Laws of the iterated logarithm on covering graphs of polynomial volume growth

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Introduction

- $\triangleright$ N : a finitely generated group.
- $\triangleright$ $\mathcal{S} = \{\gamma_1^{\pm 1}, \gamma_2^{\pm 1}, \dots, \gamma_K^{\pm 1}\}$: a symmetric generating set of N.
- $\triangleright$ $\mathcal{B}(n) := \left\{ \gamma_{i_1}^{\varepsilon_1} \cdots \gamma_{i_n}^{\varepsilon_n} \,\middle|\, \begin{array}{c} i_k = 1, 2, \dots, K, \ \varepsilon_k = \pm 1 \\ k = 1, 2, \dots, n \end{array} \right\}, \quad n \in \mathbb{N}.$
- $\triangleright$ $n \mapsto \#\mathcal{B}(n)$: the growth function of N (with $\mathcal{S}$).

Definition.

N is said to be of polynomial volume growth if

$$\#\mathcal{B}(n) \le C n^A, \qquad n \in \mathbb{N}$$

holds for some constant $C > 0$ and some integer $A \in \mathbb{N}$.

Introduction

- ▷ N : finitely generated, of polynomial volume growth
- ▷ $X_0 = (V_0, E_0)$: a finite graph
- ▷ $X = (V, E)$: a covering graph of X_0 whose covering transformation group is N ($\rightarrow$ our model)

Theorem. [Gromov ('81)]

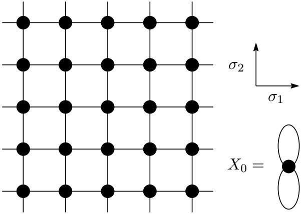
If N is a group of polynomial volume growth, then

$\exists\, \Gamma \subset N$: nilpotent normal subgroup s.t. $[N : \Gamma] < +\infty$.

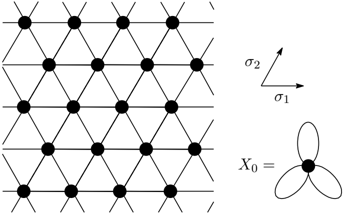
♠ By employing Gromov's theorem, we may regard X as a covering graph of a finite graph $X_0 = \Gamma \backslash X$ whose covering transformation group is Γ.

In the following, X : a nilpotent covering graph with Γ.

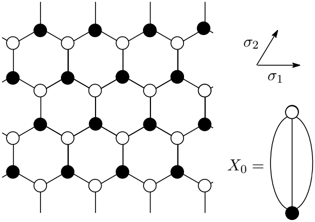
Introduction



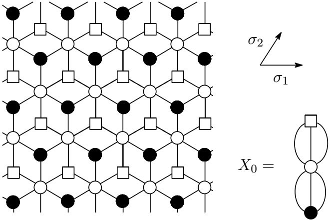
Square lattice



Triangular lattice



Hexagonal lattice



Dice lattice

Introduction

- Consider

$$N = \mathbb{H}^3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}$$

with $\mathcal{S} = \left\{ \begin{pmatrix} 1 & \pm 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pm 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \pm 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$

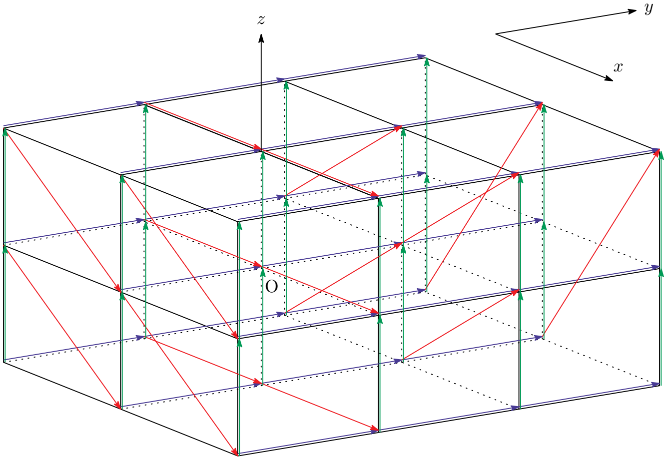
The group N is called the 3D discrete Heisenberg group.

$\longrightarrow$ the simplest example of nilpotent groups!

It is known that

$$\#\mathcal{B}(n) \leq Cn^4, \quad n \in \mathbb{N}.$$

Introduction



Problem

Interest Long time behaviors of RWs on X.

- Since any spatial scalings cannot be defined on X, we need to realize X in an appropriate continuous model periodically.

  (1) Γ : abelian $\Longrightarrow \Phi : X \longrightarrow \Gamma \otimes \mathbb{R} (\cong \mathbb{R}^d)$.

