Diffusion in Coulomb environment and a phase transition.

2019/9/3/Tue: Fukuoka University

Japanese-German Open Conference on Stochastic Analysis 2019

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- Homogenization in periodic Coulomb environments $d \geq 2$.
- A phase transition of effective constant in two dimensions.
- Homogenization in periodic Coulomb environments $d \geq 3$.

Coulomb potentials in $\mathbb{R}^d$

- Let $d \geq 2$ and $\sigma(d)$ be the surface volume of the unit ball:

$$\sigma(d) = 2\pi^{d/2}/\Gamma(d/2).$$

- Let Ψ_d be the $\frac{\sigma(d)}{2}$ times fundamental sol of $-\frac{1}{2}\Delta$ in $\mathbb{R}^d$:

$$\Psi_d(x) = \begin{cases} \frac{1}{d-2}|x|^{2-d} & (d \geq 3) \\ -\log|x| & (d = 2) \end{cases}$$

- We call Ψ the d-dimensional Coulomb potential.
- The Coulomb force given by Ψ is then

$$\nabla\Psi_d(x) = -\frac{x}{|x|^d}.$$

Lattice:

- For $\mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathbb{R}^d$ we define the d-dim lattice $\mathbb{L}$ and torus $\mathbb{T}$:

$$\mathbb{L} = \{\sum_{i=1}^{d} n_i \mathbf{v}_i \,;\, n_i \in \mathbb{Z} \ (i = 1, \ldots, d)\}, \tag{1}$$

$$\mathbb{T} = \{\sum_{i=1}^{d} t_i \mathbf{v}_i \,;\, t_i \in [0, 1) \ (i = 1, \ldots, d)\}.$$

- We take $\mathbf{v}_i$ such that $|\mathbb{T}| = 1$.

Total Coulomb force

- We put one particle with unit charge at **each** site on the lattice $\mathbb{L}$.
- The total Coulomb force acting at $x \in \mathbb{R}^d$ is then

$$\mathrm{b}(x) = \lim_{q\to\infty} \sum_{\substack{|x-s_i|<q,\\ s_i\in\mathbb{L}}} -\nabla\Psi_d(x-s_i) = \lim_{q\to\infty} \sum_{\substack{|x-s_i|<q,\\ s_i\in\mathbb{L}}} \frac{x-s_i}{|x-s_i|^d}$$

- b is a periodic function with singularity at each $s_i \in \mathbb{L}$.

- More precisely,

$$\mathrm{b}(x) = \lim_{q\to\infty} \sum_{s_i\in\mathbb{L}} \varphi_q(x-s_i)\frac{x-s_i}{|x-s_i|^d}. \qquad (2)$$

Here $\varphi_q(x) = \varphi(x/q)$, $0 \le \varphi \le 1$, $\varphi(x) = \varphi(|x|)$, and $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$\varphi(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| \ge 2. \end{cases}$$

Homogenization of diffusion in periodic Coulomb environment

- We put a particle X_t with the same charge as each site s_i.
- For $\mathsf{x} \in \mathbb{R}^d$, let $X_t^{\mathsf{x}} \in \mathbb{R}^d$ be the solution of

$$dX_t^{\mathsf{x}} = dB_t + \frac{\beta}{2} \lim_{q\to\infty} \sum_{\substack{|X_t^{\mathsf{x}}-s_i|<q \\ s_i\in\mathbb{L}}} \frac{X_t^{\mathsf{x}} - s_i}{|X_t^{\mathsf{x}} - s_i|^d} dt, \qquad X_0^{\mathsf{x}} = \mathsf{x}.$$

Here B is d-dimensional Brownian motion, β is inverse temperature.

