Random walks in random environment as rough paths

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Approximating SDEs

- SDEs often justified by universality of Brownian motion:
  X random walk,

  $$Y_{t+1} = Y_t + n^{-1} f(Y_t)(X_{t+1} - X_t),$$

  then $Y_{n^2 t} \to dZ_t = f(Z_t) dW_t$.
- Generally: Z^n, Z semimartingales with $Z^n \to Z$.
  Do we have $dY_t^n = f(Y_{t-}^n) dZ_t^n \to dY_t = f(Y_{t-}) dZ_t$?
- Answer: NO! *Uniform convergence not sufficient.*
- Maybe if $Z^n \to Z$ in stronger topology?

(In-)stability of integration

- "Best" stability result: using Young integral:
  $Z^n \to Z$ in $C^\alpha([0,T])$ for $\alpha > 1/2$, implies

$$dY^n = f(Y^n_t)dZ^n_t \to dY_t = f(Y_t)dZ_t.$$

- But Brownian motion is in $C^{\frac{1}{2}-}([0,T]) \setminus C^{\frac{1}{2}}([0,T])$ a.s.!
- A negative result Lyons '91:
  There exists no Banach space $\mathcal{X} \subset \mathbb{R}^{[0,T]}$ s.t.
  (i) *Brownian motion a.s. in $\mathcal{X}$;*
  (ii) *$C^\infty([0,T]) \subset \mathcal{X}$ dense;*
  (iii) *$\mathcal{X}^2 \ni (f,g) \mapsto \int_0^T f(s)dg(s) \in \mathbb{R}$ extended continuously.*
- Positive development: stability if we have extra information - iterated integral.

Rough paths

Definition (Lyons '98)

Let $\alpha \in (\frac{1}{3}, \frac{1}{2})$. A α-Hölder rough path is a pair $(Z, \mathbb{Z})$ defined by increments $(Z_{s,t}, \mathbb{Z}_{s,t}) \in \mathbb{R}^d \times \mathbb{R}^{d \otimes d}$ for $0 \leq s < t \leq T$ so that

(i) $Z \in C^\alpha$, $\mathbb{Z} \in C^{2\alpha}$ and

(ii) $\mathbb{Z}_{s,t} - \mathbb{Z}_{s,u} - \mathbb{Z}_{u,t} = Z_{s,u} \otimes Z_{u,t}$ ("Chen's relation").

Morally: $\mathbb{Z}_{s,t} = \int_s^t \int_s^{r_1} dZ_{r_2} \otimes dZ_{r_1}$.

Theorem (Lyons '98, Gubinelli '04)

rough integral $(Y, Y', Z, \mathbb{Z}) \mapsto \int_0^\cdot Y_s dZ_s$ *is continuous (if* Y *controlled);*
Itô-Lyons map $(Z, \mathbb{Z}) \mapsto Y_t = Y_0 + \int_0^t f(Y_s) dZ_s$ *is continuous.*

In particular, if $(Z^n, \mathbb{Z}^n) \to (Z, \mathbb{Z})$ in rough path topology then

$$Y_t^n = Y_0^n + \int_0^t f(Y_s^n) dZ_s^n \to Y_t = Y_0 + \int_0^t f(Y_s) dZ_s.$$

Program. Functional CLT for RWRE in rough path topology

Convergence of RWRE in rough path topology?

- Invariance principles of random walks in random environment were established in a few interesting classes.
- Goal: convergence of the lift $(X^n, \mathbb{X}^n)$ of the RWRE in the rough path topology.
- Program which has various aspects:
  - S(P)DEs: what are the effects on the solutions of non-trivial approximations of the noise?
  - RWRE / particle systems: richer picture of the structure of the model on large scales: homogenization theory, area anomaly.

Random walks in random environment

Random walks in random environment on $\mathbb{Z}^d$

- Environment $\omega \in \Omega$ by $\begin{cases} \omega_x(y) \geq 0, \\ \sum_{y:|x-y|=1} \omega_x(y) = 1 \end{cases}$ for every $x \in \mathbb{Z}^d$.
- For fixed ω, let $(X_n)_{n\geq 0}$ be a Markov chain on $\mathbb{Z}^d$ starting at the origin, with

$$P_\omega(X_{n+1} = y | X_n = x) = \omega_x(y), \text{ for every } |x-y| = 1, \ n \geq 0.$$

  $P_\omega(\cdot)$ is called the **quenched** law.
- For a probability P on Ω the **annealed** law $\mathbb{P}$ on the random walk is

$$\mathbb{P}(\cdot) = \int_\Omega P_\omega(\cdot) \mathrm{d}P(\omega).$$

Important classes

Ballistic RWRE: the measure P on Ω satisfies

- $(\omega_x)_{x\in\mathbb{Z}^d}$ is i.i.d. and uniformly elliptic ($\mathbb{P}(\kappa \leq \omega \leq 1-\kappa) = 1$ for some $\kappa > 0$) and
- Sznitman's T' ballisticity condition holds (ballisticity: $\frac{X_n}{n} \to v \neq 0$ $\mathbb{P}$-a.s).

