Gaussian fields and potential theory for Dirichlet forms

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September 6, 2019

Japan-German Open Conference on Stochastic Analysis 2019

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$(\mathcal{E}, \mathcal{F})$: a regluar Dirichlet form on $L^2(E; m)$
$(\mathcal{F}_e, \mathcal{E})$: its extended Dirichlet space
There are two stochastic objects associated with the form $\mathcal{E}$

One is an m-symmetric Hunt process $\mathbb{M} = (\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in E})$ on E whose transition function $\{P_t; t > 0\}$ generates the strongly continuous contraction semigroup on $L^2(E; m)$ associated with $\mathcal{E}$.

Another is the centered Gaussian field $\mathbb{G}(\mathcal{E}) = \{X_u; u \in \mathcal{F}_e\}$ indexed by $\mathcal{F}_e$ with covariance $\mathbb{E}[X_u X_v] = \mathcal{E}(u, v),\ u, v \in \mathcal{F}_e$.

In this talk, I shall consider an intrinsic concept pseudo-Markov property for the Gaussian field $\mathbb{G}(\mathcal{E})$:

I like to explain how this concept can be understood by means of the probabilistic potential theory for the regular Dirichlet form $\mathcal{E}$ formulated in terms of the Hunt process $\mathbb{M}$.

A pseudo Markov property of $\mathbb{G}(\mathcal{E})$

According to the books by J.L.Doob (1953) and K. Itô (1953, in Japanese), we have the following:

Given a linear space Λ equipped with $C(\lambda,\mu) \in \mathbb{R}$, $\lambda,\mu \in \Lambda$, such that $C(\lambda,\mu) = C(\mu,\lambda)$ and $\{C(\lambda_i,\lambda_j)\}$ is non-negative definite for any finite $\{\lambda_i\} \subset \Lambda$,
there exists uniquely Gaussian distributed random variables $\mathbb{G}(\Lambda) = \{X_\lambda; \lambda \in \Lambda\}$ defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with

$$\mathbb{E}[X_\lambda \cdot X_\mu] = C(\lambda,\mu), \quad \mathbb{E}[X_\lambda] = 0, \quad \forall \lambda,\mu \in \Lambda,$$

$\mathbb{G}(\Lambda)$ is called the Gaussian system with index set Λ.

When Λ is an Euclidean space $\mathbb{R}^d$ (resp. a function space), we may call $\mathbb{G}(\Lambda)$ a Gaussian process (resp. Gaussian field).

We recall that, in the study of the Markov property of Gaussian processes, the following notion and criterion were presented in H.P. McKean (1963) and L.D. Pitt (1971), respectively:

For sub σ-algebras $\mathcal{F}$, $\mathcal{G}$, Σ of $\mathcal{B}$, Σ is said to be a splitting σ-algebra for $\mathcal{F}$ and $\mathcal{G}$ if

$$\mathbb{P}(\Gamma_1 \cap \Gamma_2 | \Sigma) = \mathbb{P}(\Gamma_1 | \Sigma) \cdot \mathbb{P}(\Gamma_2 | \Sigma), \quad \forall \Gamma_1 \in \mathcal{F}, \quad \forall \Gamma_2 \in \mathcal{G}. \tag{1.1}$$

If $\mathcal{F} = \sigma(X_\lambda, \lambda \in \Lambda_1)$ and $\mathcal{G} = \sigma(X_\lambda, \lambda \in \Lambda_2)$ for $\Lambda_1,\ \Lambda_2 \subset \Lambda$ and if $\Sigma \subset \mathcal{F}$, then (1.1) is equivalent to the condition that

$$\sigma\{\mathbb{E}[X_\lambda \mid \mathcal{F}];\ \lambda \in \Lambda_2\} \subset \Sigma. \tag{1.2}$$

We may think of $\mathcal{F}$ (resp. $\mathcal{G}$) as the future (resp. past) events.

