Asymptotic expansion of the density for hypoelliptic rough differential equation

This talk is based on a joint work with Yuzuru Inahama

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Setting and known result

♣ Let w be a d-dim. fractional Bm with Hurst parameter $1/4 < H \leq 1/2$.

♣ We study an RDE driven by w:

$$dy_t = \sum_{i=1}^{d} V_i(y_t)\, dw_t^i + V_0(y_t)\, dt, \qquad y_0 = a \in \mathbf{R}^n.$$

♣ It is known that, under the Hörmander condition (A1), y_t admits a smooth density $p_t(a, a')$ for every $t > 0$, i.e.,

- $a' \mapsto p_t(a, a')$ is smooth,
- $\boldsymbol{P}(y_t \in A) = \int_A p_t(a, a')\, da'$.

Main result (Inahama and N.)

Let $a \neq a'$. Assume (A1)–(A4). Then, we have the following asymptotic expansion as $t \searrow 0$:

$$p_t(a, a') \sim \exp\left(-\frac{\|\bar{\gamma}\|_{\mathfrak{H}}^2}{2t^{2H}}\right) \frac{1}{t^{nH}} \left\{\alpha_0 + \alpha_{\lambda_1} t^{\lambda_1 H} + \cdots\right\}.$$

Here,

- $\bar{\gamma}$: "unique minimizer" in the CM space $\mathfrak{H}$,
- $0 = \lambda_0 < \lambda_1 < \cdots$:
  indexes belonging to $\Lambda_4 \subset (\mathbf{Z} + H^{-1}\mathbf{Z}) \cap [0, \infty)$,
- α_0 is a certain positive constant,
- α_{λ_j} $(j = 1, 2, \dots)$ are certain real constants.

(A1) Hörmander cond. at the initial point

♣ Set

$$\mathcal{V}_m = \begin{cases} \{V_i \mid 1 \le i \le d\}, & m = 0, \\ \{[V_i, U] \mid U \in \mathcal{V}_{m-1}, 0 \le i \le d\}, & m \ge 1, \end{cases}$$

$$\mathcal{V} = \bigcup_{m=0}^{\infty} \mathcal{V}_m,$$

$$\mathcal{V}(x) = \{W(x) \in \mathbf{R}^n \mid W \in \mathcal{V}\}.$$

♣ Assume

(A1) $\mathcal{V}(a)$ linearly spans $\mathbf{R}^n$.

(A2) Unique minimizer

♣ Recall that $\mathfrak{H} \hookrightarrow C^{q\text{-var}}$ for some $q \in [1,2)$.

For $\gamma \in \mathfrak{H}$, let $\phi_t^0 = \phi_t^0(\gamma)$ be a sol. to Young ODE

$$d\phi_t^0 = \sum_{i=1}^{d} V_i(\phi_t^0)\, d\gamma_t^i, \qquad \phi_0^0 = a \in \mathbf{R}^n.$$

♣ Set $K_a^{a'} = \{\gamma \in \mathfrak{H} \mid \phi_1^0(\gamma) = a'\}$ for $a' \neq a$.

♣ Assume

(A2) $\exists!\, \bar{\gamma} \in K_a^{a'}$ which minimizes $\mathfrak{H}$-norm $\| \bullet \|_{\mathfrak{H}}$,

$$\{\bar{\gamma}\} = \operatorname*{arg\,min}_{\gamma \in K_a^{a'}} \|\gamma\|_{\mathfrak{H}}.$$

(A3) Non-degeneracy for Malliavin mat.

