex analysis**
   - Classical Loewner theory
3. **Extension to multiply connected domains**
   - Komatu–Loewner equation and Brownian motion with darning
   - My results and proof
4. **Concluding remarks**

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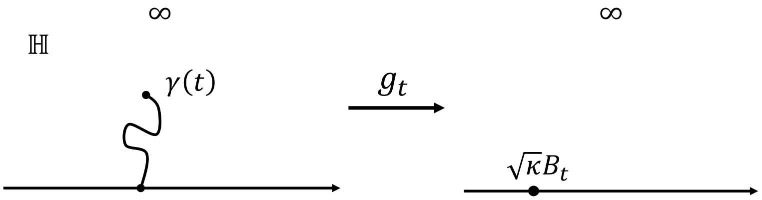
- (Chordal) $\mathbf{SLE}_\kappa$: $\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}$

- Multiple(-paths) $\mathbf{SLE}_\kappa$: $\frac{dg_{n,t}(z)}{dt} = \frac{1}{n}\sum_{k=1}^{n} \frac{2}{g_{n,t}(z) - V_k(t)}$

$$dV_k(t) = \sqrt{\frac{\kappa}{n}}\, dB_k(t) + \frac{1}{n}\sum_{l \neq k} \frac{4}{V_k(t) - V_l(t)}\, dt$$

$g_{n,t}(z)$ defines the conf. map. $g_{n,t}\colon \mathbb{H} \setminus \bigcup_{k=1}^{n} \gamma_k(0,t] \to \mathbb{H}$.
[e.g. Bauer, Bernard & Kytölä ('05), Kozdron & Lawler ('07)]



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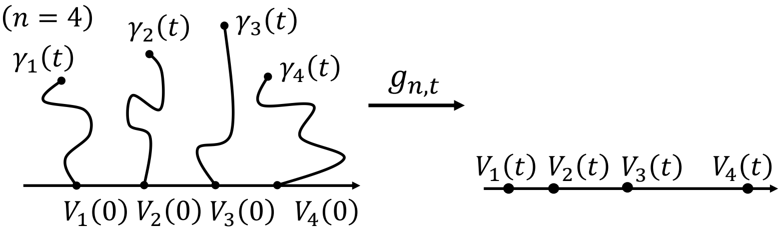

Hydrodynamic limit of multiple SLE

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- $(\, V_k(t)\ ;\ k = 1, \ldots, n,\ t \geq 0\,)$ is a linear time-change of **Dyson's Brownian motion**.
- $\mu_{n,t} := \frac{1}{n} \sum_{k=1}^{n} \delta_{V_k(t)}$ (empirical measure, configuration on $\mathbb{R}$)

Hydrodynamic limit of Dyson (e.g. Rogers & Shi (1992))

Let **initial configuration** satisfy $\mu_{n,0} \xrightarrow{\text{weakly}} \mu_0$ with assumptions.

$$(\, \mu_{n,t}\ ;\ t \geq 0\,) \xrightarrow{\text{law}} (\, \mu_t\ ;\ t \geq 0\,) \quad \text{(deterministic)}$$

as $\mathrm{Prob}(\mathbb{R})$-valued stochastic processes.

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$$\frac{dg_{n,t}(z)}{dt} = \frac{1}{n}\sum_{j=1}^{n} \frac{2}{g_{n,t}(z) - V_j(t)} = \int_{\mathbb{R}} \frac{2}{g_{n,t}(z) - u}\, \mu_{n,t}(du)$$

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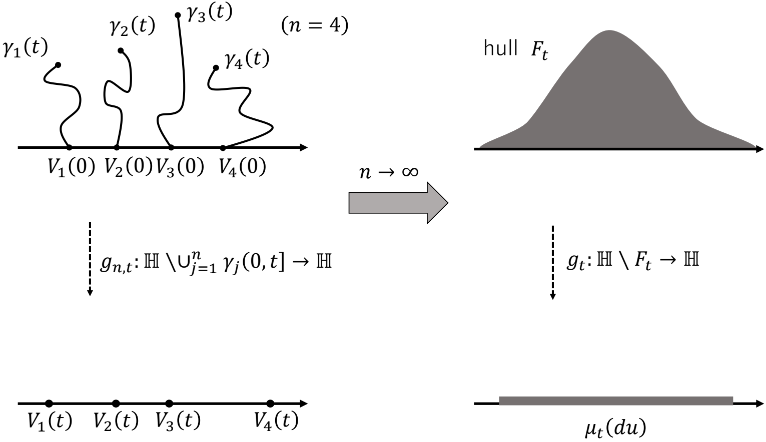
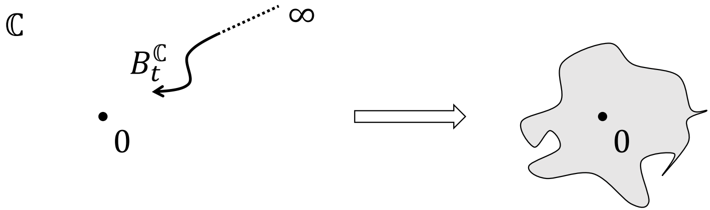