  (2) Γ : nilpotent $\Longrightarrow \Phi : X \longrightarrow \boxed{\quad ? \quad}$.

Theorem. [Malcév ('51)]

If Γ : finitely generated, nilpotent, then

$\exists G$: nilpotent Lie group s.t. $\Gamma \cong$ cocpt lattice in G.

Remark $\Gamma = \mathbb{H}^3(\mathbb{Z}) \Longrightarrow G = \mathbb{H}^3(\mathbb{R})$ (3D Heisenberg group).

Problem

♠ Large deviation principles (LDPs) on X have been discussed by a few authors in a geometric point of view.

▷ [Baldi & Caramellino ('99)] : LDP on nilpotent Lie groups.

▷ [Kotani & Sunada ('06)] LDP in the case of crystal lattices.

▷ [Tanaka ('11)] LDP in the case of nilpotent covering graphs.

$$\mathbb{P}_x\Big(\tau_{1/n}\big(\underbrace{\Phi(w_n)\cdot\rho^{-n}}_{\text{centered RW on } G}\big)\in A\Big) \underset{n\to\infty}{\sim} \exp\Big(-n\inf_{g\in A} I_L(g)\Big).$$

Problem

♠ Moderate deviation principles (MDPs)?

$$\mathbb{P}_x\Big(\tau_{1/a_n}\big(\Phi(w_n)\cdot\rho^{-n}\big)\in A\Big) \underset{n\to\infty}{\sim} \exp\Big(-\frac{a_n^2}{n}\inf_{g\in A} I_M(g)\Big),$$

where $\{a_n\}_{n=1}^{\infty}\subset(0,\infty)$ satisfies

$$\lim_{n\to\infty}\frac{a_n}{\sqrt{n}}=+\infty \quad \text{and} \quad \lim_{n\to\infty}\frac{a_n}{n}=0.$$

♠ MDPs deal with what occurs at any intermediate scalings between n (LLN-type) and $\sqrt{n}$ (CLT-type).

(Example.) a scaling for laws of the iterated logarithm (LILs):

$$\sqrt{n}\ll b_n:=\sqrt{n\log\log n}\ll n.$$

$\longrightarrow$ LILs on X (by applying MDPs with $\{b_n\}_{n=1}^{\infty}$)?

RW on X

- ▷ $p : E \longrightarrow (0,1)$: Γ-invariant transition probability, i.e.,

$$p(\gamma e) = p(e), \quad \gamma \in \Gamma,\, e \in E.$$

This induces an RW on X : $(\Omega_x(X), \mathbb{P}_x, \{w_n\}_{n=0}^{\infty})$.

- ▷ $m : V_0 \longrightarrow (0,1]$: normalized invariant measure on V_0.

- ▷ $\Phi : X \longrightarrow G$: periodic realization of X.
- ▷ The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ satisfies

$$\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}^{(i)}; \quad [\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \begin{cases} \subset \mathfrak{g}^{(i+j)} & (i+j \leq r), \\ = \{0_{\mathfrak{g}}\} & (i+j > r), \end{cases}$$

and $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(i)}]$ $(i = 1, \ldots, r-1)$.

Nilpotent Lie groups

▷ global coordinates of G (through $\exp : \mathfrak{g} \longrightarrow G$):

$$G \ni g \longleftrightarrow (x^{(1)}, x^{(2)}, \dots, x^{(r)}) \in \mathbb{R}^M,$$

where $M = \sum_{k=1}^{r} \dim \mathfrak{g}^{(k)}$.

▷ dilations (scalar multiplications on G): for $\varepsilon > 0$,

$$\tau_\varepsilon(x^{(1)}, x^{(2)}, \dots, x^{(r)}) := (\varepsilon x^{(1)}, \varepsilon^2 x^{(2)}, \dots, \varepsilon^r x^{(r)}).$$

RW on $\mathfrak{g}^{(1)}$

- ▷ $\Phi : X \longrightarrow G$: Γ-periodic realization.
- ▷ $\xi_n := \Phi(w_n)\ (n = 0, 1, 2, \dots)$: RW on G.
- ▷ $\Xi_n := \log \Phi(w_n)|_{\mathfrak{g}^{(1)}}\ (n = 0, 1, 2, \dots)$: RW on $\mathfrak{g}^{(1)}$.