♣ The map $\mathfrak{H} \ni \gamma \mapsto \phi_1^0(\gamma) \in \mathbf{R}^n$ is Fréchet diff.
$D\phi_1^0(\gamma)$ stands for the derivative, i.e.,

$$D\phi_1^0(\gamma) = (D[\phi_1^0(\gamma)]^1, \dots, D[\phi_1^0(\gamma)]^d) \in (\mathfrak{H}^*)^d.$$

♣ Define the deterministic Malliavin covariance matrix
$Q(\gamma) = (Q(\gamma)_{kl})_{1 \le k,l \le n}$ by

$$Q(\gamma)_{kl} = \langle D[\phi_1^0(\gamma)]^k, D[\phi_1^0(\gamma)]^l \rangle_{\mathfrak{H}^*}.$$

♣ Assume

(A3) $\exists c > 0$ s.t. $Q(\bar{\gamma}) \ge cI$.

(A4) Strictly positive

♣ The Hessian of the functional

$$K_a^{a'} \ni \gamma \mapsto \frac{\|\gamma\|_{\mathfrak{H}}^2}{2}$$

at $\bar{\gamma} \in K_a^{a'}$ is strictly positive in the form sense, i.e.,

(A4) If $(-\epsilon_0, \epsilon_0) \ni u \mapsto f(u) \in K_a^{a'}$ is

- a smooth curve
- $f(0) = \bar{\gamma}$ and $f'(0) \neq 0$,

then

$$\frac{d^2}{du^2} \frac{\|f(u)\|_{\mathfrak{H}}^2}{2}\bigg|_{u=0} > 0.$$

Index set Λ_1

♣ Write $\mathbf{N} = \{0, 1, \ldots, \}$, $\mathbf{N}_+ = \{1, \ldots, \}$.

♣ Set

- $\Lambda_1 = \mathbf{N} + H^{-1}\mathbf{N} = \left\{ n_1 + \frac{n_2}{H} \,\middle|\, n_1, n_2 \in \mathbf{N} \right\}$.
- $0 = \kappa_0 < \kappa_1 < \kappa_2 < \cdots$ are all the elements of Λ_1.

♣ For $1/3 < H < 1/2$, $\kappa_0, \kappa_1, \ldots, \kappa_6, \ldots$ are equal to

$$0, \quad 1, \quad 2, \quad \frac{1}{H} \quad, 3, \quad 1 + \frac{1}{H}, \quad 4, \quad \ldots$$

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Index sets Λ_2, Λ_2', Λ_3, Λ_3' and Λ_4

♣ Set

- $\Lambda_2 = \{\kappa_i - 1 \mid i \geq 1\}$.
- $\Lambda_2' = \{\kappa_i - 2 \mid i \geq 2\}$.

♣ Set

- $\Lambda_3 = \{a_1 + \cdots + a_m \mid a_i \in \Lambda_2\}$.
- $\Lambda_3' = \{a_1 + \cdots + a_m \mid a_i \in \Lambda_2'\}$.

♣ Set

- $\Lambda_4 = \Lambda_3 + \Lambda_3' = \{\nu + \rho \mid \nu \in \Lambda_3, \rho \in \Lambda_3'\}$.
- $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ are all the elements of Λ_4.

Remark and example

Main result (review)

Let $a \neq a'$. Assume (A1)–(A4). Then, we have the following asymptotic expansion as $t \searrow 0$:

$$p_t(a, a') \sim \exp\left(-\frac{\|\bar{\gamma}\|_{\mathfrak{H}}^2}{2t^{2H}}\right) \frac{1}{t^{nH}} \left\{\alpha_0 + \alpha_{\lambda_1} t^{\lambda_1 H} + \cdots\right\}.$$

Here,

- $\bar{\gamma}$: "unique minimizer" in the CM space $\mathfrak{H}$,
- $0 = \lambda_0 < \lambda_1 < \cdots$:
  indexes belonging to $\Lambda_4 \subset (\mathbf{Z} + H^{-1}\mathbf{Z}) \cap [0, \infty)$,
- α_0 is a certain positive constant,
- α_{λ_j} $(j = 1, 2, \dots)$ are certain real constants.

Remark on the index sets

♣ Assume that $V_0 \equiv 0$ or $H = 1/2, 1/3$.