(**del Monaco & Schleissinger (2016)**, Hotta & Katori (2018))

Diffusion Limited Aggregation (DLA)



Question

Is DLA conformally invariant?

Conformally invariant versions:

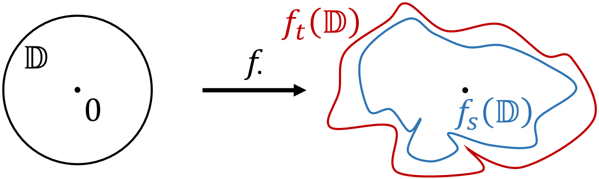
- **Hastings–Levitov clusters** (e.g. Johansson Viklund, Sola & Turner (2012))
  Iteration of $f : \overline{\mathbb{D}}^c \to \overline{\mathbb{D}}^c \setminus P$ and rotations
- **Quantum Loewner evolution** (Miller & Sheffield (2016))

Radial case by Pommerenke (1975)

$(f_t)_{t\geq 0}$ is called a **Loewner chain** if

- $\forall t,\ e^{-t} f_t \in \mathcal{S}_{\mathbb{D}}$
- $0 \leq s < t \Rightarrow f_s(\mathbb{D}) \subset f_t(\mathbb{D})$

$\mathcal{S}_{\mathbb{D}} = \{\, f : \mathbb{D} \to \mathbb{C}\ ;$
$f(0) = 0,\ f'(0) = 1,$
f is univalent $\}$



$$\frac{\partial f_t(z)}{\partial t} = z f_t'(z) \int_{\mathbb{R}} \frac{1 + z e^{-i\xi}}{1 - z e^{-i\xi}}\, \nu_t(d\xi), \quad \text{a.e. } t \in [0, \infty).$$

The **evolution family** $\phi_{t,s} := f_t^{-1} \circ f_s \in \mathrm{Hol}(\mathbb{D})$ of this chain plays an important role in the proof.

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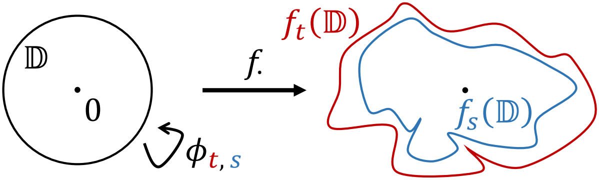

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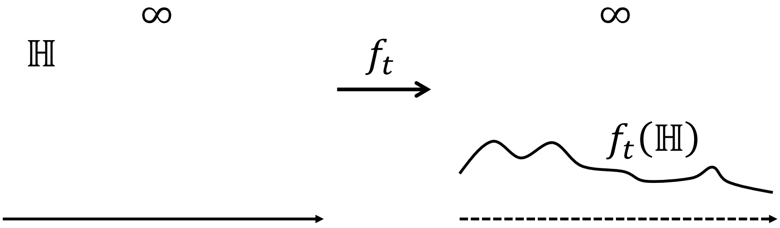
Chordal case (boundary fixed point)

Let $\mathcal{S}_{\mathbb{H}}$ be the set of all univalent $f\colon \mathbb{H} \to \mathbb{H}$ s.t.

- $\forall \eta > 0$, $\lim_{z\to\infty,\ \Im z>\eta} (f(z) - z) = 0$;
- $\exists c_f \in [0, \infty)$ s.t. $\forall \theta \in (0, \pi/2)$,
  $$\lim_{z\to\infty,\ \theta<\arg z<\pi-\theta} z(f(z) - z) = -c_f.$$

Remark. c_f is the "half-plane capacity" of the hull $\mathbb{H} \setminus f(\mathbb{H})$.