♠ **LLN** on $\mathfrak{g}^{(1)}$:

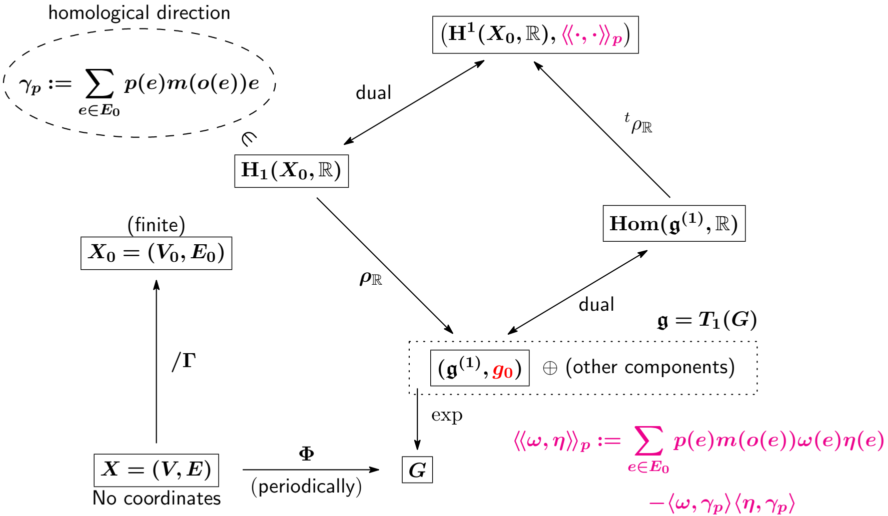
$$\lim_{n\to\infty} \frac{1}{n}\Xi_n = \rho_{\mathbb{R}}(\gamma_p) \quad \mathbb{P}_x\text{-a.s.}$$

♠ For $i, j = 1, 2, \dots, d$, we have

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}^x\Big[(\Xi_n - \rho_{\mathbb{R}}(\gamma_p))_i(\Xi_n - \rho_{\mathbb{R}}(\gamma_p))_j\Big] = \langle\!\langle \omega_i, \omega_j \rangle\!\rangle_p,$$

where $\{\omega_1, \omega_2, \dots, \omega_d\} \subset \mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R})$: a fixed basis.

Construction of the Albanese metric on $\mathfrak{g}^{(1)}$



MDP on a path space

For $n = 1, 2, \ldots$ and $0 \leq t \leq 1$, we define

$$Z_t^{(n)} := \frac{1}{a_n}(\Xi_{[nt]} - n\rho_{\mathbb{R}}(\gamma_p)) + \frac{nt - [nt]}{a_n}(\Xi_{[nt]+1} - \Xi_{[nt]} - \rho_{\mathbb{R}}(\gamma_p)).$$

Proposition (N. '19)

The sequence of $\mathrm{AC}_0([0,1], \mathfrak{g}^{(1)})$-valued r.v.'s $\{Z^{(n)}\}_{n=1}^{\infty}$ satisfies an MDP on $\mathrm{AC}_0([0,1], \mathfrak{g}^{(1)})$ with the good rate function

$$I(h) := \int_0^1 \alpha^*(\dot{h}(t))\, dt, \quad h \in \mathrm{AC}_0([0,1], \mathfrak{g}^{(1)}),$$

where

$$\alpha^*(\lambda) = \frac{1}{2}\langle \Sigma^{-1}\lambda, \lambda \rangle, \quad \lambda \in \mathfrak{g}^{(1)}.$$

▷ $\Sigma = \left(\langle\langle \omega_i, \omega_j \rangle\rangle_p\right)_{i,j=1}^d \longleftrightarrow \Sigma^{-1} = \left(\langle X_i, X_j \rangle_{g_0}\right)_{i,j=1}^d$

From $\mathrm{AC}_0([0,1],\mathfrak{g}^{(1)})$ to G

▷ For $n = 1, 2, \dots,$ we set

$$\overline{\zeta}_n := \exp(Z_1^{(n)}) = \tau_{1/a_n}\Big(\exp\big(\Xi_n - n\rho_{\mathbb{R}}(\gamma_p)\big)\Big).$$

♠ By the contraction principle, the sequence $\{\overline{\zeta}_n\}_{n=1}^{\infty}$ satisfies an MDP with a good rate function $I_M : G \longrightarrow [0,\infty]$ defined by

$$I_M(g) := \inf\big\{I(h) \mid \exp(h_1) = g,\ h \in \mathrm{AC}_0([0,1],\mathfrak{g}^{(1)})\big\}.$$

♠ However, our target process is $\{\overline{\xi}_n\}_{n=1}^{\infty}$ given by

$$\overline{\xi}_n := \tau_{1/a_n}\Big(\xi_n \cdot \exp(-n\rho_{\mathbb{R}}(\gamma_p))\Big).$$

$\longrightarrow$ $\{\overline{\zeta}_n\}_{n=1}^{\infty}$ and $\{\overline{\xi}_n\}_{n=1}^{\infty}$ are very "close" due to

$$\log(\overline{\zeta}_n)\big|_{\mathfrak{g}^{(1)}} = \log(\overline{\xi}_n)\big|_{\mathfrak{g}^{(1)}},\ \ n \in \mathbb{N}.$$

MDP on X

▷ The transfer lemma now implies the desired MDPs on X.