♣ Then we can replace Λ_4 by $2\mathbf{N}$ in our main thm, i.e.,

$$p_t(a, a') \sim \exp\left(-\frac{\|\bar{\gamma}\|_{\mathfrak{H}}^2}{2t^{2H}}\right)\frac{1}{t^{nH}} \times \left\{\alpha_0 + \alpha_2 t^{2H} + \alpha_4 t^{4H} + \cdots\right\}$$

as $t \searrow 0$.

Remark on the preceding results

♣ When $H = 1/2$, our main theorem holds. Thus, we reprove the result by Ben Arous (1988) via rough path theory and Malliavin calc.

♣ Our main thm improves a result by Inahama (2016). Because

1. we are working under the Hörmander condition instead of the ellipticity assumption,
2. the case $1/4 < H \leq 1/3$ is also treated. (We must use the third level rough paths.)

Examples

♣ Elliptic case

- $\{V_1(a), \ldots, V_d(a)\}$ linearly spans $\mathbf{R}^n$,
- a' is sufficiently close to a,

♣ The "fractional diff. proc." on the Heisenberg group

- $d = 2$, $n = 3$.
- $V_0 = 0, \ V_1 = \dfrac{\partial}{\partial x^1} + 2x^2 \dfrac{\partial}{\partial x^3}, \ V_2 = \dfrac{\partial}{\partial x^2} - 2x^1 \dfrac{\partial}{\partial x^3}.$

Proof of main result

Malliavin calculus

♣ Let

- $1 < r < \infty$, $s \in \mathbf{R}$, K: real sep. Hilbert space.
- $\mathbf{D}_{r,s}(K)$: the K-valued Gaussian-Sobolev space.

♣ The spaces of test functions are defined by

- $\mathbf{D}_{\infty}(K) = \bigcap_{1<r<\infty} \bigcap_{s=1}^{\infty} \mathbf{D}_{r,s}(K)$,
- $\tilde{\mathbf{D}}_{\infty}(K) = \bigcup_{1<r<\infty} \bigcap_{s=1}^{\infty} \mathbf{D}_{r,s}(K)$,

♣ The spaces of Watanabe distributions are defined by

- $\mathbf{D}_{-\infty}(K) = \bigcup_{1<r<\infty} \bigcup_{s=1}^{\infty} \mathbf{D}_{r,-s}(K)$,
- $\tilde{\mathbf{D}}_{-\infty}(K) = \bigcap_{1<r<\infty} \bigcup_{s=1}^{\infty} \mathbf{D}_{n,-s}(K)$.

Malliavin calc. for sol. to RDE

♣ Inahama ('14), Cass-Hairer-Litterer-Tindel ('15):

- $y_t \in \mathbf{D}_\infty(\mathbf{R}^n)$,
- Under the Hörmander cond. (A1), for every $t > 0$, y_t is non-degenerate in the sense of Malliavin, i.e.,

$$\det(\text{Malliavin cov. matrix of } y_t)^{-1} \in L^{\infty-}.$$

- Hence, y_t has a smooth density $p_t(a, a')$ for $t > 0$.

♣ Due to Watanabe's distributional Malliavin cal.,

$$p_t(a, a') = \boldsymbol{E}[\delta_{a'}(y_t)] = {}_{\tilde{\mathbf{D}}_\infty}\langle 1, \delta_{a'}(y_t)\rangle_{\tilde{\mathbf{D}}_{-\infty}}.$$

Scaled (and shifted) RDE

- Let $\epsilon \in (0,1]$ and $\bar{\gamma} \in \mathfrak{H}$ as in (A2).
- The scaled RDE

$$dy_t^\epsilon = \sum_{i=1}^{d} V_i(y_t^\epsilon)\, \epsilon dw_t^i + V_0(y_t^\epsilon)\, \epsilon^{1/H} dt.$$

- The scaled and shifted RDE

$$d\tilde{y}_t^\epsilon = \sum_{i=1}^{d} V_i(\tilde{y}_t^\epsilon)\, d(\epsilon w + \bar{\gamma})_t^i + V_0(\tilde{y}_t^\epsilon)\, \epsilon^{1/H} dt.$$

Expresion of $p_t(a, a')$

♣ $\delta_{a'}(y_1^\epsilon)$ and $\delta_{a'}(\tilde{y}_1^\epsilon)$ are well-defined for the same reason as the case without ϵ.

