Gaussian fluctuations for the partition functions of directed polymers.

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Summary

- In this talk, we discuss the CLT for the partition function in directed polymers.
- When the temperature is sufficiently high, the CLT was proved in [Comets-Liu 2018 JMAA].
- We proved the CLT in the whole $\mathbb{L}^2$-region.

Conjecture

$$\text{Central Limit Theorem} \Leftrightarrow \mathbb{L}^2\text{-region}$$

Setting

- $(\omega(i,x))_{i\in\mathbb{N},x\in\mathbb{Z}^d}$: I.I.D. random variables. (random weights)
- Let $\Pi_n = \{(x_i)_{i=1}^n \subset \mathbb{Z}^d \mid x_0 = 0,\ |x_i - x_{i-1}|_1 = 1\ \forall i\}$.
- Given a path $\mathbf{x}_n \in \Pi_n$,

$$H(\mathbf{x}_n) := -\sum_{i=1}^n \omega(i, x_i). \quad \text{(Hamiltonian)}$$

- For $\beta \geq 0$, we assume:

$$e^{\lambda(\beta)} := \mathbb{E} e^{\beta\omega(0,0)} < \infty.$$

- For $\beta \geq 0$ (inverse temperature),

$$W_n := (2d)^{-n} \sum_{\mathbf{x}_n \in \Pi_d} e^{-\beta H(\mathbf{x}_n) - n\lambda(\beta)}, \quad \text{(partition function)}.$$

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- Given $\mathbf{x}_n \in \Pi_n$,

$$\mathcal{G}_n^{\beta}(\mathbf{x}_n) = \frac{1}{W_n}(2d)^{-n} e^{-\beta H(\mathbf{x}_n) - n\lambda(\beta)}, \quad (\text{Gibbs measure}).$$

$\mathbb{L}^2$-region

Notation

- Let $\mathbb{P}$ and $\mathbb{E}$ be the probability measure and its expectation with respect to $(\omega(i,x))$.
- Let P_x and E_x be the probability measure and its expectation with respect to the SRW $(S_n)_{n\in\mathbb{N}}$ starting at x.

Let $\pi_d := \mathrm{P}_0(\exists n \in \mathbb{N}_{>0}$ such that $S_n = 0)$.

Proposition 1 ($d \geq 3$)

There exists $\beta_2(d) > 0$ such that:

$$\beta < \beta_2(d) \Leftrightarrow \lambda(2\beta) - 2\lambda(\beta) < \log\left(\pi_d^{-1}\right)$$

Definition 1

$\mathbb{L}^2\text{-region} = \{\beta \geq 0 |\ \beta < \beta_2(d)\}$.

The convergence of the partition function and positivity

Proposition 2 (Bolthausen '89)

- *For any* $\beta \geq 0$, *the following limit exists almost surely:*

$$\lim_{n\to\infty} W_n \ (=: W_\infty).$$

- *If* $\beta < \beta_2(d)$, *then*

$$\mathbb{P}(W_\infty > 0) = 1.$$

We will prove it later.

Main results

Theorem 3 (Comets-Liu. '18, Cosco-N. 19+)

Suppose $d \geq 3$ and $\beta < \beta_2(d)$. Then there exists $\sigma(\beta) > 0$ such that,

$$n^{\frac{d-2}{4}} \frac{W_n - W_\infty}{W_n} \overset{distr}{\Rightarrow} \mathcal{N}(0, \sigma(\beta)^2).$$

Remark

$$\lim_{\beta \to \beta_2(d)} \sigma(\beta) = \infty.$$

Main resuls

Theorem 4 (Comets-Liu. '18, Cosco-N. 19+)

Suppose $d \geq 3$ and $\beta < \beta_2(d)$. Then there exists $\sigma(\beta) > 0$ suchh that,

$$n^{\frac{d-2}{4}} \frac{W_n - W_\infty}{W_n} \stackrel{distr}{\Rightarrow} \mathcal{N}(0, \sigma(\beta)^2).$$

Corollary 1 (Cosco-N. 19+)

Suppose $d \geq 3$ and $\beta < \beta_2(d)$. Then, for the same $\sigma(\beta) > 0$,

$$n^{\frac{d-2}{4}} (\log W_n - \log W_\infty) \stackrel{distr}{\Rightarrow} \mathcal{N}(0, \sigma(\beta)^2).$$

W_n is a martingale

Let $\mathcal{F}_k = \sigma(\omega(l,x) |\ x \in \mathbb{Z}^d,\ l \leq k)$.

Proposition 5

For any $l < k$, $\mathbb{E}[W_k] = 1$ and

$$W_l = \mathbb{E}[W_k |\ \mathcal{F}_l].$$

In particular, by the martingale convergence theorem, the following limit exists,

$$\lim_{k\to\infty} W_k = W_\infty \quad a.s.$$

Properties of $\mathbb{L}^2$-region

In the $\mathbb{L}^2$ region, we can compute

$$\lim_{n\to\infty} \mathbb{E}W_n^2 = \frac{1-\pi_d}{1-\pi_d e^{\lambda(2\beta)-2\lambda(\beta)}} < \infty.$$

In particular, by the Martingale convergence theorem,

$$\mathbb{E}W_\infty = \lim_{n\to\infty} \mathbb{E}W_n = 1.$$

Proposition 6

If $\beta < \beta_2(d)$, then

$$\mathbb{P}(W_\infty > 0) = 1.$$

Proof.