As is well known, (1.1) is also equivalent to the condition that

$$P(\Gamma \mid \mathcal{G}) = \mathbb{P}(\Gamma \mid \Sigma), \text{ for any } \Gamma \in \mathcal{F}.$$

Given a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$,

let $\mathbb{G}(\mathcal{E}) = \{X_u : u \in \mathcal{F}_e)$ be the centered Gaussain field defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with covariance

$$\mathbb{E}[X_u X_v] = \mathcal{E}(u, v), \qquad u, v \in \mathcal{F}_e.$$

A function $u \in \mathcal{F}_e$ is called $\mathcal{E}$-harmonic on an open set $G \subset E$ if $\mathcal{E}(u, v) = 0$ for any $v \in \mathcal{F} \cap C_c(E)$ with supp$[v] \subset G$, where $C_c(E)$ is the family of continuous functions on E with compact support. Following

[BD] A.Beurling and J.Deny, Dirichlet spaces, Proc.Nat.Acad.U.S.A. 45(1959), 208-215

the complement of the largest open set where u is harmonic will be called the spectrum of u and denoted by $s(u)$.

For any set $A \subset E$, we define the sub σ-algebra $\sigma(A)$ of $\mathcal{B}$ by

$$\sigma(A) = \sigma\{X_u : \ u \in \mathcal{F}_e, \ s(u) \subset A\}. \tag{1.3}$$

When $(\mathcal{E}, \mathcal{F})$ is transient, it is known that

$$s(U\mu) = \text{supp}[\mu] \quad \text{for any} \quad \mu \in \mathcal{S}_0^{(0)},$$

where $U\mu$ denotes the 0-order potential of μ.

A similar identity holds in the irreducible recurrent case under the condition (AC) stated later.

For any closed set $B \subset E$, let $\mathcal{F}_{e,E\setminus B}$ be a linear subspace of $\mathcal{F}_e$ defined by

$$\mathcal{F}_{e,E\setminus B} = \{u \in \mathcal{F}_e : \widetilde{u} = 0 \text{ q.e. on } B\}, \tag{1.4}$$

where $\widetilde{u}$ denotes a quasi-continuous version of u.

It can be verified that $s(u) \subset B$ if and only if

$$\mathcal{E}(u, v) = 0, \qquad \forall v \in \mathcal{F}_{e,E\setminus B}. \tag{1.5}$$

Let $\mathbb{M} = (X_t, \mathbb{P}_x)$ be the Hunt process on E associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$.
$\mathcal{B}(E)$ will denote the totality of Borel subsets of E.

For any $B \in \mathcal{B}(E)$, the hitting distribution $H_B(x, \cdot)$ of $\mathbb{M} = (X_t, \mathbb{P}_x)$ for B is defined by

$$H_B f(x) = \mathbb{E}_x[f(X_{\sigma_B})],\ x \in E,$$

for any bounded Borel function f on E where $\sigma_B = \inf\{t > 0 : X_t \in B\}$.

Generally it holds for a closed set B and $u \in \mathcal{F}_e$ that $H_B|\widetilde{u}|(x) < \infty$ for q.e. $x \in E$ and $H_B\widetilde{u}$ is a quasi-continuous element of $\mathcal{F}_e$ satisfying (1.5).

Hence

$$s(H_B\widetilde{u}) \ \subset \ B,\ \text{for any closed set } B \subset E \text{ and for any } u \in \mathcal{F}_e. \quad (1.6)$$

Lemma 1 (Fundamental identity for $\mathbb{G}(\mathcal{E})$)

The Gaussian field $\mathbb{G}(\mathcal{E}) = \{X_u : u \in \mathcal{F}_e\}$ enjoys the following property: For any closed set $B \subset E$ and any $u \in \mathcal{F}_e$,

$$X_u - X_{H_B \tilde{u}} \quad \text{is independent of} \quad \sigma(B) \tag{1.7}$$

and, equivalently

$$\mathbb{E}[X_u \mid \sigma(B)] = X_{H_B \tilde{u}}. \tag{1.8}$$

Proof. Take any $v \in \mathcal{F}_e$ with $s(v) \subset B$.
Since $u - H_B\tilde{u} \in \mathcal{F}_{e,E\setminus B}$, $\mathcal{E}(u - H_B\tilde{u}, v) = 0$ by (1.5).
Hence $\mathbb{E}\left[(X_u - X_{H_B\tilde{u}})X_v\right] = 0$ so that (1.7) holds as all random variables involved are centered Gaussian.
Consequently

$$\mathbb{E}\left[X_u - X_{H_B\tilde{u}} \middle| \sigma(B)\right] = \mathbb{E}\left[X_u - X_{H_B\tilde{u}}\right] = 0,$$

and so (1.8) is valid by (1.6). □

(1.7) or (1.8) has been noticed by

[Dy] E.B. Dynkin, Markov proceses and random fields, Bull.Amer.Math.Soc. 3(1980),957-999