♣ $y_{\epsilon^{1/H}} = y_1^\epsilon$ in law follows from self-similarity of fBm.

♣ Note $\delta_{a'}(\tilde{y}_1^\epsilon) = \delta_0\left(\epsilon \cdot \frac{\tilde{y}_1^\epsilon - a'}{\epsilon}\right) = \frac{1}{\epsilon^n}\delta_0\left(\frac{\tilde{y}_1^\epsilon - a'}{\epsilon}\right)$.

♣ From the above and the CM formula,

$$
\begin{aligned}
&p_{\epsilon^{1/H}}(a, a') = \boldsymbol{E}[\delta_{a'}(y_{\epsilon^{1/H}})] = \boldsymbol{E}[\delta_{a'}(y_1^\epsilon)] \\
&= \exp\Big(- \frac{\|\bar{\gamma}\|_{\mathfrak{H}}^2}{2\epsilon^2}\Big) \boldsymbol{E}\Big[\exp\Big(- \frac{1}{\epsilon}\langle \bar{\gamma}, w\rangle\Big)\delta_{a'}(\tilde{y}_1^\epsilon)\Big] \\
&= \exp\Big(- \frac{\|\bar{\gamma}\|_{\mathfrak{H}}^2}{2\epsilon^2}\Big) \frac{1}{\epsilon^n}\boldsymbol{E}\Big[\exp\Big(- \frac{1}{\epsilon}\langle \bar{\gamma}, w\rangle\Big)\delta_0\Big(\frac{\tilde{y}_1^\epsilon - a'}{\epsilon}\Big)\Big].
\end{aligned}
$$

Expansion of $\tilde{y}_1^\epsilon$

Taylor expansion of Lyons-Itô map around $\bar{\gamma}$

♣ Recall $0 = \kappa_0 < \kappa_1 < \cdots$ belong to $\Lambda_1 = \mathbf{N} + H^{-1}\mathbf{N}$.
♣ Let $\phi^0 \equiv \phi^0(\bar{\gamma})$ and $\phi^{\kappa_i} \equiv \phi^{\kappa_i}(w, \bar{\gamma})$ for $i \geq 1$ solutions to some differential equations (explained later).

Proposition

It holds that

$$\tilde{y}_1^\epsilon \sim \phi_1^0 + \epsilon^{\kappa_1}\phi_1^{\kappa_1} + \epsilon^{\kappa_2}\phi_1^{\kappa_2} + \cdots \quad \textit{in } \mathbf{D}_\infty \textit{ as } \epsilon \searrow 0.$$

What are ϕ^{κ_i}?

♣ Recall the scaled and shifted RDE

$$d\tilde{y}^{\epsilon} = \sigma(\tilde{y}^{\epsilon})\, d(\epsilon w + \bar{\gamma}) + b(\tilde{y}^{\epsilon})\, \epsilon^{1/H} dt.$$

♣ Let ϕ^0 be a sol to the above eq. with $\epsilon = 0$, i.e.,

$$d\phi^0 = \sigma(\phi^0)\, d\bar{\gamma}.$$

♣ Set $\triangle\phi = \tilde{y}^{\epsilon} - \phi^0$.