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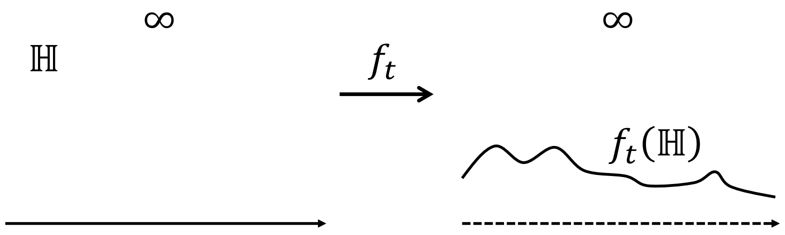

Pick representation

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Key lemma

$f \in \mathcal{S}_{\mathbb{H}}$ if and only if it is univalent on $\mathbb{H}$ and

$$f(z) = z - \int_{\mathbb{R}} \frac{1}{z-\xi}\,\mu(f; d\xi)$$

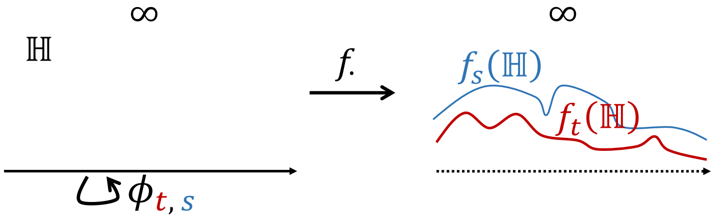
for a finite measure $\mu(f;\cdot)$. In this case,

$$\mu(f; d\xi) = \pi^{-1}\Im f(\xi + i0+)\,d\xi$$
$$\mu(f;\mathbb{R}) = c_f$$

Chordal case by Goryainov & Ba (1992)

Assume that $(f_t)_{0\leq t\leq T} \subset \mathcal{S}_{\mathbb{H}}$ satisfies

- $\mu(f_t;\mathbb{R}) = 2(T-t)$
- $0 \leq s < t \leq T \Rightarrow f_s(\mathbb{H}) \subset f_t(\mathbb{H})$



Theorem

For each fixed s, the mapping $\phi_{t,s} := f_t^{-1} \circ f_s$ satisfies

$$\frac{d\phi_{t,s}(z)}{dt} = -\int_{\mathbb{R}} \frac{2}{\phi_{t,s}(z)-\xi}\, \nu_t(d\xi), \quad \text{a.e. } t \in [s,T).$$

Very rough sketch of the proof

Note that $\phi_{t,s} \in \mathcal{S}_{\mathbb{H}}$. For $s \le u < v$, we have

$$\begin{aligned}\phi_{v,s}(z) - \phi_{u,s}(z) &= \phi_{v,u}(\phi_{u,s}(z)) - \phi_{u,s}(z) \\ &= -\int_{\mathbb{R}} \frac{1}{\phi_{u,s}(z) - \xi}\, \mu(\phi_{v,u}; d\xi) \\ &= -2(v-u) \int_{\mathbb{R}} \frac{1}{\phi_{u,s}(z) - \xi}\, \frac{\mu(\phi_{v,u}; d\xi)}{2(v-u)}.\end{aligned}$$

Let $u, v \to t$. Then

$$\frac{\mu(\phi_{v,u}; d\xi)}{2(v-u)} = \frac{\mu(\phi_{v,u}; d\xi)}{\mu(\phi_{v,u}; \mathbb{R})} \xrightarrow{\text{vaguely}} \exists \nu_t(d\xi).$$

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$$\phi_{v,s} = \phi_{v,u} \circ \phi_{u,s}, \quad \phi_{v,u}(z) = z - \int_{\mathbb{R}} \frac{1}{z - \xi}\, \mu(\phi_{v,u}; d\xi)$$

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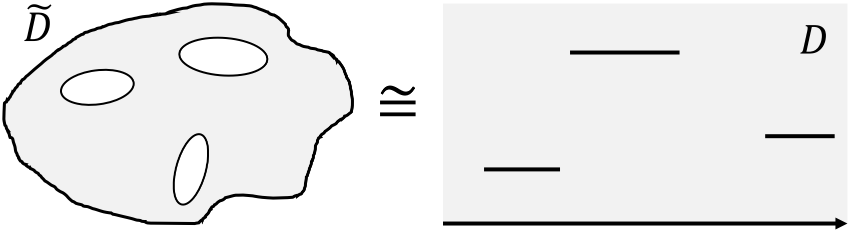
Uniformization theorem

Any **finitely (multiply) connected domain** $\tilde{D}$ is conformally equivalent to a **parallel slit half-plane** $D = \mathbb{H} \setminus \bigcup_{j=1}^{N} C_j$.