[R] M. Röckner, Generalized Markov fields and Dirichlet forms, Acta.Appl.Math. 3(1985), 285-311

[Sh] S.Sheffield, Gaussian free fields for mathematicians, Probab.Theory Related Fields 139(2007), 521-541

[Sz] A.S. Sznitman, Topics in Occupation Times and Gaussian Free Fields, European Mathematical Society 2012

[Dy] and [R] consider the subfamily of $\mathbb{G}(\mathcal{E})$ indexed by measures $\mu \in \mathcal{S}_0^{(0)} - \mathcal{S}_0^{(0)}$ for a transient $\mathcal{E}$.

[Sh] and [Sz] treat the cases where E is discrete.

[Sh] called (1.7) a 'Markov property' of GFF.

Fix a closed subset B of E,
For a positive Radon measure μ, the measure μ_B defined by

$$\mu_B(C) = \int_E \mu(dx) H_B(x, C). \qquad C \in \mathcal{B}(E),$$

is called the swept measure of μ (balayage). $\quad$ supp$[\mu_B] \subset B$.

When $(\mathcal{E}, \mathcal{F})$ is transient, it holds that

$$H_B(U\mu) = U\mu_B, \qquad \mu \in \mathcal{S}_0^{(0)}.$$

An analogous identity is also valid in the irreducible recurent case under the condition (AC) stated later.

For $u \in \mathcal{F}_e$, the function $H_B u \in \mathcal{F}_e$ is called the reduced function of u.

For a subfamily $\mathcal{H}$ of $\mathcal{F}_e$, define $\quad \sigma(\mathcal{H}) := \sigma\{X_u : u \in \mathcal{H}\}$.

Lemma 1 and Pitt's criterion (1.2) then imply

Proposition 1.1 (pseudo-Markov property of $\mathbb{G}(\mathcal{E})$)

(i) *For any open set $A \subset E$ and any $\mathcal{H} \subset \mathcal{F}_e$, any σ-algebra Σ such that*

$$\sigma\{H_{\bar{A}}u : u \in \mathcal{H}\} \subset \Sigma \subset \sigma(\bar{A}) \tag{1.9}$$

is a splitting σ-algebra for $\sigma(\mathcal{H})$ and $\sigma(\bar{A})$.
(ii) *For any open set $A \subset E$, any σ-algebra Σ such that*

$$\sigma\{H_{\bar{A}}u : u \in \mathcal{F}_e,\ s(u) \in E \setminus A\} \subset \Sigma \subset \sigma(\bar{A}) \tag{1.10}$$

is a splitting σ-algebra for $\sigma(E \setminus A)$ and $\sigma(\bar{A})$.

(ii) is just a special case of (i) for $\mathcal{H} = \{u \in \mathcal{F}_e : s(u) \subset E \setminus A.\}$

We call $\sigma\{H_{\bar{A}}u : u \in \mathcal{H}\}$ in (i) the reduced σ-algebra of $\sigma(\mathcal{H})$.
Sometimes, for a special choice of $\mathcal{H}$, its swept σ-algebra becomes trivial yielding the independence of $\sigma(\mathcal{H})$ and $\sigma(\bar{A})$; see for instance

X.Hu, J.Millar and Y. Peres, Thick points of the Gaussian free field, Ann.Probab. 38(2010), 896-920

The Gaussian field $\mathbb{G}(\mathcal{E})$ is said to possess the Markov property with respect to a set $A \subset E$ if

$$\sigma(\partial A) \text{ is a splitting } \sigma\text{-algebra for } \sigma(\overline{E \setminus A}) \text{ and } \sigma(\bar{A}). \qquad (1.11)$$

This property was studied by

S. Albeverio-R. Hoegh-Krohn, Uniqueness and the global Markov property for Euclidean Fields, The case of trigonometric interactions, Commun. Math. Phys 68(1979), 95-128,

E.B. Dynkin([Dy] 1980) and M. Röckner([R] 1985).

When the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is transient, M. Röckner([R]) established that $\mathbb{G}(\mathcal{E})$ enjoys the Markov property for any $A \subset E$ if and only if $\mathcal{E}$ is local.