♣ Substituting it into the the scaled and shifted RDE,

$$\begin{aligned}
&d(\phi^0 + \triangle\phi) \\
&\qquad = \sigma(\phi^0 + \triangle\phi)\, d(\epsilon w + \bar{\gamma}) + b(\phi^0 + \triangle\phi)\, \epsilon^{1/H} dt \\
&\qquad = \sum_{k=0}^{\infty} \frac{\nabla^k \sigma(\phi^0)}{k!} \langle \underbrace{\triangle\phi, \ldots, \triangle\phi}_{k}; \epsilon dw + d\bar{\gamma} \rangle \\
&\qquad\qquad + \sum_{k=0}^{\infty} \frac{\nabla^k b(\phi^0)}{k!} \langle \underbrace{\triangle\phi, \ldots, \triangle\phi}_{k} \rangle\, \epsilon^{1/H} dt
\end{aligned}$$

♣ If $\triangle\phi \sim \epsilon^{\kappa_1}\phi^{\kappa_1} + \epsilon^{\kappa_2}\phi^{\kappa_2} + \cdots$, then

$$\langle \underbrace{\triangle\phi, \ldots, \triangle\phi}_{k} \rangle = \sum_{i_1,\ldots,i_k=0}^{\infty} \epsilon^{\kappa_{i_1} + \cdots + \kappa_{i_k}} \left\langle \phi^{\kappa_{i_1}}, \ldots, \phi^{\kappa_{i_k}} \right\rangle$$

♣ Picking up the terms of order ϵ^{κ_i}, we see ϕ^{κ_i} satisfy

$$\begin{aligned} d\phi^0 &= \sigma(\phi^0)\, d\bar{\gamma}, & \phi_0^0 &= a, \\ d\phi^1 - \nabla\sigma(\phi^0)\langle\phi^1, d\bar{\gamma}\rangle &= \sigma(\phi^0)\, dw, & \phi_0^1 &= 0, \\ d\phi^{\kappa_i} - \nabla\sigma(\phi^0)\langle\phi^{\kappa_i}, d\bar{\gamma}\rangle &= \cdots, & \phi_0^{\kappa_i} &= 0. \end{aligned}$$

Expansion of $\delta_0\left(\frac{\tilde{y}_1^\epsilon - a'}{\epsilon}\right)$

Expansion of $\delta_0\left(\frac{\tilde{y}_1^\epsilon - a'}{\epsilon}\right)$

$$\delta_0\left(\frac{\tilde{y}_1^\epsilon - a'}{\epsilon}\right) \sim \delta_0(\phi_1^1) + \epsilon^{\nu_1}\Phi_{\nu_1} + \cdots \quad \text{in } \tilde{\mathbf{D}}_{-\infty} \text{ as } \epsilon \searrow 0.$$

follows from

- $\dfrac{\tilde{y}_1^\epsilon - a'}{\epsilon} \sim \epsilon^{\kappa_1 - 1}\phi_1^{\kappa_1} + \epsilon^{\kappa_2 - 1}\phi_1^{\kappa_2} + \cdots$ in $\mathbf{D}_\infty$ as $\epsilon \searrow 0$.
    - We have already shown this expansion.
    - The index set is $\Lambda_3 = \{\kappa_1 - 1, \kappa_2 - 1, \dots\}$.
- uniform non-degeneracy of $\dfrac{\tilde{y}_1^\epsilon - a'}{\epsilon}$

Estimate of $\det$ of Mallavin martrix of $\tilde{y}_1^\epsilon$

Defined the Malliavin covariance matrix $Q^\epsilon = (Q^\epsilon_{kl})_{1\le k,l\le n}$ of y_1^ϵ and $\tilde{Q}^\epsilon = (\tilde{Q}^\epsilon_{kl})_{1\le k,l\le n}$ of $\tilde{y}_1^\epsilon$ by

$$Q^\epsilon_{kl} = \langle Dy_1^{\epsilon,k}, Dy_1^{\epsilon,l}\rangle_{\mathfrak{H}}, \qquad \tilde{Q}^\epsilon_{kl} = \langle D\tilde{y}_1^{\epsilon,k}, D\tilde{y}_1^{\epsilon,l}\rangle_{\mathfrak{H}}.$$