Theorem. (N, '19)

The sequence of G-valued r.v.'s $\{\overline{\xi}_n\}_{n=0}^{\infty}$ satisfies an MDP with the rate a_n^2/n and a good rate function $I_M : G \longrightarrow [0, \infty]$. Namely,

$$-\inf_{g \in A^\circ} I_M(g) \leq \varliminf_{n \to \infty} \frac{n}{a_n^2} \log \mathbb{P}_x\Big(\overline{\xi}_n \in A\Big) \leq \varlimsup_{n \to \infty} \frac{n}{a_n^2} \log \mathbb{P}_x\Big(\overline{\xi}_n \in A\Big) \leq -\inf_{g \in \overline{A}} I_M(g)$$

for $A \in \mathcal{B}(G)$.

Application of MDPs to LILs

- We aim to show LILs on X by applying MDPs for

$$b_n := \sqrt{n \log \log n}, \quad n = 1, 2, \ldots .$$

Related works

- [Crépel–Roynette ('77)] : LILs on $\mathbb{H}^3(\mathbb{R})$.
- [Caramellino–Vincenzo ('01)] : LILs on nilpotent Lie groups.

- However, LILs on nilpotent covering graphs have not been obtained (even in the case of crystal lattices!).
- We state LILs on X by characterizing the set of all $\mathbb{P}_x$-a.s. limit points of

$$\overline{\xi}_n = \tau_{1/b_n}\big(\xi_n \cdot \exp(-n\rho_{\mathbb{R}}(\gamma_p))\big), \quad n = 1, 2, \ldots$$

as $n \to \infty$.

Application of MDPs to LILs

♠ We can show that

$$K := \{h \in \mathrm{AC}_0([0,1]; \mathfrak{g}^{(1)}) \mid I(h) \leq 1\}$$
$$= \{\mathbb{P}_x\text{-a.s. limit points of } \{Z^{(n)}\}_{n=1}^{\infty}\}.$$

$\longrightarrow$ Key : $\lim_{n\to\infty} \mathrm{dist}(Z^{(n)}, K) = 0$, $\mathbb{P}_x$-a.s.

▷ $K_\varepsilon := \{h \in \mathrm{AC}_0([0,1]; \mathfrak{g}^{(1)}) \mid \mathrm{dist}(h, K) \geq \varepsilon\}, \quad \varepsilon > 0.$

▷ Since K is cpt and I is lower-semiconti., we know

$$\exists\, \delta = \delta(\varepsilon) > 0 \text{ s.t. } \inf_{h \in K_\varepsilon} I(h) > 1 + \delta.$$

Application of MDPs to LILs

♠ Then one has

$$\sum_{m=1}^{\infty} \mathbb{P}_x\Big(\mathrm{dist}(Z^{(2^m)}, K) > \varepsilon\Big) = \sum_{m=1}^{\infty} \mathbb{P}_x(Z^{(2^m)} \in K_\varepsilon)$$
$$\leq \sum_{m=1}^{\infty} e^{-(1+\delta)\log\log 2^m}$$
$$= \frac{1}{(\log 2)^{1+\delta}} \sum_{m=1}^{\infty} \frac{1}{m^{1+\delta}} < \infty,$$

where we used the upper estimate of MDP for the 2nd line.

♠ The Borel–Cantelli lemma leads to the desired a.s. convergence.

Main theorem

- It follows from the continuity of $\exp : \mathfrak{g} \longrightarrow G$ that

$$\mathcal{K} := \{g \in G \mid I_M(g) \leq 1\}$$
$$= \{\mathbb{P}_x\text{-a.s. limit points of } \{\overline{\zeta}_n = \exp(Z_1^{(n)})\}_{n=1}^{\infty}\}.$$

- Since $\{\overline{\zeta}_n\}_{n=1}^{\infty}$ and $\{\overline{\xi}_n\}_{n=1}^{\infty}$ are very "close", we obtain

Theorem. (N, '19)

Let $\overline{\mathcal{K}}$ be the set of all $\mathbb{P}_x$-a.s. limit points of

$$\overline{\xi}_n = \tau_{1/\sqrt{n \log \log n}}\big(\xi_n \cdot \exp(-n\rho_{\mathbb{R}}(\gamma_p))\big), \quad n = 1, 2, \ldots.$$

Then we obtain

$$\overline{\mathcal{K}} = \{g \in G \mid I_M(g) \leq 1\}.$$