Random conductance model:

- environment coming from i.i.d. uniformly bounded weights (conductances) on edges of $\mathbb{Z}^d$.
- reversibility, no ballisticity: $v = 0$.

Regeneration structure, strong LLN, functional CLT

Sznitman-Zerner '99:

- Regeneration structure. $\mathbb{P}$- a.s. $\exists$ times $0 =: \tau_0 < \tau_1 < \tau_2 < \ldots < \infty$ so that

$$\left(\{X_{\tau_k,\tau_k+m}\}_{0\le m\le \tau_{k+1}-\tau_k}, \tau_{k+1}-\tau_k\right)_{k\ge 1} \text{ are i.i.d.,}$$

- Strong LLN. $\frac{X_n}{n} \to \frac{\mathbb{E}[X_{\tau_1,\tau_2}]}{\mathbb{E}[\tau_2-\tau_1]} =: v$ $\mathbb{P}$-a.s.

Sznitman '00:

- Functional CLT.

$$X^n(\cdot) \Rightarrow B \text{ in } C([0,T],\mathbb{R}^d),$$

a Brownian motion with covariance $\Sigma^2_{i,j} = \frac{\mathbb{E}[\bar{X}^i_{\tau_1,\tau_2}\bar{X}^j_{\tau_1,\tau_2}]}{\mathbb{E}[\tau_2-\tau_1]}$ under $\mathbb{P}$. With the recentering & rescaling

$$X^n(t) := \frac{1}{n}(\bar{X}_{\lfloor n^2 t\rfloor} + (n^2 t - \lfloor n^2 t\rfloor)(\bar{X}_{\lfloor n^2 t\rfloor+1} - \bar{X}_{\lfloor n^2 t\rfloor}), \quad \bar{X}_n := X_n - nv.$$

Homogenization

- Although the walk is non-homogeneous, on large scales walk feels the environment in an “averaged” sense: its fluctuations have a (Gaussian) limit.
- Covariance is different than in homogenous case.

Ballistic RWRE as rough paths

Ballistic RWRE: convergence & area anomaly

$(X^n{}_t, \mathbb{X}^n_{s,t})$ - rescaled & recentred linearly interpolated lift.

Theorem (Lopusanschi - O '18)

Ballistic RWRE, $d \geq 2$. Under $\mathbb{P}$:

$$(X^n, \mathbb{X}^n) \Rightarrow (B, \mathbb{B}^{Str} + \Gamma \cdot)$$

in the α-Hölder rough path topology, for every $\alpha < 1/2$.

- *B - Brownian motion, $\mathbb{B}^{Str}$ - its iterated integral w.r.t. Stratonovich integration*
- *Γ is a deterministic $d \times d$ matrix with the explicit form*

$$\Gamma = \frac{\mathbb{E}[\mathrm{Antisym}(\bar{\mathbb{X}}_{\tau_1,\tau_2})]}{\mathbb{E}[\tau_2 - \tau_1]},$$

where $\mathrm{Antisym}(\bar{\mathbb{X}}_{s,t})^{i,j} = \frac{1}{2}(\bar{\mathbb{X}}^{i,j}_{s,t} - \bar{\mathbb{X}}^{j,i}_{s,t})$.

Ballistic RWRE: remarks

$$\Gamma = \frac{\mathbb{E}[\mathrm{Antisym}(\bar{\mathbb{X}}_{\tau_1,\tau_2})]}{\mathbb{E}[\tau_2 - \tau_1]}.$$

- The solution to an SDE approximated by $\bar{X}$ converges to a solution of an SDE with an explicit correction in terms of Γ.
- Area anomaly has a geometrical interpretation: Γ is the expected signed stochastic area of $\bar{X}$, the re-centered walk, on a regeneration interval, normalized by its expected size.

Important classes

Ballistic RWRE: the measure P on Ω satisfies

- $(\omega_x)_{x\in\mathbb{Z}^d}$ is i.i.d. and uniformly elliptic and
- Sznitman's T' ballisticity condition holds.

Random conductance model:

- environment coming from i.i.d. uniformly bounded weights (conductances) on edges of $\mathbb{Z}^d$.
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Random walk in random conductances

- Random conductances:
  $\{\omega_{x,y} = \omega_{y,x} : x, y \in \mathbb{Z}^d, |x - y| = 1\}$ *i.i.d. with values in* $[c, C]$*;*
  for fixed ω
  $$\mathbb{P}(X_{n+1} = y | X_n = x) = \frac{\omega_{x,y}}{\sum_z \omega_{x,z}}.$$
- *Functional CLT:* $X^n(t) = n^{-1} X_{n^2 t}$. *Künnemann '83:*
  Under $\mathbb{P}$*:* X^n *converges weakly in* $D([0, T], \mathbb{R}^d)$ *to a Brownian motion with a covariance matrix* $\Sigma = \Sigma(\mathrm{law}(\omega))$.
- Again, homogenization: on large scales walk feels the environment in an "averaged" sense, covariance is different than in homogenous case.

Random conductances and rough paths

Two strategies to tackle random conductances

1. Developing a Kipnis-Varadhan additive functionals theory in the rough path topology.
2. Lifting to the rough path space the abstract process from the point of view of the environment decomposed according to the harmonic and corrector coordinates of the cocycle space from homogenization theory.