Remark

$$\epsilon^{-2}\tilde{Q}^\epsilon_{kl} = \left\langle D\left(\frac{\tilde{y}_1^{\epsilon,k} - (a')^k}{\epsilon}\right), D\left(\frac{\tilde{y}_1^{\epsilon,l} - (a')^l}{\epsilon}\right)\right\rangle_{\mathfrak{H}}.$$

By borrowing idea in [CHLT15], we can obtain

Proposition

Suppose that (A1) holds. $\exists \mu > 0$*.* $\exists c = c(r)$ *for every* $1 < r < \infty$*. Then, for every* $0 < \epsilon < 1$*,*

$$\boldsymbol{E}[|\det Q^{\epsilon}|^{-r}]^{1/r} < c(r)\epsilon^{-\mu}.$$

Proposition

Suppose that (A1), (A2) and (A3) holds. Then, for every $1 < r < \infty$*, we have*

$$\sup_{0<\epsilon<1} \boldsymbol{E}[|\det \epsilon^{-2}\tilde{Q}^{\epsilon}|^{-r}]^{1/r} < \infty.$$

Proof. Let

- $\lambda_t = t$,
- $O \subset G\Omega_p(\mathbf{R}^d) \times \mathbf{R}\langle\lambda\rangle$: a small nbhd of $(0,0)$,
- $U_\epsilon = \{w \in \Omega \mid (\epsilon \boldsymbol{w}, \epsilon^{1/H}\lambda) \in O\}$.

♣ (A2) and (A3) imply $\boldsymbol{E}[\{\det \tilde{Q}^\epsilon\}^{-r}; U_\epsilon] \le (c\epsilon^2)^{-nr}$.

♣ The CM formula, the Schilder-type LD, the previous proposition imply

$$\boldsymbol{E}[\{\det \tilde{Q}^\epsilon\}^{-r}; U_\epsilon^\complement] \le \boldsymbol{E}[\{\det \tilde{Q}^\epsilon\}^{-2r}]^{1/2}\boldsymbol{P}(U_\epsilon^\complement)$$
$$\le \boldsymbol{E}\left[\{\det Q^\epsilon\}^{-2r}\exp\left(\left\langle w, \frac{\bar{\gamma}}{\epsilon}\right\rangle - \frac{\|\bar{\gamma}\|_{\mathfrak{H}}^2}{2\epsilon^2}\right)\right]^{\frac{1}{2}}\exp\left(-\frac{c}{2\epsilon^2}\right)$$
$$\le c\epsilon^{-2nr}.$$

Proposition

As $\epsilon \searrow 0$ *in* $\tilde{\mathbf{D}}_{-\infty}(\mathbf{R}^n)$*-topology, we have*

$$\delta_0\Big(\frac{\tilde{y}_1^\epsilon - a'}{\epsilon}\Big) \sim \Phi_{\nu_0} + \epsilon^{\nu_1}\Phi_{\nu_1} + \epsilon^{\nu_2}\Phi_{\nu_2} + \cdots .$$

Here,

- $0 = \nu_0 < \nu_1 < \nu_2 < \cdots$ *are elements in* Λ_3
- $\Phi_{\nu_0} = \delta_0(\phi_1^1)$*,* $\Phi_{\nu_i} \in \tilde{\mathbf{D}}_{-\infty}(\mathbf{R}^n)$*.*

More precisely, $\forall i \in \mathbf{N}$, $\exists k \in \mathbf{N}_+$ s.t.