Lem 1. *There exists a symmetric matrix $\gamma_{\mathsf{eff}}^{\beta}$ such that for all x*

$$\lim_{\varepsilon\to\infty} \varepsilon X_{t/\varepsilon^2}^{\mathsf{x}} = \sqrt{\gamma_{\mathsf{eff}}^{\beta}} B_t \quad \textit{weakly in } C([0,\infty);\mathbb{R}^d) \tag{3}$$

$$0 < \gamma_{\mathsf{eff}}^{\beta} < E \textit{ for all } \beta \geq 0.$$

- The constant matrix $\gamma_{\mathsf{eff}}^{\beta}$ is called effective conductivity.
- $\gamma_{\mathsf{eff}}^{\beta}$ is given by a sol of *Poisson* equation and variational formula.
- $0 < \gamma_{\mathsf{eff}}^{\beta}$ follows from comparison with periodic homogenization of reflecting Brownian m.

Homogenization of diffusion in periodic Coulomb environment

- We assume that $\gamma_{\mathrm{eff}}^{\beta}$ is a scaler matrix.
- If $\mathbb{L}$ is a d-dim cubic lattice or the triangular lattice in $d = 2$, then this is the case.

- We remove m-particle $t_1, \ldots, t_m$ from $\mathbb{L}$.
- Let $\mathbb{L}_\diamond$ be the **defect** lattice:

$$\mathbb{L}_\diamond = \mathbb{L} \backslash \{t_1, \ldots, t_m\}.$$

- For $\mathsf{x} \in \mathbb{T}$, let $Y_t^{\mathsf{x}} \in \mathbb{R}^d$ be the solution of

$$dY_t^{\mathsf{x}} = dB_t + \frac{\beta}{2} \lim_{q \to \infty} \sum_{\substack{|Y_t^{\mathsf{x}} - s_i| < q \\ s_i \in \mathbb{L}_\diamond}} \frac{Y_t^{\mathsf{x}} - s_i}{|Y_t^{\mathsf{x}} - s_i|^2} dt, \quad Y_0^{\mathsf{x}} = \mathsf{x}.$$

- Simulation!!

Homogenization of diffusion in a defect lattice: a phase transition

- We remove m-particle $t_1, \ldots, t_m$ from $\mathbb{L}$.
- Let $\mathbb{L}_\diamond$ be the **defect** lattice:

$$\mathbb{L}_\diamond = \mathbb{L} \backslash \{t_1, \ldots, t_m\}.$$

- For $\mathsf{x} \in \mathbb{T}$, let $Y_t^{\mathsf{x}} \in \mathbb{R}^d$ be the solution of

$$dY_t^{\mathsf{x}} = dB_t + \frac{\beta}{2} \lim_{q \to \infty} \sum_{\substack{|Y_t^{\mathsf{x}} - s_i| < q \\ s_i \in \mathbb{L}_\diamond}} \frac{Y_t^{\mathsf{x}} - s_i}{|Y_t^{\mathsf{x}} - s_i|^2} dt, \qquad Y_0^{\mathsf{x}} = \mathsf{x}.$$

Thm 1 (a phase transition). *Assume $d = 2$. Let*

$$\gamma_0(\beta) = \mathrm{trace}(\gamma_{\mathrm{eff}}^{\beta}).$$

Then $\gamma_0(\beta)/m$ is a critical point in the following sense:

$$\lim_{\varepsilon \to \infty} \varepsilon Y_{t/\varepsilon^2}^{\mathsf{x}} = \begin{cases} \textit{not } 0 & \textit{if } \beta < \gamma_0(\beta)/m \\ 0 & \textit{if } \beta \geq \gamma_0(\beta)/m. \end{cases} \tag{4}$$

Moreover, $0 < \gamma_0(\beta) < 2$.

- $\gamma_0(0) = 2$. So if $\gamma_0(\beta)$ is strictly decreasing in β, then these exists a unique $\gamma(\mathbb{L})$ such that $\gamma(\mathbb{L}) = \gamma_0(\beta)$.
- $\gamma(\mathbb{L})$ depends only on the lattice.