Strategy 1. Additive functionals of Markov processes as rough paths

Kipnis-Varadhan theory in rough path topology

Theorem (Deuschel - O - Perkowski '19)

X Markov with generator $\mathcal{L}$, μ stationary and ergodic for $\mathcal{L}$,$\mathcal{L}^*$.
$F : E \to \mathbb{R}^d$ bounded and measurable with $\int F d\mu = 0$ and
$Z^n_t = n^{-1} \int_0^{n^2 t} F(X_s) ds$. *Assume $\mathcal{H}^{-1}$ condition. Then*

$$(Z^n, \mathbb{Z}^n) \to \left(B, \mathbb{B}^{Str} + \cdot \lim_{\lambda \to 0} \mathbb{E}[\Phi_\lambda \otimes \mathcal{L}_A \Phi_\lambda] \right)$$

in (p-variation) rough path topology (for all $p > 2$)), where B is a Brownian motion with covariance

$$\langle B, B \rangle_t = 2t \lim_{\lambda \to 0} \mathbb{E}[\Phi_\lambda \otimes (-\mathcal{L}_S) \Phi_\lambda].$$

Note: correction vanishes if $\mathcal{L} = \mathcal{L}^*$.

$\mathcal{H}^{-1}$ condition. For $(\lambda - \mathcal{L})\Phi_\lambda = F$: $\lambda \int |\Phi_\lambda|^2 d\mu + \int (\Phi_\lambda - \Phi_{\lambda'}) \otimes (-\mathcal{L})(\Phi_\lambda - \Phi_{\lambda'}) d\mu \to 0$.

Application to random conductances

Convergence in rough path topology and area anomaly

Theorem (Deuschel - O - Perkowski '19)

For the Itô lift $(X^n, \mathbb{X}^n)$ *we have a rough path convergence*

$$(X^n, \mathbb{X}^n) \to \left(B, \int_0^{\cdot} B_s \otimes dB_s + \frac{1}{2}\langle B, B \rangle_{\cdot} - \frac{1}{2}\langle B^M, B^M \rangle_{\cdot} \right)$$

- Started from Itô, so would not expect Stratonovich integrals.
- For linear interpolations we get converges to Stratonovich (without anomaly!).

Strategy 2. Cocycle space and the corrector process

Cocycle spaces - quenched FCLT in a nutshell

Progress of last two decades:

- Can find an Hilbert space H of cocycle functions on $\mathbb{Z}^d \times \Omega$ containing the projection $\Pi(x, \omega) = x$, with a decomposition $H = M \oplus G$. Here G is a closure of gradient functions.
- In particular, for $\Pi = \Psi + \chi$,

$$X_t = \Psi(X_t, \omega) + \chi(X_t, \omega)$$

  is a decomposition to a martingale (the harmonic process) plus a corrector.
- Analytical tools $\rightsquigarrow$ corrector is sublinear $\rightsquigarrow$ vanishes in limit (e.g. Sidoravicius and Sznitman '04).
- Deduce quenched CLT for X from martingale CLT for $\Psi(X_t, \omega)$.

Rough path treatment

- Consider the decomposition

$$X_t = \Psi(X_t, \omega) + \chi(X_t, \omega)$$

  to the harmonic process and the corrector.
- With Martin Slowik: we have seen the corrector $\chi(X_t, \omega)$ vanishes in limit. We show: its iterated integral converges to a non-zero linear function, explicitly,

$$(\text{time}) \times (L^2_{\mathrm{cov}}(\mathbb{P}) \text{ norm the corrector}).$$

- In fact, this is exactly the area anomaly we got using rough path Kipnis-Varadhan.

On-going projects and summary

Related on-going projects

- Reversible case:
  - Quenched, RWs in random conductances using gradient spaces. Hard point - quenched convergence of the corrector's iterated integral. On-going with Jean-Dominique Deuschel, Nicolas Perkowski and Martin Slowik.
- Non-reversible case:
  - Quenched analog for rough FCLT for random walks in ballistic random environment. On-going with Noam Berger (TU Munich).
  - Quenched result for additive functionals assuming better approximating rate of resolvent - with Johannes Bäumler (currently a Master student with Noam Berger at TU Munich).

Summary

- Goal: Rough invariance principle for "noise".
    - Approximation of SDEs.
    - Richer understanding of path structure in RWRE models.
- Ballistic RWRE (non-reversible case).
    - Identification of area anomaly in terms of a stochastic area in regeneration interval.
- Developed a Kipnis-Varadhan theory in rough path topology.
    - Get no area anomaly if the process is reversible.
    - Application to random conductances: canonical limit for linear interpolations, correction for Itô rough path.
    - Method extends to many other models, not necessary reversible, e.g. periodic diffusions.
- Cycycle space and corrector for random conductances.
    - Area anomaly in Itô case, identified as the $L^2_{\mathrm{cov}}(\mathbb{P})$ norm the corrector.
    - Potential to be extended to the much stronger quenched measure.

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Thank you!