The 'if' part can be derived by the above Proposition 1.2(ii) combined with the celebrated spectral synthesis theorem:

(SS) For any $u \in \mathcal{F}_e$, there exists a sequence $\mu_n \in \ \mathcal{M}_0 = \mathcal{S}_0^{(0)} - \mathcal{S}_0^{(0)}$ such that supp$[|\mu_n|] \subset s(u)$ and $U\mu_n \in \mathcal{F}_e$ is $\mathcal{E}$-convergent to u.

(SS) was announced by A.Beurling-J.Deny in 1959. Its potential theoretic proof was given by J.Deny in 1970 for a transient regular Dirichlet form. Its α-order version for a general regular Dirichlet form was adopted by [F] in 1980 $(\alpha > 0)$,
which was then shown in [CF] (2012) only by using the associated Hunt process and the notion of $\mathcal{E}$-nest due to [MR] (1992).

In recent papers

[F] M. Fukushima, Logarithmic and linear potentials of signed measures and Markov property of associated Gaussian fields, Potential Anal. 49(2018), 359-379

[FO-1] M. Fukushima and Y. Oshima, Recurrent Dirichlet forms and Markov property of associated Gaussian fields, Potential Anal. 49(2018), 609-633

[FO-2] M. Fukushima and Y. Oshima, Gaussian fields, equilibrium potentials and Liouville random measures for Dirichlet forms, Preprint

Röckner's result in [R] has been extended to any irreducible recurrent Dirichlet form under the condition that the transition function $P_t(x, dy)$ of the associated Hunt process $\mathbb{M}$ satisfies

(AC) There exists a certain Borel properly exceptional set $N \subset E$ such that $P_t(x, \cdot)$ is absolutely continuous with respect to m for each $t > 0$ and $x \in E \setminus N$.

Support of jumping measure and splitting σ-algebra

Let J be the jumping measure in the Beurling-Deny representation of the regular Dirichlet form $\mathcal{E}$.

J is a symmetric positive Radon measure on $E \times E \setminus d$.

Proposition 2.1

Let A be an open subset of E with $E \setminus A \neq \emptyset$. Suppose there exists a closed set A_1 with

$$\partial A \subset A_1 \subset \bar{A}, \quad J(E \setminus \bar{A}, A \setminus A_1) = 0. \tag{2.1}$$

Then $\sigma(A_1)$ is a splitting σ-algebra for $\sigma(E \setminus A)$ and $\sigma(\bar{A})$.

If $\mathcal{E}$ is local, then $J = 0$ and one can take $A_1 = \partial A$ in the above in getting the Markov property for A.

Proof. By virtue of the pseudo-Markov property (1.11), it suffices to show that

$$\text{for any } u \in \mathcal{F}_e \text{ with } s(u) \subset E \setminus A, \quad s(H_{\bar{A}}u) \subset A_1. \tag{2.2}$$

It has been shown in [FOT] that,

for any non-negative h vanishing on A and $v \in \mathcal{F} \cap C_c(E)$ with supp$[v] \subset A$,

$$\mathbb{E}_{h\cdot m}\left[e^{-\alpha\sigma_{\bar{A}}}v(X_{\sigma_{\bar{A}}})\right] = \int R_\alpha^{E\setminus\bar{A}}h(x)v(y)J(dxdy),$$

which combined with the assumption (2.1) implies

$$H_{\bar{A}}(x, \bar{A} \setminus A_1) = 0, \quad \text{q.e. } x \in E \setminus A. \tag{2.3}$$

Assume the transience of $(\mathcal{E}, \mathcal{F})$.

For any $u \in \mathcal{F}_e$ with $s(u) \subset E \setminus A_1$,
use (SS) to find $\mu_n \in \mathcal{M}_0 = \mathcal{S}_0^{(0)} - \mathcal{S}_0^{(0)}$ such that

supp$[|\mu_n|] \subset E \setminus A$ and $U\mu_n$ is $\mathcal{E}$-convergent to u. Then, by (2.3)

$$\mu_{n,\bar{A}}(E \setminus A_1) = \mu_{n,\bar{A}}(\bar{A} \setminus A_1) = \int_{E\setminus A} \mu_n(dx) H_{\bar{A}}(x, \bar{A} \setminus A_1) = 0,$$

and, for any $f \in \mathcal{F} \cap C_c(E \setminus A_1)$,

$$\begin{aligned} \mathcal{E}(H_{\bar{A}}u, g), f) &= \lim_{n\to\infty} \mathcal{E}(H_{\bar{A}}(U\mu_n), f) \\ &= \lim_{n\to\infty} \mathcal{E}(U\mu_{n,\bar{A}}, f) = \lim_{n\to\infty} \langle \mu_{n,\bar{A}}, f \rangle = 0. \end{aligned}$$

Hence, $H_{\bar{A}}u$ is $\mathcal{E}$-harmonic on $E \setminus A_1$ yielding desired $s(H_{\bar{A}}u) \in A_1$.