- $\Phi_{\nu_0}, \dots, \Phi_{\nu_i} \in \bigcap_{1<p<\infty} \mathbf{D}_{p,-k}$,
- $\|\delta_0\big(\frac{\tilde{y}_1^\epsilon - a'}{\epsilon}\big) - (\Phi_{\nu_0} + \cdots + \epsilon^{\nu_i}\Phi_{\nu_i})\|_{\mathbf{D}_{p,-k}} = O(\epsilon^{\epsilon^{\nu_{i+1}}})$

Expansion of $\exp\left(-\frac{1}{\epsilon}\langle\bar{\gamma}, w\rangle\right)$

Expansion of $\exp\big(-\frac{1}{\epsilon}\langle\bar{\gamma}, w\rangle\big)$

♣ Set $r^{2,\epsilon} = \tilde{y}_1^\epsilon - (\phi_1^0 + \epsilon\phi_1^1)$. Then

$$\frac{r^{2,\epsilon}}{\epsilon^2} \sim \epsilon^{-2}\left(\epsilon^{\kappa_2}\phi_1^{\kappa_2} + \epsilon^{\kappa_3}\phi_1^{\kappa_3} + \cdots\right).$$

Note the index set for it is $\Lambda_3 = \{\kappa_2 - 2, \kappa_3 - 2, \dots\}$.

♣ $\exists\bar{\nu} \equiv \bar{\nu}(\bar{\gamma}) \in \mathbf{R}^n$ s.t. $\langle\bar{\gamma}, w\rangle = \langle\bar{\nu}, \phi_1^1(w, \bar{\gamma})\rangle$ for all w.

♣ Under $a' = \tilde{y}_1^\epsilon$ ($\Longleftrightarrow a' = a' + \epsilon\phi_1^1 + r^{2,\epsilon}$), we have

$$-\frac{1}{\epsilon}\langle\bar{\gamma}, w\rangle = -\frac{1}{\epsilon}\langle\bar{\nu}, \phi_1^1\rangle = \left\langle\bar{\nu}, \frac{r^{2,\epsilon}}{\epsilon^2}\right\rangle.$$

Proposition

As $\epsilon \searrow 0$ *in* $\tilde{\mathbf{D}}_\infty(\mathbf{R}^n)$*-topology, we have*

$$\exp\Big(-\frac{1}{\epsilon}\langle\bar{\gamma}, w\rangle\Big) \text{“}\sim\text{”}\, e^{\langle\bar{\nu},\phi_1^2\rangle}\big(1+\epsilon^{\rho_1}\Xi_{\rho_1}+\epsilon^{\rho_2}\Xi_{\rho_2}+\cdots\big).$$

Here

- $0=\rho_0<\rho_1<\rho_2<\cdots$ *are elements in* Λ'_3,
- $\Xi_{\rho_i}\in\mathbf{D}_\infty$.

♣ To prove this proposition, we use (A4) to ensure integrability of $e^{\langle\bar{\nu},r^{2,\epsilon}\rangle/\epsilon^2}$ “=” $e^{(\text{quadratic Wiener functoinal})}$.

Conclusion

♣ From the above, we have

$$
\begin{aligned}
\epsilon^n \exp\Big(\frac{\|\bar{\gamma}\|_{\mathfrak{H}}^2}{2\epsilon^2}\Big) p_{\epsilon^{1/H}}(a, a')
&= \boldsymbol{E}\Big[\exp\Big(-\frac{1}{\epsilon}\langle\bar{\gamma}, w\rangle\Big)\delta_0\Big(\frac{\tilde{y}_1^\epsilon - a'}{\epsilon}\Big)\Big] \\
&= \boldsymbol{E}\Big[e^{\langle\bar{\nu},\phi_1^2\rangle}(1 + \epsilon^{\rho_1}\Xi_{\rho_1} + \epsilon^{\rho_2}\Xi_{\rho_2} + \cdots) \\
&\qquad\qquad \times \big(\delta_0(\phi_1^1) + \epsilon^{\nu_1}\Phi_{\nu_1} + \epsilon^{\nu_2}\Phi_{\nu_2} + \cdots\big)\Big] \\
&= \alpha_0 + \alpha_1\epsilon^{\lambda_1} + \alpha_2\epsilon^{\lambda_2} + \cdots .
\end{aligned}
$$

Here, $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ are elements in Λ_4.

♣ By setting $\epsilon = t^H$, we see the assertion.

Thank you for your attention.