By Kolgomorov's 0-1 Law, it suffices to show $\mathbb{P}(W_\infty > 0) > 0$, which follows from $\mathbb{E}W_\infty = 1$. □

High temerature region and $\mathbb{L}^2$-region

Definition 2

$\mathbb{P}(W_\infty > 0) = 1 \Leftrightarrow$ *high temperature region.*

Properties of high temperature region

- Diffusivity (Imbrie-Spencer, Bolthausen, Comets-Yoshida)
- De-localization (Comets-Shiga-Yoshida)

By the previous result,

$$\mathbb{L}^2\text{−region} \subset \text{ high temperature region.}$$

Martingale CLT

- $D_{k+1} := W_{k+1} - W_k$.
- $\mathcal{F}_k := \sigma(\omega(m,x)|\ x \in \mathbb{Z}^d,\ \ m \leq k)$.
- $\mathbb{E}_k[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_k]$.

Proposition 7

We assume that there exists $\sigma(\beta) > 0$ such that

1. $n^{\frac{d-2}{2}} \sum_{k\geq n} \mathbb{E}_k[D_{k+1}^2] \to \sigma(\beta)^2 W_\infty^2$ *in probability.*
2. $\forall \epsilon > 0,\ n^{\frac{d}{2}} \mathbb{E}[D_{n+1}^2 \mathbf{1}(n^{\frac{d-2}{4}} |D_{n+1}| > \epsilon)] \to 0.$

Then,

$$n^{(d-2)/4} \frac{W_n - W_\infty}{W_n} \overset{distr}{\Rightarrow} \mathcal{N}(0, \sigma(\beta)^2).$$

We only give the proof of ①:

$$n^{\frac{d-2}{2}}\sum_{k\geq n}\mathbb{E}_k[D_{k+1}^2]\to\sigma(\beta)^2W_\infty^2 \text{ in probability.}$$

Notations

- $\kappa(\beta)=e^{\lambda_2(\beta)}-1,\ \lambda_2(\beta)=\lambda(2\beta)-2\lambda(\beta)$.
- $e_k=e^{\beta\sum_{i=1}^k\omega(i,S_i)-k\lambda(\beta)}$.
- $\overleftarrow{W}_{k,l}^y=\mathrm{P}_y\left[\exp\left(\beta\sum_{i=1}^l\omega(k-i,S_i)-l\lambda(\beta)\right)\right]$.

Note that

$$\mathbb{E}_k[D_{k+1}^2]=\kappa_2(\beta)\sum_{x\in\mathbb{Z}^d}(\mathrm{E}[e_k\mathbf{1}_{\{S_{k+1}=x\}}])^2.$$

Let $l_k = [k^{1/3}]$. Then,

$$\sum_{k\geq n} \mathbb{E}_k[D_{k+1}^2] = \kappa_2(\beta) \sum_{k\geq n} \sum_{x\in\mathbb{Z}^d} (\mathrm{E}[e_k \mathbf{1}_{\{S_{k+1}=x\}}])^2$$
$$\approx \kappa_2(\beta) \sum_{k\geq n} \sum_{x\in\mathbb{Z}^d} W_{l_k}^2 (\overleftarrow{W}_{l_k,k+1}^x)^2 \mathrm{P}(S_{k+1}=x)^2.$$

Proposition 8 (Local Limit Theorem (Sinai, Vargas))

In the $\mathbb{L}^2$*-region, for any* $\alpha > 0$,

$$\lim_{k\to\infty} \max_{|x|\leq\alpha\sqrt{k}} \mathbb{E}\left[\left(\mathrm{E}[e_k \mid S_{k+1}=x] - W_{l_k}\overleftarrow{W}_{l_k,k+1}^x\right)^2\right] = 0.$$

$$\kappa_2(\beta) W_{l_k}^2 \sum_{k \geq n} \sum_{x \in \mathbb{Z}^d} (\overleftarrow{W}_{l_k,k+1}^x)^2 \mathrm{P}(S_{k+1} = x)^2$$

$$\approx \kappa_2(\beta) W_{l_k}^2 \sum_{k \geq n} \sum_{x \in \mathbb{Z}^d} \mathbb{E}[(\overleftarrow{W}_{l_k,k+1}^x)^2] \mathrm{P}(S_{k+1} = x)^2 \quad (\textit{homogenization})$$

$$\approx \kappa_2(\beta) W_{\infty}^2 \sum_{k \geq n} \sum_{x \in \mathbb{Z}^d} \mathbb{E}[W_{\infty}^2] \mathrm{P}(S_{k+1} = x)^2$$

$$\approx n^{-(d+1)/2} \sigma(\beta)^2 W_{\infty}^2,$$

with some $\sigma(\beta) > 0$.

$\square$

CLT for the free energy

Using the CLT for the partition function,

$$
\begin{aligned}
n^{(d-2)/4}(\log W_\infty - \log W_n) &= n^{(d-2)/4} \log W_\infty / W_n \\
&= n^{(d-2)/4} \log \left(1 + \frac{W_\infty - W_n}{W_n}\right) \\
&\approx n^{(d-2)/4} \frac{W_\infty - W_n}{W_n} \\
&\overset{distr}{\Rightarrow} \mathcal{N}(0, \sigma(\beta)^2),
\end{aligned}
$$

where we have used the Taylor expansion:

$$
\log(1+x) \approx x \quad \text{if } |x| \ll 1.
$$