Finally assume that $(\mathcal{E}, \mathcal{F})$ is irreducible recurrent and $\mathbb{M}$ satisfies (AC).

Then, the resolvent kernel $\{R_\alpha, \alpha > 0\}$ of $\mathbb{M}$ admits a density function $r_\alpha(x, y),\ x, y \in E \setminus N$, with respect to m such that it is strictly positive, symmetric Borel measurable, α-excessive relative to $\mathbb{M}$ in each variable.

A set $F \subset E \setminus N$ is called an admissible reference set or a reference set if

$$\begin{cases} F \text{ is compact, } \ m(F) > 0 \quad \text{and for some} \quad c > 0 \text{ and } \frac{1}{2} < a < 1, \\ m(\{y \in F : r_1(x, y) > c\}) > a \cdot m(F) \quad \text{for every} \quad x \in F. \end{cases} \tag{2.4}$$

For any Borel set $B \subset E \setminus N$ with $m(B) > 0$,

there exists a reference set F contained in B.

For a fixed reference set F, we write $m_F(C) = m(C \cap F),\ C \in \mathcal{B}(E)$.

We consider the perturbed form

$$\mathcal{E}^{m_F}(u,v) = \mathcal{E}(u,v) + \int_F uvdm, \quad u,v \in \mathcal{F}^{m_F} = \mathcal{F} \cap L^2(F;m),$$

which is a regular transient Dirichlet form on $L^2(E;m)$.

Its extended Dirichlet space $\mathcal{F}_e^{m_F}$ equals $\mathcal{F}_e \cap L^2(F;m)$.

Let $\mathcal{S}_0^{m_F,(0)}$ be the space of positive Radon measures on E with finite 0-order energy relative to the form $\mathcal{E}^{m_F}$.

Define

$$\begin{cases} \mathcal{M}_0 = \{\mu = \mu_1 - \mu_2 : \mu_i \in S_0^{m_F,(0)}, \mu_i(E) < \infty, i = 1,2\}, \\ \mathcal{M}_{00} = \{\mu \in \mathcal{M}_0 : \mu(E) = 0\}. \end{cases} \tag{2.5}$$

The followings have been shown in [FO1]:

There exists for any $\mu \in \mathcal{M}_0$ a quasi continuous function $R\mu \in \mathcal{F}_e^{m_F}$ uniquely up to q.e. equivalence such that

$$\mathcal{E}(R\mu, u) = \left\langle \mu, \widetilde{u} - \frac{1}{m(F)}\langle m_F, u\rangle \right\rangle \quad \forall u \in \mathcal{F}_e^{m_F}, \text{ and } \langle m_F, R\mu\rangle = 0. \tag{2.6}$$

The first equation in the above determines $R\mu \in \mathcal{F}_e^{m_F}$ up to an additive constant, while the second identity is its normalization.

We call $\{R\mu : \mu \in \mathcal{M}_0\}$

the family of recurrent potentials relative to a reference set F.

Contrarily to the transient case,

the class $\mathcal{M}_0$ of measures and potentials $R\mu$, $\mu \in \mathcal{M}_0$,

depend on the choice of a reference set F, making relevent arguments more involved.

Now, for an open set $A \subset E$ with $E \setminus A \neq \emptyset$,

one can take a referene set F contained in A and consider

the family $\{R\nu : \nu \in \mathcal{M}_0\}$ of recurrent potentials with reference set F.

Then, for any $u \in \mathcal{F}_e$ with $s(u) \in E \setminus A$,

one can find $\nu_n \in \mathcal{M}_{00}$ such that

supp$[\nu_n] \subset E \setminus A$ and $R\nu_n$ is $\mathcal{E}$-convergent to u.

Furthermore

$$H_{\bar{A}} R\nu_n = R\nu_{n,\bar{A}}. \quad \text{q.e. modulo an additive constant}$$

So one can prove (2.2) in exactly the same way as in the transient case.