Homogenization of diffusion in Coulomb environment

- In the previous theorem, the limit dynamics starts at the origin. We consider a different type of initial condition. That is,

$$\mathsf{x} \neq 0.$$

- We also specify the limit SDE.
- For this it is convenient to rewrite $X^{\varepsilon,\mathsf{x}}$ and $Y^{\varepsilon,\mathsf{y}}$ as follows:

$$X_t^{\varepsilon,\mathsf{x}} = \varepsilon X_{t/\varepsilon^2}^{\mathsf{x}/\varepsilon}, \quad Y_t^{\varepsilon,\mathsf{y}} = \varepsilon Y_{t/\varepsilon^2}^{\mathsf{x}/\varepsilon}$$

Then

$$dX_t^{\varepsilon,\mathsf{x}} = dB_t + \frac{\beta}{2} \lim_{q\to\infty} \sum_{\substack{|X_t^{\varepsilon,\mathsf{x}} - \varepsilon s_i| < q \\ s_i \in \mathbb{L}}} \frac{X_t^{\varepsilon,\mathsf{x}} - \varepsilon s_i}{|X_t^{\varepsilon,\mathsf{x}} - \varepsilon s_i|^2} dt, \quad X_0^{\varepsilon,\mathsf{x}} = \mathsf{x}$$

$$dY_t^{\varepsilon,\mathsf{y}_\varepsilon} = dB_t + \frac{\beta}{2} \lim_{q\to\infty} \sum_{\substack{|Y_t^{\varepsilon,\mathsf{y}_\varepsilon} - \varepsilon s_i| < q \\ s_i \in \mathbb{L}_\diamond}} \frac{Y_t^{\varepsilon,\mathsf{y}_\varepsilon} - \varepsilon s_i}{|Y_t^{\varepsilon,\mathsf{y}_\varepsilon} - \varepsilon s_i|^2} dt, \quad Y_0^{\varepsilon,\mathsf{y}_\varepsilon} = \mathsf{y}_\varepsilon$$

Homogenization of diffusion in Coulomb environment

- Consider a subsidiary SDE in $\mathbb{R}^2$:

$$dU_t^{\mathsf{y}} = \sqrt{\gamma_{\mathsf{eff}}^{\beta}}\, dB_t - \frac{m\beta}{2}\frac{U_t^{\mathsf{y}}}{|U_t^{\mathsf{y}}|^2}dt, \quad U_0^{\mathsf{y}} = \mathsf{y}. \tag{5}$$

Thm 2. *Let $d = 2$. Assume initial starting points y_ε satisfy*

$$\lim_{\varepsilon\to 0} \mathsf{y}_\varepsilon = \mathsf{y} \in \mathbb{R}^2.$$

Then

$$\lim_{\epsilon\to 0} Y^{\varepsilon,\mathsf{y}_\varepsilon} = U^{\mathsf{y}}. \tag{6}$$

Let $\sigma = \inf\{t > 0; U_t^{\mathsf{y}} = 0\}$. Then

$$P(\sigma < \infty) = 1 \quad \textit{for all } \beta > 0.$$

Furthermore,

$$P(U_t^{\mathsf{y}} = 0 \textit{ for all } t \geq \sigma) = 1 \quad \textit{for } \beta \geq \gamma_0(\beta)/m.$$

Proof of Homogenization of diffusion in Coulomb environment $d = 2$

Proof: Recall $d = 2$. Then

$$dX_t^{\varepsilon,\mathsf{x}} = dB_t + \frac{\beta}{2}\frac{1}{\epsilon}b(\frac{1}{\epsilon}X_t^{\varepsilon,\mathsf{x}})dt,$$

$$dY_t^{\varepsilon,\mathsf{y}_\varepsilon} = dB_t + \frac{\beta}{2}\frac{1}{\epsilon}b(\frac{1}{\epsilon}Y_t^{\varepsilon,\mathsf{y}_\varepsilon})dt - \frac{\beta}{2}\frac{Y_t^{\varepsilon,\mathsf{y}_\varepsilon}}{|Y_t^{\varepsilon,\mathsf{y}_\varepsilon}|^2}dt$$