Question

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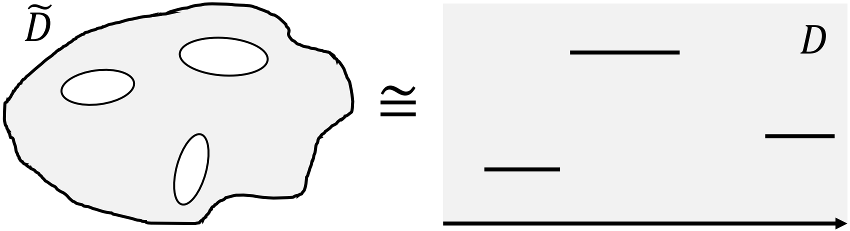

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Komatu–Loewner equation

- **Single-slit mapping case**: Komatu ('50), Bauer & Friedrich ('04, '06, '08), Lawler ('06), Drenning ('11), Chen, Fukushima & Rohde ('16), Chen & Fukushima ('18)

$$\frac{dg_t(z)}{dt} = -2\pi\Psi_{D_t}(g_t(z), \xi(t)), \quad D_t = g_t(D)$$

- **Multi-slits case**: Böhm & Lauf ('14), Böhm ('15)

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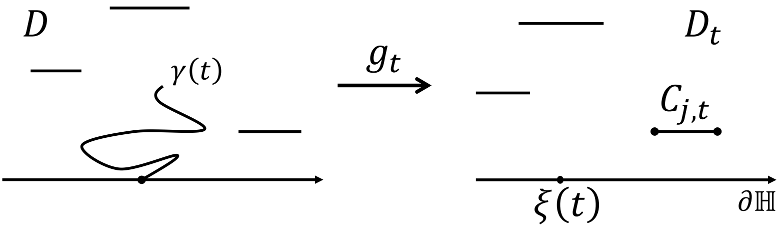

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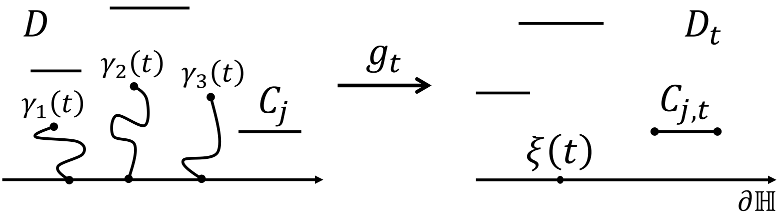


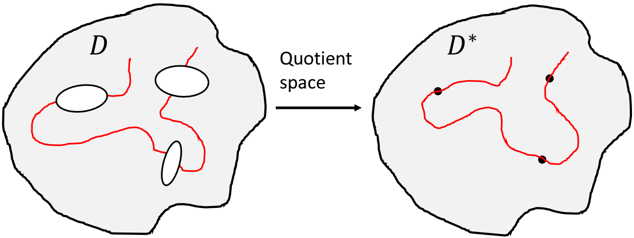
Brownian motion with darning (BMD)

- $D = \mathbb{H} \setminus \bigcup_{j=1}^{N} C_j$
- $D^* = D \cup \{c_1^*, \ldots, c_N^*\}$, $m =$ Lebesgue measure on D^*

Definition (e.g. Chen, Fukushima & Rohde ('16))

BMD is an m-symmetric **diffusion** on D^* such that

- its killed process in D is the **absorbing BM** in D;
- it admits **no killings** on $\{c_1^*, \ldots, c_N^*\}$.



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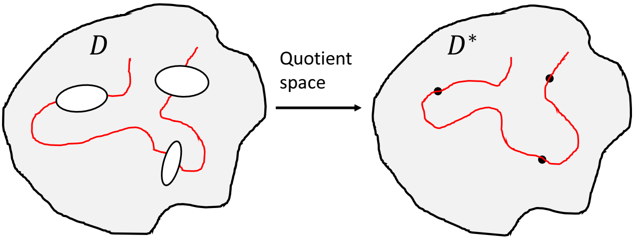

(Construction: Dirichlet form theory)

$$\frac{dg_t(z)}{dt} = -2\pi \Psi_{D_t}(g_t(z), \xi(t)), \quad D_t = g_t(D)$$

$\Psi_{D_t}(z, \xi)$ is defined by the following properties:

- $z \mapsto \Psi_{D_t}(z, \xi)$ is holomorphic;
- $\lim_{z \to \infty} \Psi_{D_t}(z, \xi) = 0$;

and

$$\Im \Psi_{D_t}(z, \xi) = K^*(z, \xi) := \text{(BMD } \textbf{Poisson kernel}\text{)}$$
$$= -\frac{1}{2} \frac{\partial}{\partial \vec{n}_\xi} \text{(BMD } \textbf{Green function}\text{)}.$$

Cf. simply connected case

$$\Psi_{\mathbb{H}}(z, \xi) = -\frac{1}{\pi} \frac{1}{z - \xi}, \quad \Im \Psi_{\mathbb{H}}(x + iy, \xi) = \frac{1}{\pi} \frac{y}{(x - \xi)^2 + y^2}$$

Problems

Multi-slits case: Böhm & Lauf ('14), Böhm ('15)

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- Böhm's method is very complicated and not enough to construct SLE-like objects.
- **There are no results known about the Komatu–Loewner equation with measure-valued driving processes.**

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Main results

If $f \in \mathcal{S}_D$, then

$$f(z) = z + \pi \int_{\mathbb{R}} \Psi_D(z,\xi)\, \mu(f; d\xi)$$

for the finite measure $\mu(f;\cdot) = \pi^{-1} \Im f(\xi + i0+)\, d\xi$.

Suppose that $\phi_{t,s} \colon D_s \hookrightarrow D_t$ satisfies the following:

- $\phi_{t,s} \in \mathcal{S}_{D_s}$
- $0 \leq s < t < u \leq T \Rightarrow \phi_{u,s} = \phi_{u,t} \circ \phi_{t,s}$
- $\mu(\phi_{t,0}; \mathbb{R}) = 2t$

For each fixed s,

$$\frac{d\phi_{t,s}(z)}{dt} = 2\pi \int_{\mathbb{R}} \Psi_{D_t}(\phi_{t,s}(z), \xi)\, \nu_t(\xi), \quad \text{a.e. } t \in [s, T).$$

Comment on the proof

$f \in \mathcal{S}_D$ if and only if it is univalent on D and

$$f(z) = z + \pi \int_{\mathbb{R}} \Psi_D(z, \xi)\, \mu(f; d\xi)$$

for a finite measure $\mu(f; \cdot)$.

- The case $D = \mathbb{H}$ follows from the **Pick representation**, but there are no such formulas on parallel slit half-planes.
- Because $\Im(f(z) - z)$ is a **BMD-harmonic function**,

$$\Im(f(z) - z) = \pi \int_{\mathbb{R}} K_D^*(z, \xi) \Im(f(\xi + i0+) - i0)\, d\xi$$

(**Poisson integral formula**).

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1. **Motivations from probability theory**
   - Multiple SLE and its “hydrodynamic limit”
   - DLA and related models
2. **Motivations from complex analysis**
   - Classical Loewner theory
3. **Extension to multiply connected domains**
   - Komatu–Loewner equation and Brownian motion with darning
   - My results and proof
4. **Concluding remarks**

Ongoing/future works

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- On other canonical domains
- Multiple SLE on multiply connected domains
  [cf. Jahangoshahi & Lawler (2018, arXiv)]
- Construction and convergence of critical models on multiply connected domains
  (Only a few results on LERW are known.)

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Appendix. Construction of BMD

- Z: absorbing BM in $\mathbb{H}$
- $K := \bigcup_{j=1}^{N} C_j$
- $\sigma_K := \inf\{\, t > 0 \,;\, Z_t \in K \,\}$
- $u^{(j)}(z) := \mathbb{E}_z \left[e^{-\sigma_K} \,;\, Z_{\sigma_K} \in C_j\right]$

We define

$$\mathcal{E}^*(u,v) := \int_D \nabla u \cdot \nabla v \, dx, \quad \|u\|_1^2 := \mathcal{E}^*(u,u) + \|u\|_{L^2(D)}^2,$$

$$\mathcal{F}^* := \overline{C_c^\infty(D) \cup \{\, u^{(j)}|_D \,;\, 1 \le j \le N \,\}}^{\|\cdot\|_1}$$

$$= \{\, u|_D \,;\, u \in W_0^{1,2}(\mathbb{H}),\ u \text{ is constant } \mathcal{E}^*\text{-q.e. on each } C_j \,\}.$$

Then $(\mathcal{E}^*, \mathcal{F}^*)$ is a **strongly local regular Dirichlet form** on $L^2(D^*; m)$.