- Periodic homo + Girsanov formula
- Mosco convergence + Lower & Upper schemes of Dirichlet forms
- Chen-Croydon-Kumagai th + heat kernel estimates
- The origin is treated as "boundary". Uniqueness of Dirichlet forms
- The modulus process $|U|_t^{\mathsf{y}} = |U_t^{\mathsf{y}}|$ satisfies the Bessel SDE:

$$d|U|_t^{\mathsf{y}} = \sqrt{\frac{\gamma_0(\beta)}{2}}dB_t - \frac{\gamma_0(\beta)}{2}\frac{1}{2}(1 - \frac{2m\beta}{\gamma_0(\beta)})\frac{|U|_t^{\mathsf{y}}}{(|U|_t^{\mathsf{y}})^2}dt \tag{7}$$

$$|U|_0^{\mathsf{y}} = |\mathsf{y}|.$$

Here $\gamma_0(\beta) = \mathrm{trace}(\gamma_{\mathrm{eff}}^{\beta})$ and B is one-dimensional Brownian motion.

- Phase transition follows from (7).

$\square$

Homogenization of diffusion in Coulomb environment: $d \geq 3$

Thm 3. *Let $d \geq 3$. Assume initial starting points* y_ε *satisfy*

$$\lim_{\varepsilon \to 0} \mathrm{y}_\varepsilon = \mathrm{y}.$$

Then for each $0 \leq \beta < \infty$

$$\lim_{\epsilon \to 0} Y_t^{\varepsilon, \mathrm{y}_\varepsilon} = \sqrt{\gamma_{\mathrm{eff}}^{\beta}} B_t + \mathrm{y}.$$

Proof: • Rewrite SDE as

$$dY_t^{\varepsilon,\mathrm{y}_\varepsilon} = dB_t + \frac{1}{\varepsilon} b(\frac{Y_t^{\varepsilon,\mathrm{y}_\varepsilon}}{\varepsilon}) - \varepsilon^{d-2} \frac{\beta}{2} \sum_{i=1}^{m} \frac{Y_t^{\varepsilon,\mathrm{y}_\varepsilon} - \varepsilon t_i}{|Y_t^{\varepsilon,\mathrm{y}_\varepsilon} - \varepsilon t_i|^d} dt.$$

□

Remark: If we replace the Coulomb potentials by Ruelle's class potentials, then the limit is always non-degenerate for $d \geq 2$.

Homogenization of diffusion in random Coulomb environment: $d = 2$

- S: Configuration space over $\mathbb{R}^d$ (Unlabeled particle space)

$$\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\,;\ s_i \in \mathbb{R}^d,\ \mathsf{s}(|s| < r) < \infty\ (\forall r \in \mathbb{N})\}$$

- μ: point process (PP) on $\mathbb{R}^d$. i.e. prob meas. on S.
- A lattice $\mathbb{L}$ can be regarded as a periodic point process, and each site s_i is regarded as a particle.

- Let μ_{gin} be the Ginibre point process.
- μ_{gin} is a translation invariant point process on $\mathbb{R}^2$.
- μ_{gin} can be regarded as a Gibbs measure with interaction potential

$$-2\log|x|.$$

- Very loosely, μ_{gin} is a translation invariant *measure* on $(\mathbb{R}^2)^{\mathbb{N}}$ such as

$$\mu_{\mathrm{gin}}(\prod_{i\in\mathbb{N}} ds_i) = \frac{1}{\mathcal{Z}} \prod_{i<j}^{\infty} |s_i - s_j|^2 \prod_{k\in\mathbb{N}} ds_k$$

This is of course not rigorous at all.

Correlation functions & determinantal point proceses

- ρ^n is called the n-correlation function of μ w.r.t. Radon m. m if

$$\int_{A_1^{k_1}\times\cdots\times A_m^{k_m}} \rho^n(\mathbf{x}_n) \prod_{i=1}^{n} m(dx_i) = \int_{\mathsf{S}} \prod_{i=1}^{m} \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i)-k_i)!} d\mu$$

for any disjoint $A_i \in \mathcal{B}(S)$, $k_i \in \mathbb{N}$ s.t. $k_1 + \ldots + k_m = n$.

- μ is called the determinantal PP generated by (K, m) if its n-corraltion fun. ρ^n is given by

$$\rho^n(\mathbf{x}_n) = \det[K(x_i, x_j)]_{1\le i,j\le n} \tag{8}$$

- Ginibre PP $S = \mathbb{C}$. μ_{gin} is generated by (K, g)

$$K(x,y) = e^{x\bar{y}} \qquad g(dx) = \pi^{-1} e^{-|x|^2} dx$$

Log derivative of μ

- Let μ_x be the (reduced) Palm m. of μ conditioned at x

$$\mu_x(\cdot) = \mu(\cdot - \delta_x | \mathsf{s}(x) \geq 1)$$

- Let $\mu^{[1]}$ be the 1-Campbell measure on $\mathbb{R}^d \times \mathsf{S}$:

$$\mu^{[1]}(A \times B) = \int_A \rho^1(x) \mu_x(B) dx \tag{9}$$

- $\mathsf{d}_\mu \in L^1(\mathbb{R}^d \times \mathsf{S}, \mu^{[1]})$ is called the log derivative of μ if

$$\int_{\mathbb{R}^d \times \mathsf{S}} \nabla_x f d\mu^{[1]} = -\int_{\mathbb{R}^d \times \mathsf{S}} f \mathsf{d}_\mu d\mu^{[1]} \tag{10}$$

for all $f \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}$. Here ∇_x is the nabla on $\mathbb{R}^d$, $\mathcal{D}$ is the space of local smooth functions on S with compact support.

- Very informally

$$\mathsf{d}_\mu = \nabla_x \log \mu^{[1]}$$

The logarithmic derivative $\mathsf{d}_{\mu_{\mathsf{gin}}}$ of the Ginibre point process is

$$\mathsf{d}_{\mu_{\mathsf{gin}}}(x, \mathsf{s}) = \lim_{q \to \infty} \sum_{|x - s_i| < q} \frac{x - s_i}{|x - s_i|^2}, \qquad \text{where } \mathsf{s} = \sum_i \delta_{s_i}$$

Homogenization of diffusion in random Coulomb environment

- For a configuration $\mathsf{s} = \sum_i \delta_{s_i}$, we write $\mathbb{L}[\mathsf{s}] = \{s_i\}$.
- For $\mu_{\mathrm{gin},0}$-a.s. s we consider

$$dX_t^{\mathsf{x}} = dB_t + \lim_{q\to\infty} \sum_{\substack{|X_t^{\mathsf{x}} - s_i| < q \\ s_i \in \mathbb{L}[\mathsf{s}]}} \frac{X_t^{\mathsf{x}} - s_i}{|X_t^{\mathsf{x}} - s_i|^2} dt, \qquad X_0^{\mathsf{x}} = \mathsf{x}.$$

Thm 4. *For $\mu_{\mathrm{gin},0}$-a.s.* $\mathsf{s} \in \mathsf{S}$

$$\lim_{\varepsilon\to\infty} \varepsilon X_{t/\varepsilon^2}^{\mathsf{x}} = 0 \qquad \textit{weakly in } C([0,\infty);\mathbb{R}^2) \tag{11}$$

- The proof is completely different from the periodic case.
- We expect that the same phase transition holds for this case.
- I have not yet prove the positivity of the effective constant for the original Ginibre point process. That is, "for μ_{gin}-a.s. s".
- If this is done, then the rest is same as the periodic case.